Special Regularized HSS Iteration Method for Tikhonov Regularization

Jingjing Cui, Guohua Peng, Quan Lu, Zhengge Huang

Abstract—**In this paper, based on the regularized Hermitian and skew-Hermitian splitting (RHSS) iterative method, a speical RHSS (SRHSS) iterative method is established for solving the augmented system derived by Tikhonov regularization for ill-posed problems and image restoration. Then, we theoretically analyze the convergence properties of the SRHSS method. Moreover, optimal iteration parameters which minimize the spectral radius of the iteration matrix of the SRHSS method are also derived in detail. Numerical experiments on a Fredholm integral equation of the first kind and image restoration show that the SRHSS iteration method significantly outperforms the newly developed ones in iteration counts and computing times and image recovering quality.**

Index Terms—**regularized Hermitian and skew-Hermitian splitting, Tikhonov regularization, ill-posed problems, image restoration, iteration method.**

I. Introduction

WITH the rapid development of computer technology and multimedia technology, a large number of images are generated for information expression and transmission. Thus, to find a specific image and obtain the desired information effectively in a massive image dataset, some image techniques have been developed, such as image retrieval [36], image segmentation [23], [31], image restoration [18], [32], [35] and so on. Image restoration is one of the most fundamental issues in imaging science and plays an important role in many mid-level and high-level image processing applications. On account of the imperfection of an imaging system, a recorded image may be inevitably degraded during the process of image capture, transmission, and storage. It is well known that image restoration belongs to a general class of problems which are rigorously classified as ill-posed problems. Besides, ill-posed problems occur frequently enough in science and engineering, such as signal processing and Fredholm integral equations of the first kind [21], to make it worthwhile to provide efficient and numerically stable methods. In this paper, we consider the ill-conditioned linear system as follows

$$Af = g, \quad A \in R^{n^2 \times n^2}, \quad f, g \in R^{n^2}, \tag{1}$$

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arising from the discretization of some linear inverse problem [9] or of the linearized system of some nonlinear inverse problem [11], [6]. The matrix A has a large condition number and may be rank-deficient, and the right-hand side vector g typically is contaminated by an error, which can be expressed as

$$g = \hat{g} + e,$$

where $\hat{g}$ is an unknown error-free vector associated with g and e represents the error in g. We will refer to e as noise. In image restoration applications, f and A in (1) represent the desired true image and the blurring matrix whose structure depends on the discrete point spread function (PSF) and the used boundary conditions (BCs), respectively. $\hat{g}$ stands for an unavailable blur-contaminated, but noise-free, image, while g is an available image that has been contaminated by both blur and noise. The noise may stem from measurement and/or discretization errors. Throughout this paper, $\|\cdot\|$ denotes the Euclidean vector norm or the associated induced matrix norm.

The large condition number of the matrix A makes that the solution of ill-posed inverse problem is usually very sensitive to high-frequency perturbations in the measurement data g. So the straightforward least-squares solution of minimal Euclidean norm of (1) generally does not yield a meaningful approximate solution of the system (1). Rather, the ill-conditioning implies that standard methods in numerical linear algebra for solving (1), such as LU, Cholesky, or QR factorization, cannot be used in a straightforward manner to compute such a solution. To tackle the ill-posed nature of the problems, regularization techniques are usually used to obtain a stable and accurate solution [11]. A common approach to determine a useful approximate solution of (1) is to employ Tikhonov regularization [30], [33], [34], which converts the solution of the system (1) into the solution of the regularized least-squares system

$$\min_f \|Af - g\|_2^2 + \mu^2 \|Lf\|_2^2, \tag{2}$$

where constant $\mu > 0$ is the so-called regularization parameter (generally small, i.e., $0 < \mu < 1$) and the matrix L is typically either the identity matrix or a discrete approximation of the derivative operation. The solution of this system (2) is considered as an approximation of the solution of noise-free linear system $Af = \hat{g}$. In this work, we limit our discussion to L being the identity matrix. The other cases can be obtained by using the similar technique. As is well known, the solution of (2) with $L = I$ (the identity matrix) can be obtained by solving its normal equation

$$(A^T A + \mu^2 I) f = A^T g. \tag{3}$$

Moreover, the problem (3) can be equivalently transformed into the following $2n^2$-by-$2n^2$ augmented system

$$\begin{pmatrix} I & A \\ -A^T & \mu^2 I \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}, \tag{4}$$

where the variable e denotes the additive noise, i.e., $e = g - Af$. By recasting equivalently the original system (1), employing the Tikhonov regularization method, into the $2n^2$-by-$2n^2$ linear system (4), the behaviour of ill-conditioned of latter system can be greatly improved.

It can be seen that the linear system (4) is a non-Hermitian positive-definite system. Solving the non-Hermitian positive-definite system efficiently, Bai et al. in [5] initially proposed the Hermitian and skew-Hermitian splitting (HSS) method, and it was demonstrated that the HSS iteration method converges unconditionally to the unique solution of the linear system. Due to the elegant mathematical properties, the HSS iteration method has attracted much attention and there are many papers devoted to the various aspects of this method, see [7], [17], [25]. Recently, Lv et al. in [26] applied the HSS iteration method to the ill-posed image restoration problem and established a special HSS (SHSS) iterative method. Its convergence properties and the optimal value of the iteration parameter were discussed. Later, inspired by the idea of [26], Cheng et al. in [8] derived a new special HSS (NSHSS) iterative method and made comparisons between the proposed new method and the SHSS one. In order to improve the convergence rate of the HSS method, Benzi in [7] developed a generalization of the HSS (GHSS) iteration method. After that, Aghazadeh et al. [1] extended the idea of the GHSS method and introduced a restricted version of the GHSS (RGHSS) iterative method for image restoration. In [2], based on a new splitting of the Hermitian part of the coefficient matrix for the GHSS method, Aminikhah and Yousefi newly presented a new special generalized Hermitian and skew-Hermitian splitting (SGHSS) method for solving ill-posed inverse problems. Then, Fan et al. in [12] presented a class of upper and lower triangular (ULT) splitting iteration method, of which convergence rate and optimal iteration parameters were derived. Lately, Bai et al. in [4] proposed a class of regularized Hermitian and skew-Hermitian splitting (RHSS) methods for the solution of large and sparse linear systems in saddle-point form by introducing a Hermitian positive semidefinite matrix, which further improves the convergence behavior of the HSS iteration method. Besides, in [3], Bai et al. extended the RHSS iteration method for standard saddle-point problems to stabilized saddle-point problems and developed the corresponding unconditional convergence theory for the resulting methods, and also showed that the RHSS iteration method significantly outperforms the HSS one. On this account, enlightened by the aforementioned iterative methods, we apply the RHSS iteration method to image restoration and ill-posed problems and propose a speical regularized Hermitian and skew-Hermitian splitting (SRHSS) iterative method to further accelerate the convergence rate of the SHSS method. It is expected that the SRHSS method may converge faster than some existing ones. Moreover, we investigate the convergence properties and obtain the optimal parameters that minimize the spectral radius of the iteration matrix of the SRHSS method.

The arrangement of this paper is organized as follows. In Section II, the HSS, SHSS and RHSS iterative methods for solving linear systems are introduced briefly. A special regularized Hermitian and skew-Hermitian splitting (SRHSS) iterative method is presented in Section III. The convergence behavior of the SRHSS iterative method and its optimal parameters are also investigated analytically here. To demonstrate the efficiency of these proposed method, numerical experiments from the ill-posed problems and image restoration are provided in Section IV. Finally, in Section V we end this paper with some brief conclusions.

II. Brief descriptions of the HSS, SHSS and RHSS

Naturally, any matrix K can be split into as the form $K = H + S$, where $H = \frac{1}{2}(K + K^*)$ and $S = \frac{1}{2}(K - K^*)$ are Hermitian and skew-Hermitian parts of K, respectively, and K^* denotes the conjugate transpose of the matrix K. On the basis of the Hermitian and skew-Hermitian (HS) splitting, the HSS iteration method was first proposed by Bai et al. in [5] for solving non-Hermitian positive definite linear systems. By choosing an initial vector $x^{(0)}$, for $k = 1, 2, \cdots$ until convergence of $x^{(k)}$, the HSS iteration method is given as follows:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b \end{cases}, \tag{5}$$

where α a given positive constant. They proved that the HSS iteration method converges unconditionally to the unique solution of linear system. Moreover, they extended the HSS convergence theory for (non-Hermitian) positive definite matrices to a large class of positive semidefinite matrices. In 2013, based on the HSS iteration method, Lv et al. [26] established a special HSS (SHSS) iteration method by substituting $\alpha = 1$ into the second step of the HSS one to solve the system (4) for image restoration problem. The brief outline about the SHSS method is presented as follows. Let

$$K = \begin{pmatrix} I & A \\ -A^T & \mu^2 I \end{pmatrix}, x = \begin{pmatrix} e \\ f \end{pmatrix}, b = \begin{pmatrix} g \\ 0 \end{pmatrix}, \tag{6}$$

then the linear system (4) can be rewritten as $Kx = b$, where K is a non-Hermitian matrix. Then the matrix K can be split into its Hermitian and skew-Hermitian parts as

$$K = H + S = \begin{pmatrix} I & 0 \\ 0 & \mu^2 I \end{pmatrix} + \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix},$$

where the Hermitian part H is a special diagonal matrix and the skew-Hermitian part S has a special structure. Lv et al. in [26] made full use of the special structures of the Hermitian part H and the skew-Hermitian part S and proposed a special HSS (SHSS) iterative method for solving the augmented system (4) as follows:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b \\ (I + S)x^{(k+1)} = (I - H)x^{(k+\frac{1}{2})} + b \end{cases},$$

which improves the convergence rate of the HSS iteration method. The convergence properties of the SHSS method were investigated with a detailed theoretical analysis and the optimal parameter was found. In order to further improve the convergence behavior of the HSS iteration method for the saddle-point linear system, Bai et al. in [4] utilized the regularized Hermitian and skew-Hermitian (RHS) splitting and established a regularized HSS (RHSS) iteration method by introducing a Hermitian positive semidefinite matrix, called the regularization matrix, in the HS splitting. For the following linear system in saddle-point form

$$\begin{pmatrix} B & E \\ -E^* & 0 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{7}$$

where $B \in C^{p\times p}$ and $E \in C^{p\times q}$ are Hermitian positive definite matrix and rectangular matrix of full column rank, respectively, the RHS splitting is as follows:

$$\begin{aligned}\begin{pmatrix} B & E \\ -E^* & 0 \end{pmatrix} &= \begin{pmatrix} B & 0 \\ 0 & Q \end{pmatrix} + \begin{pmatrix} 0 & E \\ -E^* & -Q \end{pmatrix} = H_+ + S_- \\ &= \begin{pmatrix} B & 0 \\ 0 & -Q \end{pmatrix} + \begin{pmatrix} 0 & E \\ -E^* & Q \end{pmatrix} = H_- + S_+\end{aligned}$$

with a given Hermitian positive semidefinite matrix Q. Note that when $Q = 0$, the RHS splitting automatically reduces to the HS splitting. Based on the RHS splitting, authors led to an equivalent reformulation, which is called RHSS iterative method, for the saddle-point linear system. They proved that the RHSS iteration method converges unconditionally to the unique solution of the saddle-point linear system. Subsequently, in [3] Bai et al. extended the RHSS iteration method and established the RHSS iterative method for stabilized saddle-point problems. Numerical experiments verified that the RHSS method significantly outperforms the HSS method in terms of both iteration steps and computing times and can be a useful tool for solving certain types of large sparse stabilized saddle-point problems.

Note that the RHSS iteration method is a valuable development and quality improvement of the HSS one, and with numerical experiments it was shown that the RHSS iteration method can be more efficient and robust than the HSS one. Based on the idea of the RHSS iteration method and motivated by [4] and [3], we develop a new special splitting of the coefficient matrix K of the augmented system (4) and propose a speical regularized Hermitian and skew-Hermitian splitting (SRHSS) iterative method for solving the system (4). It is expected that the SRHSS method may converge faster than some exiting ones. Moreover, the iteration matrix of the SRHSS iterative method has some similar properties with that of the SHSS one, for example, the iteration matrix of the SRHSS method has at least n^2 zero eigenvalues and the remaining n^2 eigenvalues of the iteration matrix are determined by the parameter α and the regularization matrix Q. It is convenient to select the appropriate α and Q to make the spectral radius of iteration matrix of the SRHSS method smaller. For details, see below.

III. The special RHSS method and its convergence analysis

In this section, we derive the speical RHSS (SRHSS) iteration method for solving the augmented system (4). The convergence properties of the SRHSS method are analyzed and its optimal iteration parameters are given. Inspired by the ideas of [4], we first develop a new splitting of the coefficient matrix K of the augmented system (4). To this end, for a given symmetric positive definite matrix $Q \in R^{n^2\times n^2}$ we can split the coefficient matrix K of the augmented system (4), obtaining the speical regularized Hermitian and skew-Hermitian (SRHS) splitting:

$$\begin{aligned} K &= \begin{pmatrix} I & 0 \\ 0 & \mu^2 I + Q \end{pmatrix} + \begin{pmatrix} 0 & A \\ -A^T & -Q \end{pmatrix} = H_1 + S_1 \\ &= \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix} + \begin{pmatrix} 0 & A \\ -A^T & \mu^2 I - Q \end{pmatrix} = H_2 + S_2. \end{aligned} \tag{8}$$

Here the matrix Q plays a regularization role in the splitting (8), so it is called the regularization matrix. It can be observed from the splitting forms (8) that the first one in (8) is the same with the one of the RHSS method with $\omega = 1$ for stabilized saddle-point problems [3]. Whereas, the second one in (8) is different from that of (2.1) in [3]. Such the splitting may lead to establish a special iterative method for (4) and the spectral radius of its iteration matrix may be smaller. Based on the SRHS splitting, in this paper we present a special regularized Hermitian and skew-Hermitian splitting iterative method to solve (4). Utilizing the idea of the SHSS iterative method in [26], the splitting (8) of the matrix K naturally leads to equivalent reformulations of the augmented system (4) as following:

$$\begin{cases} (\alpha I + H_1)x^{(k+\frac{1}{2})} = (\alpha I - S_1)x^{(k)} + b \\ (I + S_2)x^{(k+1)} = (I - H_2)x^{(k+\frac{1}{2})} + b \end{cases}, \tag{9}$$

where $\alpha > 0$ is a prescribed iteration parameter and I is the identity matrix. The iteration method (9) is referred to as the special regularized Hermitian and skew-Hermitian (SRHSS) iteration method. Note that H_2 has n^2 ones in the diagonal and $I - H_2$ must have n^2 zeros in the diagonal. This will make that the iteration matrix of the SRHSS iteration method has at least n^2 zero eigenvalues and the remaining n^2 eigenvalues of the iteration matrix are determined by the parameter α and regularization matrix Q, which may make the spectral radius of the iteration matrix smaller. And the SRHSS method can have a fast convergence rate by choosing the appropriate parameter α and the matrix Q.

The SRHSS iteration method (9) can be rewritten as a standard stationary iteration scheme as follows

$$x^{(k+1)} = L(\alpha; Q)x^{(k)} + c, \ k = 0, 1, 2, \cdots,$$

where

$$\begin{cases} L(\alpha; Q) = (I + S_2)^{-1}(I - H_2)(\alpha I + H_1)^{-1}(\alpha I - S_1) \\ c = (I + S_2)^{-1}[(1 + \alpha)I + H_1 - H_2](\alpha I + H_1)^{-1}b \end{cases}. \tag{10}$$

Note that $L(\alpha; Q)$ is the iteration matrix of the SRHSS method. We know that the SRHSS method is convergent if and only if the spectral radius of its iteration matrix $L(\alpha; Q)$ is less than one (i.e., $\rho(L(\alpha; Q)) < 1$). Next, we study the convergence of the SRHSS iteration method and derive its optimal parameters minimizing the spectral radius of the iteration matrix. Firstly, the eigenvalues of the iteration matrix of the iteration scheme (9) are given as follows:

Theorem 3.1: Let $K \in R^{2n^2\times 2n^2}$ be defined as in (6) and α be a positive scalar. Assume $Q \in R^{n^2\times n^2}$ is a symmetric positive definite matrix such that this matrix $(1 + \mu^2)I + A^TA - Q$ is symmetric positive definite. If λ is an eigenvalue of the iteration matrix $L(\alpha; Q)$ of the SRHSS method, then $\lambda = 0$ with algebraic multiplicity at least n^2, and other n^2 eigenvalues of the matrix $L(\alpha; Q)$ are those of the matrix Ψ, where $\Psi = (I - Q)\left[(\alpha + \mu^2)I + Q\right]^{-1}\left(\alpha I + Q - A^TA\right)\left[(1 + \mu^2)I + A^TA - Q\right]^{-1}$.

Proof. Since $L(\alpha;Q)$ defined as in (10) is similar to

$$\hat{L}(\alpha;Q) = (I-H_2)(\alpha I+H_1)^{-1}(\alpha I-S_1)(I+S_2)^{-1}.$$

That is, $\hat{L}(\alpha;Q)$ and $L(\alpha;Q)$ have same spectrum. Next, we only need to consider the spectral radius of the matrix $\hat{L}(\alpha;Q)$. After some calculations, it follows that

$$I-H_2 = \begin{pmatrix} 0 & 0 \\ 0 & I-Q \end{pmatrix},$$
$$(\alpha I+H_1)^{-1} = \begin{pmatrix} \frac{1}{\alpha+1}I & 0 \\ 0 & \left[(\alpha+\mu^2)I+Q\right]^{-1} \end{pmatrix},$$
$$\alpha I - S_1 = \begin{pmatrix} \alpha I & -A \\ A^T & \alpha I+Q \end{pmatrix},$$
$$(I+S_2)^{-1} = \begin{pmatrix} I-A\Phi^{-1}A^T & -A\Phi^{-1} \\ \Phi^{-1}A^T & \Phi^{-1} \end{pmatrix},$$

where $\Phi = (1+\mu^2)I + A^TA - Q$. From the above formulas, one may deduce the following equation

$$\begin{aligned}
&\hat{L}(\alpha;Q)\\
&= (I-H_2)(\alpha I+H_1)^{-1}(\alpha I-S_1)(I+S_2)^{-1}\\
&= \begin{pmatrix} 0 & 0 \\ 0 & I-Q \end{pmatrix}\begin{pmatrix} \frac{1}{\alpha+1}I & 0 \\ 0 & \left[(\alpha+\mu^2)I+Q\right]^{-1} \end{pmatrix}\\
&\quad\times\begin{pmatrix} \alpha I & -A \\ A^T & \alpha I+Q \end{pmatrix}\begin{pmatrix} I-A\Phi^{-1}A^T & -A\Phi^{-1} \\ \Phi^{-1}A^T & \Phi^{-1} \end{pmatrix}\\
&= \begin{pmatrix} 0 & 0 \\ 0 & (I-Q)[(\alpha+\mu^2)I+Q]^{-1} \end{pmatrix}\\
&\quad\times\begin{pmatrix} \alpha I-(\alpha+1)A\Phi^{-1}A^T & -(\alpha+1)A\Phi^{-1} \\ A^T+\Upsilon\Phi^{-1}A^T & \Upsilon\Phi^{-1} \end{pmatrix}\\
&= \begin{pmatrix} 0 & 0 \\ \Theta & \Psi \end{pmatrix},
\end{aligned}$$

where $\Psi = (I-Q)\left[(\alpha+\mu^2)I+Q\right]^{-1}\left(\alpha I+Q-A^TA\right)\left[(1+\mu^2)I+A^TA-Q\right]^{-1}$ and $\Upsilon = -A^TA+\alpha I+Q$. Here we do not need to write the precise form of Θ because it is not the focus in the following argument. From the structure of the matrix $\hat{L}(\alpha;Q)$ and the similarity invariance of the matrix spectrum, it can be seen that $\lambda=0$ is the eigenvalue of the iteration matrix $L(\alpha;Q)$ with algebraic multiplicity at least n^2, and other n^2 eigenvalues of the matrix $L(\alpha;Q)$ are those of the matrix Ψ. This completes the proof.

With Theorem 3.1, we can get the following important theorem which shows the convergence of the SRHSS iteration method with the regularization matrix $Q=sI$ for solving the augmented system (4) and derives the optimal parameters minimizing the spectral radius of the iteration matrix $L(\alpha;Q)$. In particular, we choose the regularization matrix $Q=sI$ with $0<s<1+u^2$ and $s\neq 1$. The condition $0<s<1+u^2$ ensures that the matrix $(1+\mu^2)I+A^TA-Q$ is symmetric positive definite. And if $s=1$, then $I-H_2=0$, and the second equation of (9) is solved without $x^{(k+\frac{1}{2})}$. So for $Q=sI$, we always assume that $0<s<1+u^2$ and $s\neq 1$.

Theorem 3.2: Let $K\in R^{2n^2\times 2n^2}$ be defined as in (6), σ_i $(i=1,2,\cdots,n^2)$ be the singular values of A with $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_{n^2}$ and regularization matrix $Q=sI$ in (9) with $0<s<1+u^2$ and $s\neq 1$, then iteration matrix $L(\alpha;Q)$ has zero eigenvalues of algebraic multiplicity at least n^2, and the remaining n^2 eigenvalues satisfy the relation

$$\frac{(1-s)(\alpha+s-\sigma_i^2)}{(\alpha+\mu^2+s)(1+\mu^2-s+\sigma_i^2)},\ i=1,2,\cdots,n^2.$$

In addition, iteration matrix $L(\alpha;Q)$ has the following properties:

(i) If $s\leq\sigma_{n^2}^2$, we have $\rho(L(\alpha;Q))<1$ with the parameters α and s lying on the following regions

$$(\alpha,s)\in\bigcup_{i=1}^{4}D_i,$$

where

$$\begin{aligned}
D_1 &:= \{(\alpha,s)|0<\alpha\leq\hat{\alpha}(s), 0<s<1, f_1(\alpha,s)<0\},\\
D_2 &:= \{(\alpha,s)|0<\alpha\leq\hat{\alpha}(s), 1<s<1+\mu^2\},\\
D_3 &:= \{(\alpha,s)|\alpha\geq\hat{\alpha}(s), 1<s<1+\mu^2, f_2(\alpha,s)<0\},\\
D_4 &:= \{(\alpha,s)|\alpha\geq\hat{\alpha}(s), 0<s<1\}
\end{aligned}$$

with the functions $\hat{\alpha}(s)$, $f_1(\alpha,s)$ and $f_2(\alpha,s)$ being defined by

$$\hat{\alpha}(s) = \frac{(1+u^2-2s)(\sigma_1^2+\sigma_{n^2}^2)+2\sigma_1^2\sigma_{n^2}^2-2s(1+u^2-s)}{2(1+u^2-s)+\sigma_1^2+\sigma_{n^2}^2},$$
$$f_1(\alpha,s) = (1-s)(\sigma_1^2-2\alpha-2s-\mu^2)-(\alpha+\mu^2+s)(\mu^2+\sigma_1^2),$$
$$f_2(\alpha,s) = (s-1)(2\alpha+2s+\mu^2-\sigma_{n^2}^2)-(\alpha+\mu^2+s)(\mu^2+\sigma_{n^2}^2).$$

In this case, the optimal parameter α^* which minimizes the spectral radius $L(\alpha;Q)$ is

$$\begin{aligned}
&\alpha^* = \hat{\alpha}(s^*)\\
&= \frac{(1+u^2-2s^*)(\sigma_1^2+\sigma_{n^2}^2)+2\sigma_1^2\sigma_{n^2}^2-2s^*(1+u^2-s^*)}{2(1+u^2-s^*)+\sigma_1^2+\sigma_{n^2}^2}.
\end{aligned}$$

(ii) If $\sigma_1^2\leq s$. When $0<s<1$, we obtain $\rho(L(\alpha;Q))<1$ for $\forall\alpha>0$. Otherwise, for $1<s<1+\mu^2$, we have $\rho(L(\alpha;Q))<1$ with the parameters α and s satisfying $f_2(\alpha,s)<0$. In this case, the optimal parameter $\alpha^*\to 0$.

(iii) If $\sigma_{n^2}^2\leq s\leq\sigma_1^2$. As for $\hat{\alpha}(s)>0$ and $\hat{\alpha}(s)\leq 0$, the convergent conditions and optimal iteration parameter α are same with those in case (i) and case (ii), respectively.

Furthermore, the optimal parameter s^* of the SRHSS method should be $s^*\to 1$.

Proof. From Theorem 3.1, it is not difficult to verify that $L(\alpha;Q)$ has n^2 zero eigenvalues and the other n^2 eigenvalues satisfy the relation

$$\frac{(1-s)(\alpha+s-\sigma_i^2)}{(\alpha+\mu^2+s)(1+\mu^2-s+\sigma_i^2)}$$

with σ_i $(i=1,2,\cdots,n^2)$ being the singular values of the matrix A. It follows from the above expression of the eigenvalues that the spectral radius of the iterative matrix $L(\alpha;Q)$ is as follows

$$\rho(L(\alpha;Q)) = \frac{|1-s|}{\alpha+\mu^2+s}\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2}, \quad (11)$$

where $\sigma(A)$ denotes the set of the singular values of the matrix A. Next, we study the conditions of α and s such that $\rho(L(\alpha;Q))<1$ by distinguishing three cases as follows.

(i) If $0<s\leq\sigma_{n^2}^2$, then $s-\sigma_i^2\leq 0, i=1,2,\cdots,n^2$. It follows from the properties of the function that there exists an $\hat{\alpha}(s)$ such that

$$\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2} = \begin{cases} \frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2}, & \alpha\leq\hat{\alpha}(s),\\ \frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}, & \alpha\geq\hat{\alpha}(s).\end{cases}$$

From (11), we get

$$\rho(L(\alpha;Q)) = \begin{cases} \frac{|1-s|}{\alpha+\mu^2+s}\frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2}, & \alpha \le \hat{\alpha}(s), \\ \frac{|1-s|}{\alpha+\mu^2+s}\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}, & \alpha \ge \hat{\alpha}(s). \end{cases} \quad (12)$$

We divide the region $D = \{(\alpha,s)|\alpha > 0, 0 < s < 1+\mu^2; s \ne 1\}$ into four subregions $D = \bigcup_{i=1}^{4} \bar{D}_i$, where

$$\begin{aligned} \bar{D}_1 &:= \{(\alpha,s)|0<\alpha\le\hat{\alpha}(s), 0<s<1\}, \\ \bar{D}_2 &:= \{(\alpha,s)|0<\alpha\le\hat{\alpha}(s), 1<s<1+\mu^2\}, \\ \bar{D}_3 &:= \{(\alpha,s)|\alpha\ge\hat{\alpha}(s), 1<s<1+\mu^2\}, \\ \bar{D}_4 &:= \{(\alpha,s)|\alpha\ge\hat{\alpha}(s), 0<s<1\}. \end{aligned}$$

From the form of the function $\rho(L(\alpha;Q))$, we have
(I) For $(\alpha,s) \in D_1$, we deduce that $\rho(L(\alpha;Q)) = \frac{(1-s)(\sigma_1^2-\alpha-s)}{(\alpha+\mu^2+s)(1+\mu^2-s+\sigma_1^2)}$. It follows from straightforward derivations that $\rho(L(\alpha;Q)) < 1$ can be simplified to

$$\begin{aligned} f_1(\alpha,s) &= (1-s)(\sigma_1^2-2\alpha-2s-\mu^2) \\ &\quad -(\alpha+\mu^2+s)(\mu^2+\sigma_1^2) < 0. \end{aligned}$$

(II) For $(\alpha,s) \in D_2$, in terms of (12), it is not difficult to check that

$$\rho(L(\alpha;Q)) = \frac{s-1}{s+\alpha+\mu^2}\frac{\sigma_1^2-\alpha-s}{\sigma_1^2+1+\mu^2-s} < 1.$$

(III) For $(\alpha,s) \in D_3$, we can easily express (12) as $\rho(L(\alpha;Q)) = \frac{(s-1)(\alpha+s-\sigma_{n^2}^2)}{(\alpha+\mu^2+s)(1+\mu^2-s+\sigma_{n^2}^2)}$. By direct computations, $\rho(L(\alpha;Q)) < 1$ directly leads to the following result

$$\begin{aligned} f_2(\alpha,s) &= (s-1)(2\alpha+2s+\mu^2-\sigma_{n^2}^2) \\ &\quad -(\alpha+\mu^2+s)(\mu^2+\sigma_{n^2}^2) < 0. \end{aligned}$$

(IV) For $(\alpha,s) \in D_4$, $\rho(L(\alpha;Q)) = \frac{(1-s)(\alpha+s-\sigma_{n^2}^2)}{(\alpha+\mu^2+s)(1+\mu^2-s+\sigma_{n^2}^2)} < 1$ is equivalent to

$$-(\mu^2+\sigma_{n^2}^2)(1+\alpha+\mu^2) < 0,$$

which holds naturally.

According to (12), $\rho(L(\alpha;Q))$ can be given as follows:

$$\begin{aligned} \rho(L(\alpha;Q)) = \max \Bigg\{ &\frac{|1-s|}{\alpha+\mu^2+s}\frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2}, \\ &\frac{|1-s|}{\alpha+\mu^2+s}\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2} \Bigg\}. \end{aligned}$$

It is well known that the convergence rate of the SRHSS iteration method with $Q = sI$ is determined by the spectral radius of its iteration matrix. Hence it makes sense to choose the parameters to minimize the spectral radius of iteration matrix $L(\alpha;Q)$. If optimal parameter α^* is such minimum point, then it should satisfy the following equation

$$\frac{|1-s|}{\alpha+\mu^2+s}\frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2} = \frac{|1-s|}{\alpha+\mu^2+s}\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}.$$

The solution α^* of the above equation is given as follows:

$$\alpha^* = \frac{(1+u^2-2s)(\sigma_1^2+\sigma_{n^2}^2)+2\sigma_1^2\sigma_{n^2}^2-2s(1+u^2-s)}{2(1+u^2-s)+\sigma_1^2+\sigma_{n^2}^2}. \quad (13)$$

Now we validate that $\alpha^* > 0$ in (13). It can be seen that the denominator $2(1+u^2-s)+\sigma_1^2+\sigma_{n^2}^2 > 0$ under the condition $s < 1+\mu^2$. Inasmuch as $\sigma_1^2 \ge \sigma_{n^2}^2 \ge s$, we derive

$$\begin{aligned} &(1+u^2-2s)(\sigma_1^2+\sigma_{n^2}^2)+2\sigma_1^2\sigma_{n^2}^2-2s(1+u^2-s) \\ &\ge 2s(1+u^2-2s)+2s^2-2s(1+u^2-s) \\ &= 0, \end{aligned}$$

which shows $\alpha^* > 0$ in (13).

(ii) If $\sigma_1^2 \le s$, then $s-\sigma_i^2 \ge 0, i = 1,2,\cdots n^2$. In this case, it is not difficult to check that

$$\begin{aligned} \rho(L(\alpha;Q)) &= \frac{|1-s|}{\alpha+\mu^2+s}\max_{\sigma_i\in\sigma(A)}\frac{\alpha+s-\sigma_i^2}{1+\mu^2-s+\sigma_i^2} \\ &= \frac{|1-s|}{\alpha+\mu^2+s}\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}. \end{aligned} \quad (14)$$

When $0 < s < 1$, from the above expression, we have $\rho(L(\alpha;Q)) = \frac{1-s}{\alpha+\mu^2+s}\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}$. It can be seen that $\rho(L(\alpha;Q)) < 1$ always holds for $\forall\alpha > 0$. If $1 < s < 1+\mu^2$, the spectral radius of the matrix $L(\alpha;Q)$ is $\rho(L(\alpha;Q)) = \frac{s-1}{\alpha+\mu^2+s}\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}$. Similar to the derivations in (III) of case (i), $\rho(L(\alpha;Q)) < 1$ if and only if the parameters α and s satisfy $f_2(\alpha,s) < 0$, where $f_2(\alpha,s)$ is defined as in case (i).

Now, the optimal parameter α^* is discussed. It follows from (14) that

$$\begin{aligned} \alpha^* &= \arg\min_{\alpha}\rho(L(\alpha;Q)) \\ &= \frac{|1-s|}{1+\mu^2-s+\sigma_{n^2}^2}\arg\min_{\alpha}\frac{\alpha+s-\sigma_{n^2}^2}{\alpha+\mu^2+s} \\ &= \frac{|1-s|}{1+\mu^2-s+\sigma_{n^2}^2}\arg\min_{\alpha}\left\{1-\frac{\mu^2+\sigma_{n^2}^2}{\alpha+\mu^2+s}\right\}. \end{aligned}$$

Then global optimal parameter α^* can be found, that is, $\alpha^* \to 0$. The optimal parameter $\alpha^* \to 0$ can be achieved for the situation $0 < s < 1$ because the convergence interval of the parameter α is $(0,+\infty)$. And as $1 < s < 1+\mu^2$, for a fixed sufficiently small α we can select parameter s such that the adopted parameters α and s satisfy $f_2(\alpha,s) < 0$.

(iii) If $\sigma_{n^2}^2 \le s \le \sigma_1^2$. Without loss of generality, we assume that there exists an k such that $s-\sigma_1^2 \le \cdots \le s-\sigma_k^2 \le 0 < s-\sigma_{k+1}^2 \le \cdots \le s-\sigma_{n^2}^2$. Similar to discussion of the case (i), for $i \le k$, there exists an $\hat{\alpha}_k(s)$ such that

$$\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2} = \begin{cases} \frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2}, & \alpha\le\hat{\alpha}_k(s), \\ \frac{\alpha+s-\sigma_k^2}{1+\mu^2-s+\sigma_k^2}, & \alpha\ge\hat{\alpha}_k(s), \end{cases} \quad (15)$$

where $\hat{\alpha}_k(s) = \frac{(1+u^2-2s)(\sigma_1^2+\sigma_k^2)+2\sigma_1^2\sigma_k^2-2s(1+u^2-s)}{2(1+u^2-s)+\sigma_1^2+\sigma_k^2}$. Furthermore, for $i \ge k+1$, it can be seen that

$$\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2} = \frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}. \quad (16)$$

Therefore, for $\alpha \le \hat{\alpha}_k(s)$, combining (15) and (16) yeilds

$$\begin{aligned} &\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2} \\ &= \max\left\{\frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2}, \frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}\right\}. \end{aligned} \quad (17)$$

As for $\hat{\alpha}(s)>0$, in terms of (17), we obtain that

$$\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2} = \begin{cases}\frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2}, & \alpha\le\hat{\alpha}(s),\\ \frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}, & \hat{\alpha}(s)\le\alpha\le\hat{\alpha}_k(s).\end{cases} \quad (18)$$

The fact $\hat{\alpha}(s)\le\hat{\alpha}_k(s)$ is easy validated and we omit it here. If $\hat{\alpha}(s)\le 0$, it follows from (17) that

$$\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2}=\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}, 0<\alpha\le\hat{\alpha}_k(s). \quad (19)$$

When $\alpha\ge\hat{\alpha}_k(s)$, by virtue of (15) and (16), it holds that

$$\begin{aligned}&\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2}\\ &=\max\left\{\frac{\alpha+s-\sigma_k^2}{1+\mu^2-s+\sigma_k^2},\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}\right\}\\ &=\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}. \end{aligned}\quad (20)$$

So, for $\hat{\alpha}(s)>0$, it follows from (18) and (20) that

$$\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2}=\begin{cases}\frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2}, & \alpha\le\hat{\alpha}(s),\\ \frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}, & \alpha\ge\hat{\alpha}(s),\end{cases}$$

consequently,

$$\rho(L(\alpha;Q))=\begin{cases}\frac{|1-s|}{\alpha+\mu^2+s}\frac{\sigma_1^2-(\alpha+s)}{1+\mu^2-s+\sigma_1^2}, & \alpha\le\hat{\alpha}(s),\\ \frac{|1-s|}{\alpha+\mu^2+s}\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}, & \alpha\ge\hat{\alpha}(s).\end{cases}$$

In the case, the convergent conditions and optimal iteration parameter α are same with those in case (i). And for $\hat{\alpha}(s)\le 0$, the form is given by

$$\max_{\sigma_i\in\sigma(A)}\frac{|\alpha+s-\sigma_i^2|}{1+\mu^2-s+\sigma_i^2}=\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2},$$

by virtue of (19) and (20). In this case $\rho(L(\alpha;Q))=\frac{|1-s|}{\alpha+\mu^2+s}\frac{\alpha+s-\sigma_{n^2}^2}{1+\mu^2-s+\sigma_{n^2}^2}$, so the convergent conditions and optimal iteration parameter α are same with those in case (ii).

Furthermore, it follows from the expression of the spectral radius $\rho(L(\alpha;Q))$ in (11) that the optimal parameter s^* of the SRHSS method should be $s^*\to 1$, making the spectral radius $\rho(L(\alpha;Q))$ fully small to obtain a rapid convergence rate. By the above analysis, the conclusions of this theorem is obtained.

As in the Theorem 3.2, the convergence results and the optimal parameters of the SRHSS iteration method are given by the following theorem when the parameter matrix is defined as $Q=sI+A^TA$ with $0<s<1+u^2$, ensuring that the matrix $(1+\mu^2)I+A^TA-Q$ is symmetric positive definite.

Theorem 3.3: Let $K\in R^{2n^2\times 2n^2}$ be defined as in (6), σ_i $(i=1,2,\cdots,n^2)$ be the singular values of A with $\sigma_1\ge\sigma_2\ge\cdots\ge\sigma_{n^2}$ and regularization matrix in (9) be chosen as $Q=sI+A^TA$ with $0<s<1+u^2$. Then iteration matrix $L(\alpha;Q)$ has zero eigenvalues of algebraic multiplicity at least n^2, and the remaining n^2 eigenvalues satisfy the relation

$$\frac{(1-s-\sigma_i^2)(\alpha+s)}{(\alpha+\mu^2+s+\sigma_i^2)(1+\mu^2-s)}, i=1,2,\cdots,n^2.$$

In addition, iteration matrix $L(\alpha;Q)$ has the following properties:
(i) If $0<s\le 1-\sigma_1^2$, then $\rho(L(\alpha;Q))<1$ holds for $\forall\alpha>0$.
(ii) If $s\ge 1-\sigma_{n^2}^2$, $\rho(L(\alpha;Q))<1$ if one of the following conditions holds true

- $s\in\left[1-\sigma_{n^2}^2,\frac{2+\mu^2-\sigma_1^2}{2}\right]$ and $\alpha\in(0,+\infty)$ or $s\in\left(\frac{2+\mu^2-\sigma_1^2}{2},1+u^2\right)$ with $(1+\mu^2)(\mu^2+\sigma_1^2)-2s(\sigma_1^2+s-1)>0$ and $\alpha\in(0,h(s))$ as $\mu^2+2\sigma_{n^2}^2-\sigma_1^2\ge 0$;
- $s\in\left[1-\sigma_{n^2}^2,1+u^2\right)$ with $(1+\mu^2)(\mu^2+\sigma_1^2)-2s(\sigma_1^2+s-1)>0$ and $\alpha\in(0,h(s))$ as $\mu^2+2\sigma_{n^2}^2-\sigma_1^2\le 0$,

where $h(s)=\frac{(1+\mu^2)(\mu^2+\sigma_1^2)-2s(\sigma_1^2+s-1)}{\sigma_1+s-2-\mu^2}$.
(iii) If $1-\sigma_1^2\le s\le 1-\sigma_{n^2}^2$, $\rho(L(\alpha;Q))<1$ holds true if and only if the conditions in the cases (i) and (ii) are both satisfied.

Besides, the global optimal parameters α^* and s^* of the SRHSS method with $Q=sI+A^TA$ are given by $\alpha^*\to 0$ and $s^*\to 0$.

Proof. It follows from Theorem 3.1 that $L(\alpha;Q)$ has n^2 zero eigenvalues and the other n^2 eigenvalues are as follows:

$$\frac{(1-s-\sigma_i^2)(\alpha+s)}{(\alpha+\mu^2+s+\sigma_i^2)(1+\mu^2-s)},\ i=1,2,\cdots,n^2,$$

in consequence, we have

$$\rho(L(\alpha;Q))=\frac{\alpha+s}{1+\mu^2-s}\max_{\sigma_i\in\sigma(A)}\frac{|1-s-\sigma_i^2|}{\alpha+\mu^2+s+\sigma_i^2}, \quad (21)$$

where $\sigma(A)$ denotes the set of the singular values of the matrix A. To prove the convergence, we have to show that $\rho(L(\alpha;Q))<1$.
(i) Now if $1-\sigma_1^2\ge s$, then $1-s-\sigma_i^2\ge 0$ $(i=1,2,\cdots,n^2)$, and we immediately have

$$\begin{aligned}\rho(L(\alpha;Q)) &= \frac{\alpha+s}{1+\mu^2-s}\max_{\sigma_i\in\sigma(A)}\frac{1-s-\sigma_i^2}{\alpha+\mu^2+s+\sigma_i^2}\\ &=\frac{\alpha+s}{1+\mu^2-s}\frac{1-s-\sigma_{n^2}^2}{\alpha+\mu^2+s+\sigma_{n^2}^2}\\ &=\frac{\alpha+s}{\alpha+\mu^2+s+\sigma_{n^2}^2}\frac{1-s-\sigma_{n^2}^2}{1+\mu^2-s}.\end{aligned}$$

It's easy to verify $\rho(L(\alpha;Q))<1$ for $\forall\alpha>0$.
(ii) Now if $s\ge 1-\sigma_{n^2}^2$, it holds that $1-s-\sigma_i^2\le 0$ $(i=1,2,\cdots,n^2)$ and therefore

$$\begin{aligned}\rho(L(\alpha;Q)) &= \frac{\alpha+s}{1+\mu^2-s}\max_{\sigma_i\in\sigma(A)}\frac{\sigma_i^2+s-1}{\alpha+\mu^2+s+\sigma_i^2}\\ &=\frac{\alpha+s}{1+\mu^2-s}\frac{\sigma_1^2+s-1}{\alpha+\mu^2+s+\sigma_1^2}.\end{aligned}\quad (22)$$

Setting $\rho(L(\alpha;Q))<1$ leads to the following equivalent condition:

$$(\alpha+s)(\sigma_1^2+2s-2-\mu^2)-(1+\mu^2-s)(\mu^2+\sigma_1^2)<0. \quad (23)$$

To solve the above equation in terms of α, we should consider the sign of the coefficient $\sigma_1^2+2s-2-\mu^2$. When $\sigma_1^2+2s-2-\mu^2\le 0$, i.e., $s\le\frac{2+\mu^2-\sigma_1^2}{2}$, then Inequality (23) holds true for all $\alpha>0$ due to $1+\mu^2-s>0$. Therefore, if $s\ge 1-\sigma_{n^2}^2$ and $s\le\frac{2+\mu^2-\sigma_1^2}{2}$, we have $\rho(L(\alpha;Q))<1$ for $\forall\alpha>0$. The condition $\mu^2+2\sigma_{n^2}^2-\sigma_1^2\ge 0$ can result in $1-\sigma_{n^2}^2\le\frac{2+\mu^2-\sigma_1^2}{2}$. That is, under the condition $\mu^2+2\sigma_{n^2}^2-\sigma_1^2\ge 0$, if $1-\sigma_{n^2}^2\le s\le\frac{2+\mu^2-\sigma_1^2}{2}$, then $\rho(L(\alpha;Q))<1$ holds true for $\forall\alpha>0$.

When $s > \frac{2+\mu^2-\sigma_1^2}{2}$, then $\sigma_1^2 + 2s - 2 - \mu^2 > 0$. Solving for α in (23) leads to

$$\alpha < \frac{(1+\mu^2)(\mu^2+\sigma_1^2) - 2s(\sigma_1^2+s-1)}{\sigma_1+s-2-\mu^2} := h(s). \quad (24)$$

Keep in mind $\alpha > 0$, so $\rho(L(\alpha;Q)) < 1$ holds for α in (24) under the condition $(1+\mu^2)(\mu^2+\sigma_1^2)-2s(\sigma_1^2+s-1) > 0$. Note that $s > \frac{2+\mu^2-\sigma_1^2}{2}, s \geq 1-\sigma_{n^2}^2$ and $s < 1+\mu^2$. Thus, if $\mu^2 + 2\sigma_{n^2}^2 - \sigma_1^2 \geq 0$, for $\frac{2+\mu^2-\sigma_1^2}{2} < s < 1+\mu^2$ satisfying $(1+\mu^2)(\mu^2+\sigma_1^2) - 2s(\sigma_1^2+s-1) > 0$ and $0 < \alpha < h(s)$, we have $\rho(L(\alpha;Q)) < 1$. Otherwise, if $\mu^2 + 2\sigma_{n^2}^2 - \sigma_1^2 \leq 0$, for $1-\sigma_{n^2}^2 \leq s < 1+\mu^2$ satisfying $(1+\mu^2)(\mu^2+\sigma_1^2) - 2s(\sigma_1^2+s-1) > 0$ and $0 < \alpha < h(s)$, we obtain $\rho(L(\alpha;Q)) < 1$.

(iii) If $1-\sigma_1^2 \leq s \leq 1-\sigma_{n^2}^2$. Without loss of generality, we assume that there exists an k such that $1-s-\sigma_1^2 \leq \cdots \leq 1-s-\sigma_k^2 \leq 0 < 1-s-\sigma_{k+1}^2 \leq \cdots \leq 1-s-\sigma_{n^2}^2$. Similar to discussion of the cases (i) and (ii), we can obtain

$$\begin{aligned}\rho(L(\alpha;Q)) = \max\Bigg\{ &\frac{\alpha+s}{1+\mu^2-s}\frac{1-s-\sigma_{n^2}^2}{\alpha+\mu^2+s+\sigma_{n^2}^2}, \\ &\frac{\alpha+s}{1+\mu^2-s}\frac{\sigma_1^2+s-1}{\alpha+\mu^2+s+\sigma_1^2}\Bigg\}. \end{aligned} \quad (25)$$

It follows form the above expression that $\rho(L(\alpha;Q)) < 1$ if and only if

$$\frac{\alpha+s}{1+\mu^2-s}\frac{1-s-\sigma_{n^2}^2}{\alpha+\mu^2+s+\sigma_{n^2}^2} < 1 \quad (26)$$

and

$$\frac{\alpha+s}{1+\mu^2-s}\frac{\sigma_1^2+s-1}{\alpha+\mu^2+s+\sigma_1^2} < 1. \quad (27)$$

The discussion and analysis of (26) and (27) are the same with the those in case (i) and case (ii), respectively. Therefore, in this case, $\rho(L(\alpha;Q)) < 1$ holds true if and only if the conditions in case (i) and case (ii) are both satisfied. For the optimal parameters α^* and s^* minimizing the spectral radius of the matrix $L(\alpha;Q)$, from (25) , they must satisfy the following equation

$$\frac{\alpha+s}{1+\mu^2-s}\frac{1-s-\sigma_{n^2}^2}{\alpha+\mu^2+s+\sigma_{n^2}^2} = \frac{\alpha+s}{1+\mu^2-s}\frac{\sigma_1^2+s-1}{\alpha+\mu^2+s+\sigma_1^2}.$$

By simplification, the above equation becomes

$$\frac{1-s-\sigma_{n^2}^2}{\alpha+\mu^2+s+\sigma_{n^2}^2} = \frac{\sigma_1^2+s-1}{\alpha+\mu^2+s+\sigma_1^2} \quad (28)$$

under the condition that $\alpha+s$ does not tend to 0. The solution α^* of Equation (28) is given as follows:

$$\alpha = \frac{(\mu^2+2s-1)(\sigma_1^2+\sigma_{n^2}^2) + 2\sigma_1^2\sigma_{n^2}^2 + 2(\mu^2+s)(s-1)}{2-2s-\sigma_1^2-\sigma_{n^2}^2}.$$

While it may not make $\rho(L(\alpha;Q))$ small enough.

Furthermore, we discuss the global optimal parameters α and s for the foregoing cases. From (21), it can be seen that the spectral radius $\rho(L(\alpha;Q))$ is monotonic increasing with respect to $\alpha > 0$. So in order to find the optimal parameter α^* minimizing the spectral radius $\rho(L(\alpha;Q))$ in (21), the optimal parameter α^* should tend to zero. Moreover, we have

$$\lim_{a\to 0}\rho(L(\alpha;Q)) = \frac{s}{1+\mu^2-s}\max_{\sigma_i\in\sigma(A)}\frac{|1-s-\sigma_i^2|}{\mu^2+s+\sigma_i^2}.$$

It follows from the above expression that the spectral radius $\rho(L(\alpha;Q))$ can be sufficiently close to zero when we adopt the optimal parameter $s^* \to 0$ under the condition $\alpha^* \to 0$. Therefore, the optimal parameters α^* and s^* should be $\alpha^* \to 0$ and $s^* \to 0$. By the above analysis, the desired results of this theorem are obtained.

Remark 3.4: From Theorems 3.2 and 3.3, the spectral radii of the iteration matrices of the SRHSS method with parameter matrices $Q = sI$ and $Q = sI + A^TA$ could be sufficiently close to zero under appropriate parameters α and s, making the versions of the SRHSS iterative method to obtain rapid convergence rates. On this account, it is expected that the versions of the SRHSS method with the parameter matrices $Q = sI$ and $Q = sI + A^TA$ may converge faster than some existing ones, which will be verified by numerical examples in Section IV.

Similar to the proposed algorithm in [26], the following algorithms of the proposed versions of the SRHSS method for augmented system (4) can be obtained. The versions of the SRHSS method with parameter matrices $Q = sI$ and $Q = sI + A^TA$ are denoted by the SRHSS-Q_1 and SRHSS-Q_2 methods, respectively.

Algorithm 3.1: the SRHSS-Q_1 method

1. Given an initial value $f^{(0)}$, the initial value of the noise is taken as $e^{(0)} = g - Af^{(0)}$. Given a very small positive value τ, and M is the maximum prescribed number of outer iterations.
2. $r^{(0)} = b - Kx^{(0)}$.
3. For $k = 0, 1, 2, \cdots$, until $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} > \tau$ or $k < M$,
4. $e^{(k+\frac{1}{2})} = \frac{\alpha e^{(k)} - Af^{(k)} + g}{\alpha+1}$;
5. $f^{(k+\frac{1}{2})} = \frac{A^Te^{(k)} + (\alpha+s)f^{(k)}}{\alpha+\mu^2+s}$;
6. $\left[(1+\mu^2-s)I + A^TA)\right] f^{(k+1)} = A^Tg + (1-s)f^{(k+\frac{1}{2})}$;
7. $e^{(k+1)} = g - Af^{(k+1)}$;
8. $r^{(k+1)} = b - Kx^{(k+1)}$;
9. end for

Algorithm 3.2: the SRHSS-Q_2 method

1. Given an initial value $f^{(0)}$, the initial value of the noise is taken as $e^{(0)} = g - Af^{(0)}$. Given a very small positive value τ, and M is the maximum prescribed number of outer iterations.
2. $r^{(0)} = b - Kx^{(0)}$.
3. For $k = 0, 1, 2, \cdots$, until $\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} > \tau$ or $k < M$,
4. $e^{(k+\frac{1}{2})} = \frac{\alpha e^{(k)} - Af^{(k)} + g}{\alpha+1}$;
5. $\left[(\alpha+\mu^2+s)I + A^TA)\right] f^{(k+\frac{1}{2})} = A^Te^{(k)} + \left[(\alpha+s) + A^TA\right] f^{(k)}$;
6. $f^{(k+1)} = \frac{A^Tg + \left[(1-s)I - A^TA\right]f^{(k+\frac{1}{2})}}{1+\mu^2-s}$;
7. $e^{(k+1)} = g - Af^{(k+1)}$;
8. $r^{(k+1)} = b - Kx^{(k+1)}$;
9. end for

In the sequel, we turn to discuss the operation cost of applying the SRHSS-Q_1 and SRHSS-Q_2 iteration methods to solve (4). Steps 4-5 in Algorithms 3.1 and 3.2 stem from the first-step iterations of the SRHSS-Q_1 and SRHSS-Q_2 methods, respectively. They require the computing of the linear system with the coefficient matrix $(\alpha + \mu^2 + s)I + A^TA$ and the matrix-vector multiplications $Af^{(k)}$, $A^Te^{(k)}$, $Af^{(k+1)}$ and $A^TAf^{(k)}$, as well as scalar-vector multiplications. Steps 6-7 in Algorithms 3.1 and 3.2 are derived from the second-step iterations of the SRHSS-Q_1 and SRHSS-Q_2 methods, respectively, which require the computing of the linear system with the coefficient matrix $(1 + \mu^2 - s)I + A^TA$ except matrix-vector multiplications. Note that the matrix $A \in R^{n^2 \times n^2}$ that arises in image restoration is highly structured, such as block circulant, block Toeplitz and block Toeplitz-plus-Hankel matrices and so forth. Hence, arithmetic operations of matrix-vector multiplications with the blurring matrix $A \in R^{n^2 \times n^2}$ is $O(n^2 \log n)$ by fast Fourier transforms (FFTs). Due to the coefficient matrices $(1 + \mu^2 - s)I + A^TA$ and $(\alpha + \mu^2 + s)I + A^TA$ with $0 < s < 1 + \mu^2$ are symmetric positive definite, the Cholesky factorization can be efficiently applied to solve the two linear sub-systems. Moreover, since for image restoration $A \in R^{n^2 \times n^2}$ is highly structured, we can employ the FFTs to solve $\left[(1 + \mu^2 - s)I + A^TA)\right] f^{(k+1)} = A^Tg + (1 - s)f^{(k+\frac{1}{2})}$ in Step 6 of Algorithm 3.1 and $\left[(\alpha + \mu^2 + s)I + A^TA)\right] f^{(k+\frac{1}{2})} = A^Te^{(k)} + \left[(\alpha + s)I + A^TA\right] f^k$ in Step 5 of Algorithm 3.2.

IV. Numerical examples

In this section, three examples are carried out to examine the feasibility and effectiveness of the SRHSS-Q_1 and SRHSS-Q_2 iteration methods for solving ill-posed problems and image restoration, and show the advantages of the proposed methods over the SHSS, NSHSS, RGHSS, SGHSS and the ULT-type ones mentioned in Section I in terms of the number of iterations (denoted by 'IT') and the total computing times in seconds (denoted by 'CPU'). All computations are carried out in MATLAB R2018a on a personal computer with 1.80-GHz central processing unit (Intel(R) Core(TM) i7-8565U), 8.00 GB of memory, and Windows 10 operating system.

While applying the SRHSS iterative method to solve ill-posed problems, the regularization parameter μ that determines the quality of the computed solution has to be chosen. Several methods for estimating the regularization parameter μ have been described in the literature; see [19], [27], [15] for details. One of the most popular methods for determining a suitable value of μ when no accurate bound for $\|e\|$ is available is the Generalized Cross Validation (GCV) method [10], [15], [16], [24], [28], [13], [14]; So in our computation, we use the GCV scheme to determine a suitable value for regularization parameter μ. The regularization parameter is given by a value which minimizes the GCV function

$$G(\mu) = \frac{\|A(A^TA + \mu^2 I)^{-1}A^Tg - g\|_2^2}{(trace(I - A(A^TA + \mu^2 I)^{-1})A^T)^2}.$$

Determining μ generally requires that the GCV function is evaluated for several μ-values. In order to evaluate this function efficiently, some algebraic methods are helpful. For example the Kronecker product approximation can be effectively used to approximate the best value of μ [29]. In addition, the parameters of the SHSS, NSHSS, RGHSS, SGHSS, SRHSS-Q_1 and SRHSS-Q_2 methods are taken to be the optimal ones determined by optimum parameter formula presented in [26], [8], [1], [2] and Theorems 3.2 and 3.3 in this paper, respectively. And the optimal values of the unknown parameters for the versions of the ULT method are given in [12]. Moreover, it follows from Theorems 3.2 and 3.3 that the optimal parameters of the versions of the SRHSS method usually tend to a certain number, then the optimal parameters are obtained by adding a very small and appropriate number to the certain number in practical examples. In actual implementations, the initial guess in Example 4.1 is the zero vector, and we adopt the initial value as $f^{(0)} = g$ in Examples 4.2-4.3. All runs are terminated if the current residual satisfies $\|r^{(k)}\|_2/\|r^{(0)}\|_2 < 10^{-6}$ or the number of the prescribed iteration steps $M = 100$ is exceeded, where $r^{(k)} = b - Kx^{(k)}$ is the residual at the kth iteration.

Qualitative measures of accuracy and efficiency are investigated. The accuracy is measured by the relative error (RES) and the peak signal-to-noise ratio (PSNR) defined by

$$\mathrm{RES} = \frac{\|f_{numerical} - f_{exact}\|_2}{\|f_{exact}\|_2},$$

$$\mathrm{PSNR} = 10\log_{10}\frac{255^2 \times n^2}{\|f_{numerical} - f_{exact}\|_2^2},$$

where the size of the image is $n \times n$ and $f_{numerical}$, f_{exact} are the numerical solutions (or restored images) and exact solutions (or the original images), respectively. The quantity PSNR provides a quantitative measure of the quality of the restored image: a larger PSNR-value usually implies that the restoration is of higher quality. However, in some cases this might not agree with visual judgment. We therefore also display the restored images.

Example 4.1: The collection of examples are from Hansens Regularization Tools [20]. All the problems are obtained by discretizing classical example of an ill-posed problem in Fredholm integral equations of the first kind with a square integrable kernel

$$\int_a^b K(s,t)f(t)dt = g(s), \quad c \le s \le d$$

and approximating integration with a quadrature rule. In the tests the kernel K and the solution f are given and discretized to yield the matrix A and the vector f, then the discrete right-hand side is determined by $\hat{g} = Af$. We consider the test examples **shaw**, **deriv2**, **foxgood**, **phillips**, **baart** and **gravity**. In all our numerical experiments, the 'noise' right-hand side g is generated from the exact data $\hat{g}$ by adding the noise in the form $g = \hat{g} + 0.001 \times rand(size(\hat{g}))$ and we set the size $n = 500$.

We apply Algorithms 3.1 and 3.2 to the six different inverse problems and compare the SRHSS-type methods with the SHSS, NSHSS, RGHSS and SGHSS ones in view of IT, CPU times and RES. The regularization parameter μ of all test problems is determined by GCV and the numerical results are reported in Table I.

From Table I, all tested methods can successfully compute approximate solutions, and either the SRHSS-Q_1 method or SRHSS-Q_2 method can achieve smallest relative error with the least IT and CPU times than other four ones as the tested iteration methods are terminated. For the test problem $phillips$, the versions of the SRHSS method can provide a regularized solution of the same accuracy as the ones by the RGHSS and SGHSS methods but require much less IT and CPU times. Moreover, The exact and numerical solutions are shown in (a) of Figures 1-6, and in order to better understand the numerical results in Table I the plots of relative error with respect to iteration k are depicted in (b) of Figures 1-6 for the six test problems. Subgraphs (b) of Figures 1-6 show that

TABLE I: Numerical results of iterative methods of Example 4.1.

Test problem	Method	Parameters	IT	CPU	RES
	SHSS	$\alpha = 0.8175$	100	0.0411	0.7551
shaw(500)	NSHSS	$\alpha = 2.7700e-6$	100	0.0686	0.9994
	RGHSS	$(\alpha, \beta) = (0.001, 0.001)$	100	0.0458	0.5164
$\mu = 0.0017$	SGHSS	$(\omega, \tau) = (0.6, 6.1662e-5)$	100	0.0429	0.0398
	SRHSS-Q_1	$(\alpha, s) = (0.001, 0.999)$	6	0.0051	0.0481
	SRHSS-Q_2	$(\alpha, s) = (1.0e-5, 1.0e-4)$	3	0.0059	0.0464
	SHSS	$\alpha = 0.0051$	100	0.0383	0.1231
deriv2(500,3)	NSHSS	$\alpha = 2.2139e-4$	100	0.0408	0.9568
	RGHSS	$(\alpha, \beta) = (0.0012, 0.001)$	36	0.0137	0.1221
$\mu = 0.0149$	SGHSS	$(\omega, \tau) = (0.6, 1.9283e-4)$	9	0.0030	0.1221
	SRHSS-Q_1	$(\alpha, s) = (1.0e-4, 0.9999)$	8	0.0031	0.1221
	SRHSS-Q_2	$(\alpha, s) = (1.0e-5, 1.0e-5)$	5	0.0125	0.1221
	SHSS	$\alpha = 0.2474$	100	0.0562	0.9523
foxgood(500)	NSHSS	$\alpha = 6.6982e-6$	100	0.0433	0.9986
	RGHSS	$(\alpha, \beta) = (0.001, 0.001)$	100	0.0636	0.2439
$\mu = 0.0026$	SGHSS	$(\omega, \tau) = (0.6, 6.4163e-5)$	100	0.0396	0.0015
	SRHSS-Q_1	$(\alpha, s) = (1.0e-4, 0.9999)$	4	0.0019	0.0012
	SRHSS-Q_2	$(\alpha, s) = (1.0e-5, 1.0e-5)$	3	0.0052	0.0011
	SHSS	$\alpha = 0.9439$	100	0.0545	0.6471
phillips(500)	NSHSS	$\alpha = 0.7414$	100	0.0419	0.8643
	RGHSS	$(\alpha, \beta) = (0.0024, 0.001)$	17	0.0073	0.0193
$\mu = 0.0272$	SGHSS	$(\omega, \tau) = (0.6, 0.0005)$	7	0.0033	0.0193
	SRHSS-Q_1	$(\alpha, s) = (0.001, 0.9999)$	3	0.0011	0.0192
	SRHSS-Q_2	$(\alpha, s) = (1.0e-5, 1.0e-4)$	3	0.0059	0.0192
	SHSS	$\alpha = 0.8390$	100	0.0409	0.7235
baart(500)	NSHSS	$\alpha = 6.13083e-5$	100	0.0416	0.9885
	RGHSS	$(\alpha, \beta) = (0.0011, 0.001)$	100	0.0397	0.1941
$\mu = 0.0078$	SGHSS	$(\omega, \tau) = (0.6, 9.6784e-5)$	17	0.0073	0.1941
	SRHSS-Q_1	$(\alpha, s) = (0.01, 0.999)$	6	0.0020	0.1721
	SRHSS-Q_2	$(\alpha, s) = (1.0e-5, 1.0e-4)$	3	0.0093	0.1849
	SHSS	$\alpha = 0.9543$	100	0.0443	0.8575
gravity(500,1)	NSHSS	$\alpha = 8.1258e-5$	100	0.0411	0.9841
	RGHSS	$(\alpha, \beta) = (0.0011, 0.001)$	100	0.0375	0.0085
$\mu = 0.0090$	SGHSS	$(\omega, \tau) = (0.6, 1.0875e-4)$	25	0.0459	0.0085
	SRHSS-Q_1	$(\alpha, s) = (0.01, 0.99)$	5	0.0020	0.0123
	SRHSS-Q_2	$(\alpha, s) = (1.0e-6, 1.0e-4)$	3	0.0060	0.0083

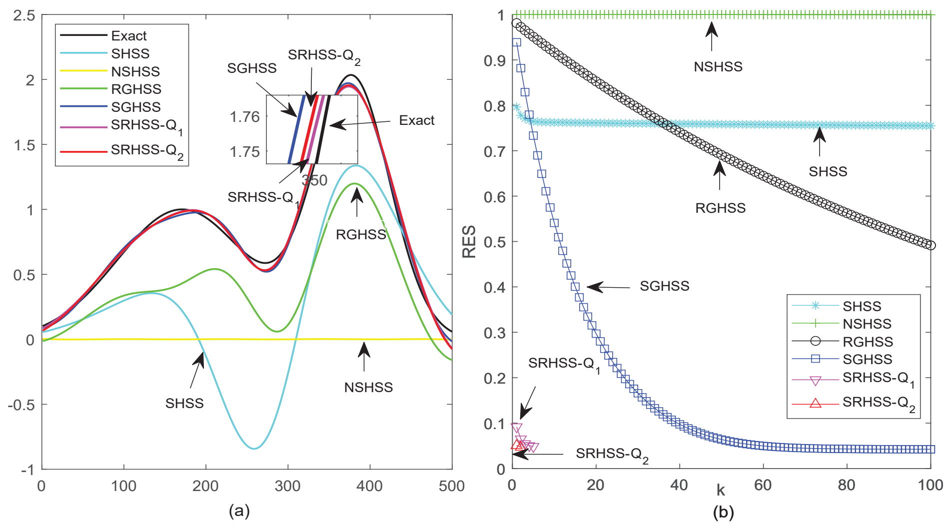

Fig. 1: Example 4.1-*shaw* test problem: (a) the exact solution and its numerical solution, (b) the relative error versus iteration k.

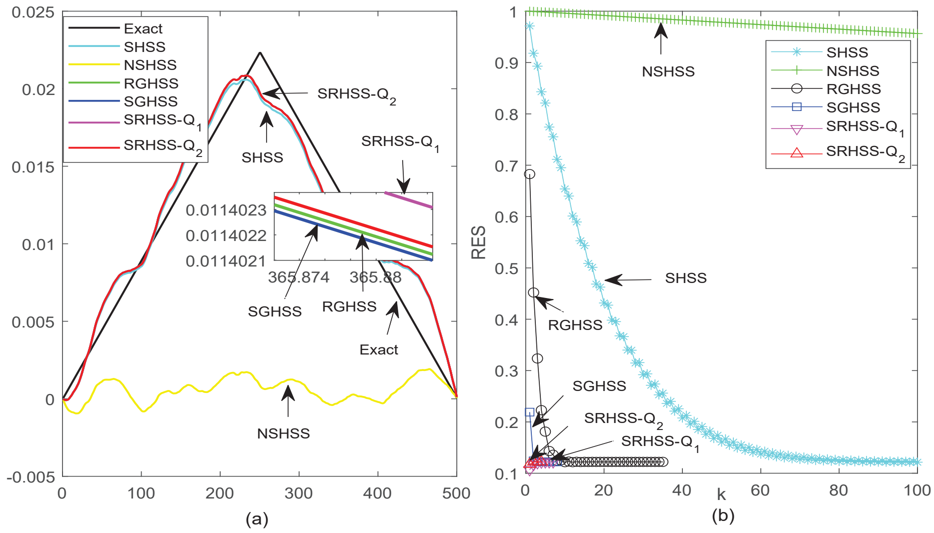

Fig. 2: Example 4.1-$deriv2$ test problem: (a) the exact solution and its numerical solution, (b) the relative error versus iteration k.

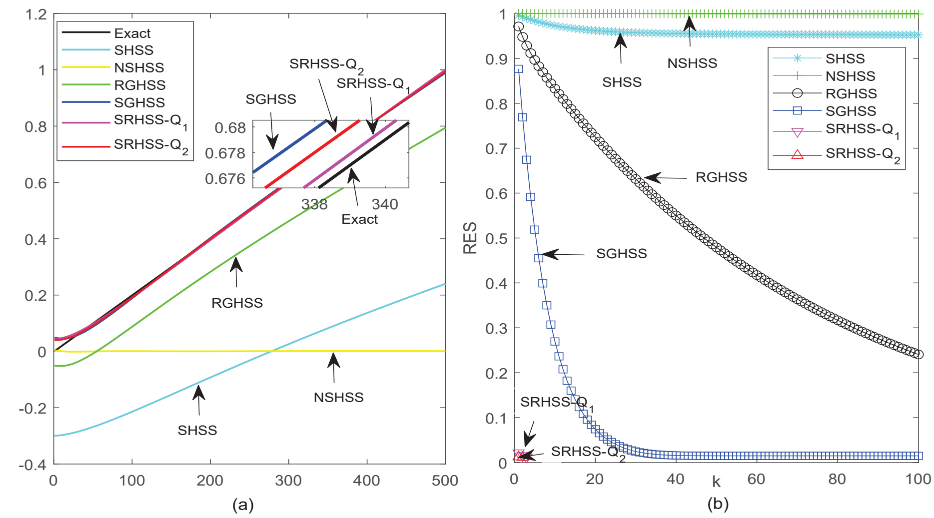

Fig. 3: Example 4.1-$foxgood$ test problem: (a) the exact solution and its numerical solution, (b) the relative error versus iteration k.

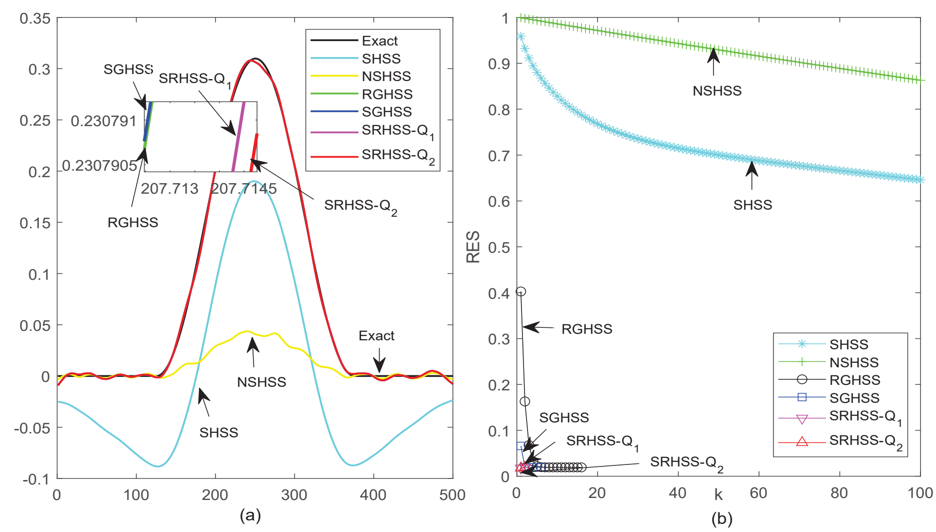

Fig. 4: Example 4.1-$phillips$ test problem: (a) the exact solution and its numerical solution, (b) the relative error versus iteration k.

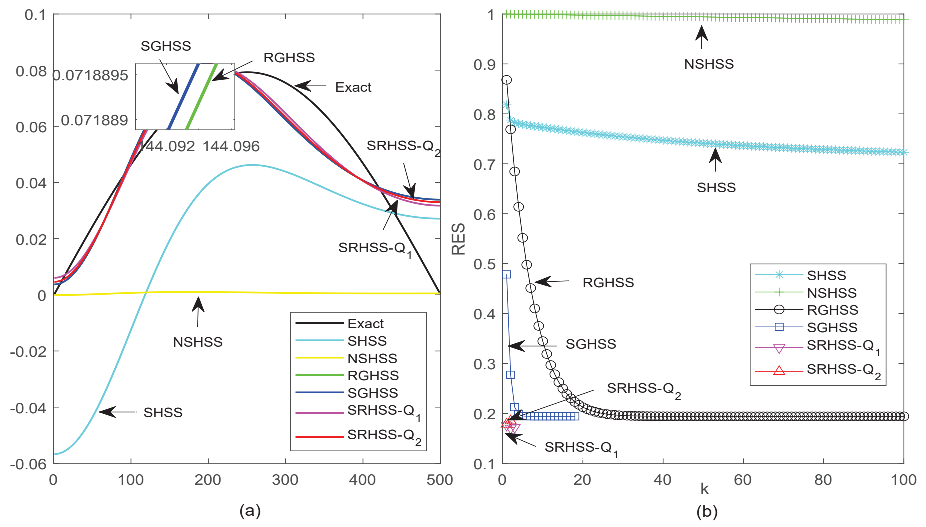

Fig. 5: Example 4.1-$baart$ test problem: (a) the exact solution and its numerical solution, (b) the relative error versus iteration k.

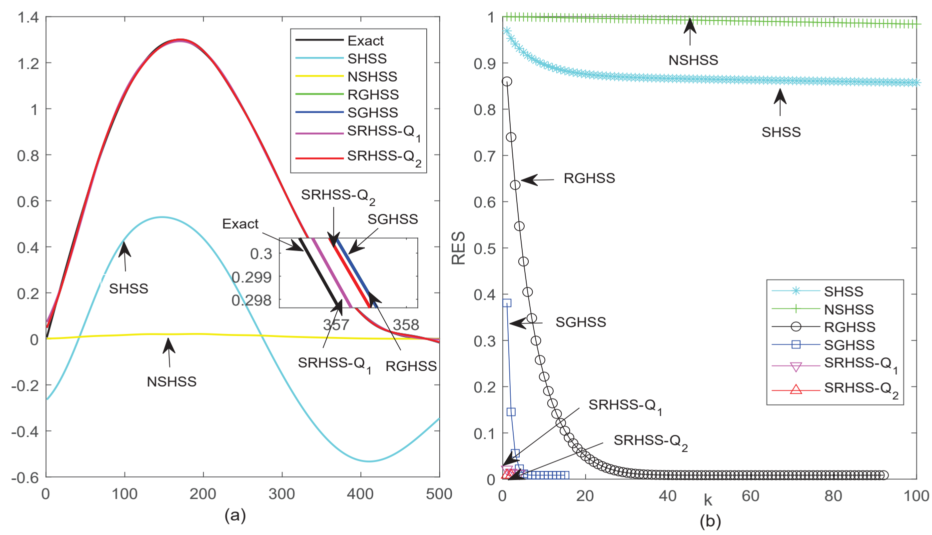

Fig. 6: Example 4.1-$gravity$ test problem: (a) the exact solution and its numerical solution, (b) the relative error versus iteration k.

among these iteration methods the proposed versions of the SRHSS one are the most effective methods as their residuals reduce in the fastest rate. Besides, we also implement the versions of the ULT method [12], requiring more IT and CPU times than our methods. Because too many lines are not good for observation, we do not present the results of the versions of the ULT method in Figures 1-6. As this results show, our proposed Algorithms 3.1 and 3.2 surpass other ones in terms of both IT and CPU times for the convergence and are more effective for solving the ill-posed problems.

Example 4.2: (Image restoration) The original image of this example is the 'Grain' image of dimension 256×256 from Matlab's image processing toolbox, and we choose $PSF = psfDefocus([7, 7], 3)$ in [22] to blur the image. The 'noisy' right-hand-side g is generated by using MATLAB code $g = \hat{g} + 0.001\|\hat{g}\|_2 \frac{w}{\|w\|_2}$, where w is a vector whose components are normally distributed with zero mean and unit variance. The true image, PSF and blurred image in the example are displayed in Figure 7.

Example 4.3: (Image restoration) We consider the restoration of an image that has been contaminated by linear motion blur and noise. The motion blur is caused by motion of a rigid object, or equivalently caused by rigid movements of the image device. The test image represented by 255×255 pixels and the PSF for motion blur are shown in (a) and (b) of Figure 8, respectively. The 'noisy' right-hand-side g contaminated by motion blur as well as noise is shown in (c) of Figure 8, and the added noise is the same as that in Example 4.2.

The purpose of Examples 4.2 and 4.3 is to illustrate that the presented versions of the SRHSS iterative method perform better than the SHSS, RGHSS, SGHSS and the versions of ULT ones when applied to the restoration of images. Furthermore, PSNR-values of the blurred images in Examples 4.2 and 4.3 are 70.5126 and 67.4780, respectively.

In two tests, we enforce the periodic BCs to construct the blurring matrix A and use the blurred and noise image as an initial guess. The regularization parameters are easily computed by the GCV method for all iterative methods with the periodic BCs. We apply Algorithms 3.1 and 3.2 to deblur the two images. In Table II we report the numerical results of the tested methods for the two examples and the restored images by the nine iterative methods are also exhibited in Figures 7-8. The reason that we do not display the image

TABLE II: Numerical results of iterative methods of Examples 4.2 and 4.3.

Test problem	Method	Parameters	IT	CPU	PSNR
	SHSS	$\alpha = 0.3333$	100	3.4064	80.4086
	RGHSS	$(\alpha, \beta) = (0.001, 0.001)$	100	3.4393	86.9805
Example 4.2	SGHSS	$(\omega, \tau) = (0.6, 2.2648e-5)$	11	0.3526	86.9849
	ULT-I$_{\mathrm{Q}_1}$	$s = 1.0023$	100	4.2084	78.8081
$\mu = 0.0046$	ULT-I$_{\mathrm{Q}_2}$	$s = 12.0000$	100	4.4053	72.5208
	ULT-II$_{\mathrm{Q}_1}$	$s = 1.0000$	100	2.9822	78.8136
PSNR(blurred)	ULT-II$_{\mathrm{Q}_2}$	$s = 0.0046$	100	4.5435	86.4305
=70.5126	SRHSS-Q$_1$	$(\alpha, s) = (0.0010, 1.0000)$	3	0.0941	86.9849
	SRHSS-Q$_2$	$(\alpha, s) = (1.e-5, 1.0e-5)$	3	0.1177	86.9763
	SHSS	$\alpha = 0.3333$	100	6.6165	86.3823
	RGHSS	$(\alpha, \beta) = (0.001, 0.001)$	100	4.2418	92.9234
Example 4.3	SGHSS	$(\omega, \tau) = (0.6, 2.0369e-5)$	13	0.5157	93.3408
	ULT-I$_{\mathrm{Q}_1}$	$s = 1.0021$	100	4.9149	84.2767
$\mu = 0.0042$	ULT-I$_{\mathrm{Q}_2}$	$s = 12.0000$	100	5.3951	72.5803
	ULT-II$_{\mathrm{Q}_1}$	$s = 1.0000$	100	3.6513	84.2211
PSNR(blurred)	ULT-II$_{\mathrm{Q}_2}$	$s = 0.0041$	100	5.8038	84.1818
=67.4780	SRHSS-Q$_1$	$(\alpha, s) = (0.001, 1.0000)$	2	0.0871	93.3717
	SRHSS-Q$_2$	$(\alpha, s) = (1.0e-5, 1.0e-5)$	2	0.1050	93.4368

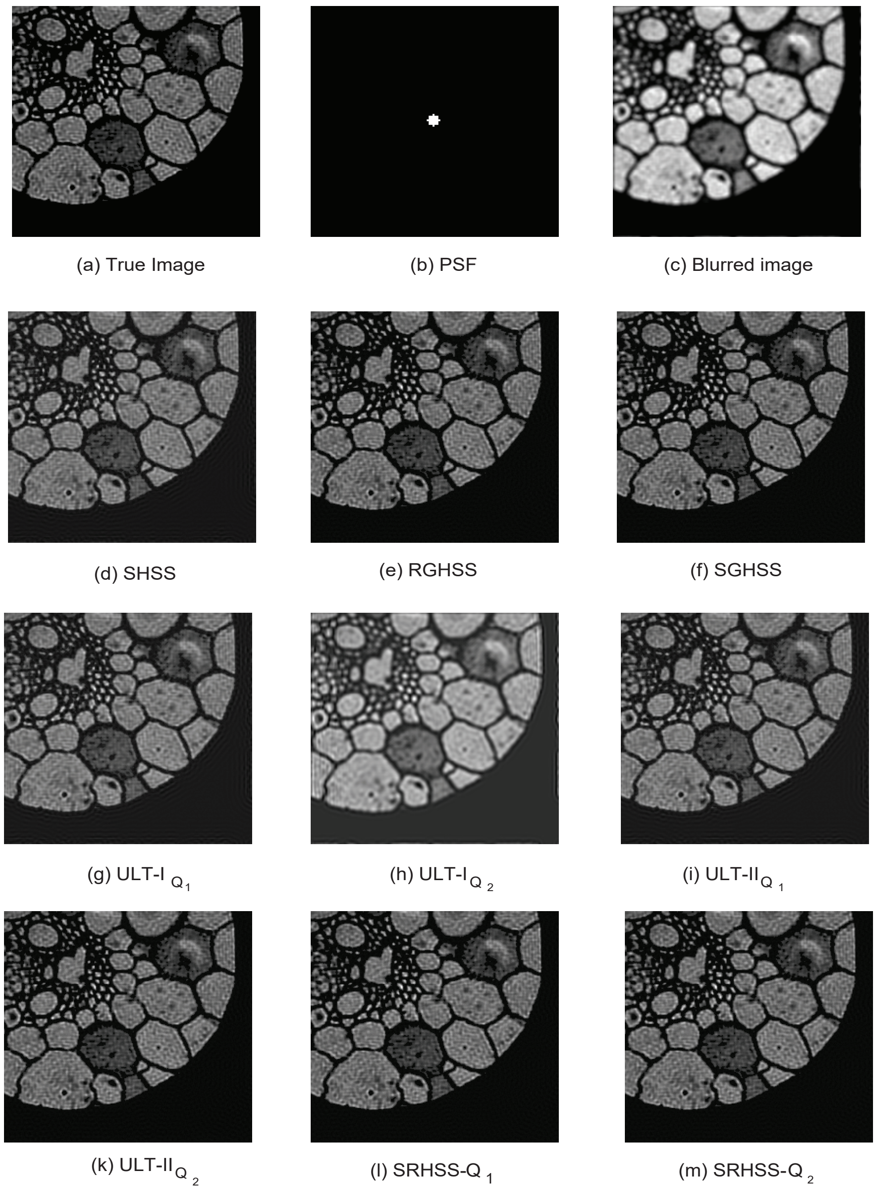

Fig. 7: True image, PSF, blurred image and restoration images with various methods for Example 4.2.

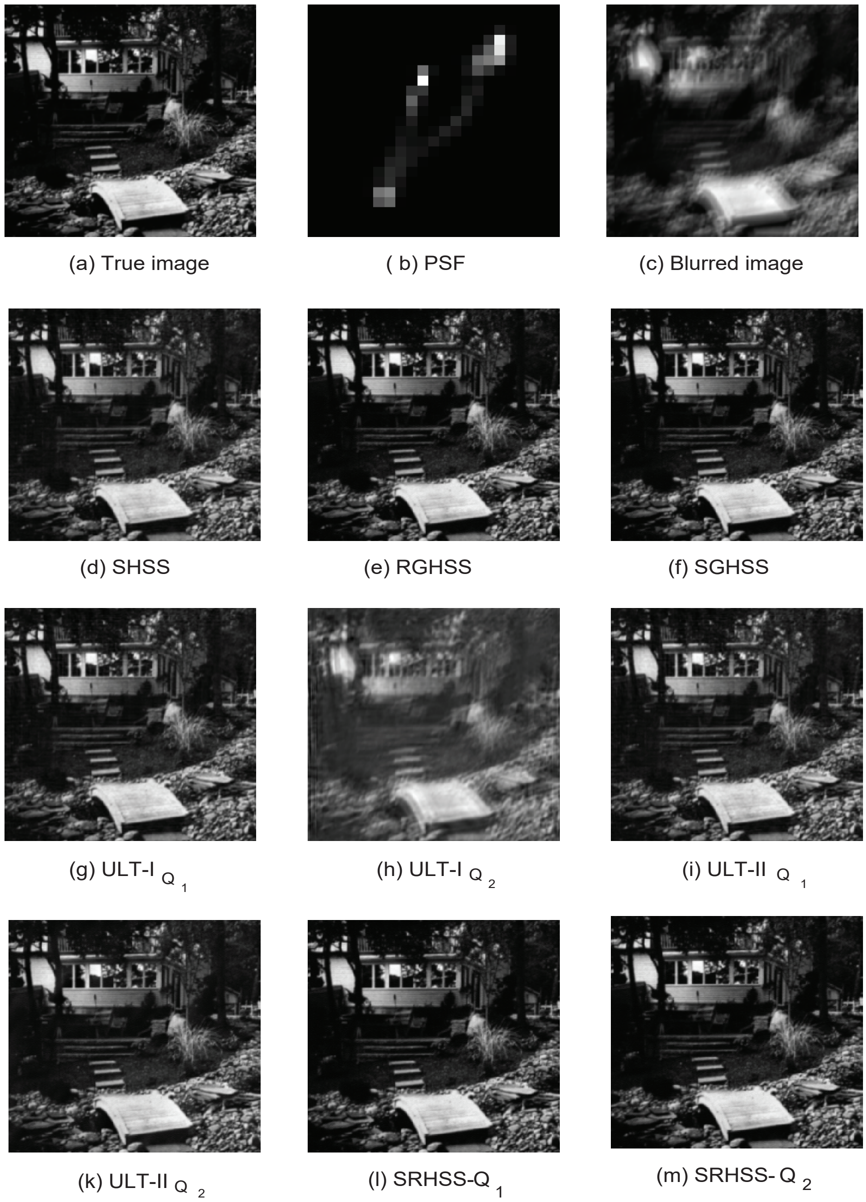

Fig. 8: True image, PSF, blurred image and restoration images with various methods for Example 4.3.

restoration of the NSHSS method is that the NSHSS method for the examples is out of effect. From Table II, it can be observed that all restorations achieved by the $\mathrm{SRHSS\text{-}Q_1}$ and $\mathrm{SRHSS\text{-}Q_2}$ methods have a larger PSNR-value than the SHSS, RGHSS, SGHSS and the versions of ULT ones. Besides, the versions of the SRHSS method require fewer IT and CPU times. In short, compared with the results of some existing methods in Table II, usually the proposed methods in the paper are able to give better restorations in very few the iteration steps and a smaller amount of time. Finally, Figures 7-8 demonstrate that the proposed versions of the SRHSS method are visually the better.

V. Conclusion

This paper puts forward a special regularized Hermitian and skew-Hermitian splitting iterative method, called the special RHSS (SRHSS) method, for solving the augmented system (4) derived by Tikhonov regularization for ill-posed problems and image restoration. We first construct a new special splitting (8) of the coefficient matrix K in (6), of which the first splitting form is the same with the one of the RHSS method with $\omega = 1$ for stabilized saddle-point problems [3]. Whereas, the second splitting form in (8) is different from that of (2.1) in [3]. Based on the new special splitting, we present a speical regularized Hermitian and skew-Hermitian splitting (SRHSS) iterative method, which can converge faster than some existing ones. Moreover, we investigate analytically the convergence behavior of the proposed versions of the SRHSS method and the optimal parameters which ensure a fast convergence rate are derived. Numerical experiments show that the proposed methods are effective and can outperform the existing ones in terms of IT, CPU times and image recovering quality.

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