Extinction in an Impulsive Nonautonomous Lotka-Volterra Competitive System with Discrete Delay and Infinite Delay

Lili Wang, Haiyan Lv

Abstract—**An impulsive nonautonomous Lotka-Volterra competitive system with discrete delay and infinite delay is studied in this paper. It is shown that if the coefficients are continuous, bounded above and below by positive constants and satisfy certain inequalities, then one of the components will be driven to extinction while the other one will stabilize at the certain positive solution of a nonlinear single species model (an impulsive logistic equation). An example together with its numerical simulations is given to illustrate the feasibility and effectiveness of the main results.**

Index Terms—**Extinction; Global attractivity; Lotka-Volterra competitive system; Delay; Impulse.**

I. Introduction

In the real world, affected by a variety of factors both naturally and manly, the inner discipline of species or environment often suffers some dispersed changes over a relatively short time interval at the fixed times. In mathematics perspective, such sudden changes could be described by impulses (see [1,2]). Owing to the theoretical and practical significance, the dynamic behaviors of impulsive differential equations have been extensively researched, see [3-8].

In recent years, the study of extinction and permanence of the species has become one of the most important topic in population dynamics. The results on the extinction and permanence of the species of impulsive population dynamic systems, see [9-12]. However, there are seldom results on the extinction and permanence of the species of impulsive population dynamic systems with delay. In fact, more realistic population dynamics should take into account the effect of delay. Noting that some studies of the dynamics of natural populations indicate that the density-dependent population regulation probably takes place over many generations, many authors have discussed the influence of many past generations on the density of species population and discussed the dynamic behaviors of competitive, predator-prey, and cooperative systems. Moreover, delay differential equations may exhibit much more complicated dynamic behaviors than ordinary differential equations since a delay could cause a stable equilibrium to become unstable and cause the population to fluctuate.

Motivated by the above statements, in this work, we shall study the following impulsive Lotka-Volterra competitive system with discrete delay and infinite delay

$$\begin{aligned}
x_1'(t) &= x_1(t)\bigg[r_1(t) - a_1(t)\int_0^{+\infty} k_1(s)x_1(t-s)ds \\
&\qquad -\frac{b_2(t)x_2(t-\tau)}{c(t)+x_2(t-\tau)}\bigg], \\
x_2'(t) &= x_2(t)\bigg[r_2(t) - a_2(t)x_2(t-\tau) \\
&\qquad -\frac{b_1(t)\int_0^{+\infty} k_1(s)x_1(t-s)ds}{c(t)+\int_0^{+\infty} k_1(s)x_1(t-s)ds}\bigg], t \neq t_j, \\
x_1(t_j^+) &= (1-h_{1j})x_1(t_j), \\
x_2(t_j^+) &= (1-h_{2j})x_2(t_j), \qquad (1)
\end{aligned}$$

where $x_1(t), x_2(t)$ are population density of species x_1 and x_2 at time t, respectively; the jump conditions $x_i(t_j^+) = (1-h_{ij})x_i(t_j), i = 1, 2$ reflects the possibility of impulsive effects on the species x_i; $j \in N, N = \{1, 2, \cdots\}$.

Given a function $f(t)$, let f^u and f^l denote $\sup_{t\in[0,\omega]} f(t)$ and $\inf_{t\in[0,\omega]} f(t)$, respectively. For system (1), throughout this paper, the following conditions are assumed:

(H_1) The functions $r_i(t), a_i(t), b_i(t), c(t)$ are positive continuous functions with period ω, and $r_i^l \le r_i(t) \le r_i^u, a_i^l \le a_i(t) \le a_i^u, b_i^l \le b_i(t) \le b_i^u, c^l \le c(t) \le c^u, i = 1, 2$;

(H_2) The impulse times $t_j, j \in N$ satisfy $0 < t_1 < t_2 < \cdots$, and $\lim_{j\to+\infty} t_j = +\infty$;

(H_3) The parameters h_{ij} are real constants satisfying $0 < h_{ij} < 1, i = 1, 2, j \in N$;

(H_4) $\prod_{0\le t_j<t}(1-h_{ij})$ are periodic functions of period ω, and there exist positive constants m_i and M_i such that $m_i \le \prod_{0\le t_j<t}(1-h_{ij}) \le M_i$ for all $t \ge 0, i = 1, 2, j \in N$;

(H_5) The delay τ is a nonnegative constant; $k_1 : [0, +\infty) \to [0, +\infty)$ is piecewise continuous and integrable on $[0, +\infty)$ with $\int_0^{+\infty} k_1(s)ds = 1$.

Definition 1. *(x_1, x_2) is said to be a solution of system (1) provided*

(i) $x_i(t), i = 1, 2$ are absolutely continuous on each interval $(0, t_1]$ and $(t_j, t_{k+1}], j \in N$;

(ii) For any $t_j, j \in N$, $x_i(t_j^+)$ and $x_i(t_j^-)$ exist and $x_i(t_j^-) = x_i(t_j)$;

(iii) $x_i(t), i = 1, 2$ satisfy (1) for almost everywhere (a.e.) in $[0, +\infty)/\{t_j\}$ and satisfy $x_i(t_j^+) = (1-h_{ij})x_i(t_j)$ for every $t = t_j, j \in N$.

The main purpose of this paper is to study the extinction and stability of system (1), and derive some sufficient conditions which guarantee one of the species will be driven to

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L. Wang and H. Lv are with the School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan, 455000 China e-mail: ay_wanglili@126.com.

extinction while the other one will be globally attractive with any positive solution of an impulsive single species model.

The initial conditions of (1) are of the form

$$x_i(s) = \varphi_i(s) > 0, s \in (-\infty, 0], \varphi_i(0) > 0, \qquad (2)$$

where φ_i, $i = 1, 2$ are bounded and continuous functions on $(-\infty, 0]$.

II. Preliminaries

In this section, we state the following lemmas which will be useful in the proof of our main results.

Lemma 1. *Let $x(t) = (x_1(t), x_2(t))^T$ be any solution of system (1) such that $x_i(0^+) > 0$, then there exists positive constants x_i^* such that*

$$\limsup_{t\to+\infty} x_i(t) \le x_i^*, i = 1, 2,$$

where

$$x_1^* = \frac{r_1^u}{a_1^l m_1 \int_0^{+\infty} k_1(s) e^{-r_1^u s} ds}, x_2^* = \frac{M_2 r_2^u}{m_2 a_2^l} e^{r_2^u \tau}.$$

Proof: By the relation between the solutions of impulsive system and the corresponding non-impulsive system. The proof of Lemma 1 is similar to that of Lemma 2.2 in [13] and [14]. So we omit here.

Lemma 2. ([15]) *Let x be a bounded nonnegative continuous function, and let $k : [0, +\infty) \to [0, +\infty)$ be a continuous kernel such that $\int_0^{+\infty} k(s)ds = 1$. Then*

$$\begin{aligned}\liminf_{t\to+\infty} x(t) &\le \liminf_{t\to+\infty} \int_{-\infty}^{t} k(t-s)ds \\ &\le \limsup_{t\to+\infty} \int_{-\infty}^{t} k(t-s)ds \le \limsup_{t\to+\infty} x(t).\end{aligned}$$

Remark 1. *If $\lim_{t\to+\infty} x(t) = x^*$, then $\lim_{t\to+\infty} \int_{-\infty}^{t} k(t-s)x(s)ds = x^*$.*

III. Extinction of x_2 and stability of x_1

In this section, we present the extinction of the species x_2.

Theorem 1. *Assume that the inequality*

$$\liminf_{t\to+\infty} \frac{\tilde{r}_1(t)}{\tilde{r}_2(t)} > \limsup_{t\to+\infty} \left\{ \frac{a_1(t)(c(t) + x_1^*)}{b_1(t)}, \frac{b_2(t)}{a_2(t)c(t)} \right\} \qquad (3)$$

or

$$\limsup_{t\to+\infty} \frac{\tilde{r}_2(t)}{\tilde{r}_1(t)} < \liminf_{t\to+\infty} \left\{ \frac{b_1(t)}{a_1(t)(c(t) + x_1^*)}, \frac{a_2(t)c(t)}{b_2(t)} \right\} \qquad (4)$$

holds, where

$$\tilde{r}_1(t) = r_1(t) + \frac{1}{\omega} \sum_{j=1}^{q} \ln(1 - h_{1j}) > 0; \qquad (5)$$

$$\tilde{r}_2(t) = r_2(t) + \frac{1}{\omega} \sum_{j=1}^{q} \ln(1 - h_{2j}) > 0; \qquad (6)$$

then the species x_2 will be driven to extinction, that is, for any positive solution $(x_1(t), x_2(t))^T$ of system (1), $x_2(t) \to 0$ exponentially as $t \to +\infty$.

Proof: We only prove one case, the proof for the other case is similar.

Let $\tilde{x}(t) = (x_1(t), x_2(t))^T$ be a solution of system (1) with initial conditions (2). By inequality (3), we can choose $\alpha, \beta, \varepsilon > 0$ such that

$$\begin{aligned}\liminf_{t\to+\infty} \frac{\tilde{r}_1(t)}{\tilde{r}_2(t)} &> \frac{\alpha}{\beta} + \varepsilon > \frac{\alpha}{\beta} \\ &> \limsup_{t\to+\infty} \left\{ \frac{a_1(t)(c(t) + x_1^*)}{b_1(t)}, \frac{b_2(t)}{a_2(t)c(t)} \right\},\end{aligned}$$

then there exists a positive constant $T_1 > 0$ such that for all $t > T_1$,

$$\tilde{r}_1(t)\beta - \tilde{r}_2(t)\alpha > \varepsilon\beta\tilde{r}_2(t) > \varepsilon\beta\tilde{r}_2^l > 0; \qquad (7)$$

$$\alpha b_1(t) - \beta a_1(t)(c(t) + x_1^*) > 0; \qquad (8)$$

$$\alpha a_2(t)c(t) - \beta b_2(t) > 0. \qquad (9)$$

From system (1) and inequalities (8)-(9), it follows that

$$\begin{aligned}&\frac{d}{dt}\left[\ln \frac{(x_2(t))^\alpha}{(x_1(t))^\beta}\right] \\ =\ & \frac{d}{dt}\left[\alpha \ln x_2(t) - \beta \ln x_1(t)\right] \\ =\ & (\alpha r_2(t) - \beta r_1(t)) \\ & -\left[\frac{\alpha b_1(t)}{c(t) + \int_0^{+\infty} k_1(s)x_1(t-s)ds} - \beta a_1(t)\right] \\ & \times \int_0^{+\infty} k_1(s)x_1(t-s)ds \\ & -\left[\alpha a_2(t) - \frac{\beta b_2(t)}{c(t) + x_2(t-\tau)}\right] x_2(t-\tau) \\ \le\ & (\alpha r_2(t) - \beta r_1(t)), t \ne t_j, \end{aligned} \qquad (10)$$

and

$$\ln\left[\frac{(x_2(t_j^+))^\alpha}{(x_1(t_j^+))^\beta}\right] = \ln\left[\frac{(1-h_{2j})^\alpha}{(1-h_{1j})^\beta}\right] + \ln\left[\frac{(x_2(t_j))^\alpha}{(x_1(t_j))^\beta}\right]. \qquad (11)$$

For any $t \in [\lambda_\sigma, \lambda_{\sigma+1})$ and $\lambda_\sigma \in [m\omega, (m+1)\omega)$, $m \in N$, integrating both sides of (10) over internals $[0, \lambda_1)$, $[\lambda_1, \lambda_2)$, $\cdots$, $[\lambda_{\sigma-1}, \lambda_\sigma)$ and $[\lambda_\sigma, t)$, respectively, and adding the σ inequalities, it follows from (5)-(7) and (11) that

$$\begin{aligned}&\ln\left(\frac{(x_2(t))^\alpha}{(x_1(t))^\beta}\right) - \ln\left(\frac{(x_2(0))^\alpha}{(x_1(0))^\beta}\right) \\ \le\ & \int_0^t (\alpha r_2(t) - \beta r_1(t))dt + \ln \frac{\prod_{0<t_j<t} (1-h_{2j})^\alpha}{\prod_{0<t_j<t} (1-h_{1j})^\beta} \\ \le\ & m\alpha\left(\omega r_2^u + \sum_{j=1}^{q} \ln(1-h_{2j})\right) \\ & -m\beta\left(\omega r_1^l + \sum_{j=1}^{q} \ln(1-h_{1j})\right) + \xi \\ =\ & m\omega\left[\alpha\left(r_2^u + \frac{1}{\omega}\sum_{j=1}^{q} \ln(1-h_{2j})\right)\right. \\ & \left. -\beta\left(r_1^l + \frac{1}{\omega}\sum_{j=1}^{q} \ln(1-h_{1j})\right)\right] + \xi \\ <\ & -m\omega\varepsilon\beta\tilde{r}_2^l + \xi, \end{aligned} \qquad (12)$$

where

$$\xi = \sup_{0\le p\le\omega}\left(\int_{m\omega}^{p}(\alpha r_2(t)-\beta r_1(t))dt + \ln\frac{\prod\limits_{0<t_j<p}(1-h_{2j})^{\alpha}}{\prod\limits_{0<t_j<p}(1-h_{1j})^{\beta}}\right).$$

This shows that

$$\begin{aligned}(x_2(t))^{\alpha} &< (x_1(t))^{\beta}\exp\{-m\omega\varepsilon\beta\tilde{r}_2^l+\xi\}\frac{(x_2(0))^{\alpha}}{(x_1(0))^{\beta}}\\ &= (x_1(t))^{\beta}\exp\{-m\omega\varepsilon\beta\tilde{r}_2^l\}e^{\xi}\frac{(x_2(0))^{\alpha}}{(x_1(0))^{\beta}}.\end{aligned}$$

If $t\to+\infty$, then $m\to+\infty$, according to Lemma 2, and noticing that $x_1(t)$ is ultimately upper bounded, hence, we obtain $x_2(t)\to 0$ exponentially as $t\to+\infty$. This completes the proof.

Consider the following impulsive logistic equations

$$\begin{cases} x'(t) = x(t)\Big[r_1(t) -a_1(t)\int_0^{+\infty}k_1(s)x(t-s)ds\Big], \\ \qquad t\ne t_j, j\in N,\\ x(t_j^+) = (1-h_{1j})x(t_j). \end{cases} \tag{13}$$

and

$$\begin{cases} x'(t)=x(t)[r_1(t)-a_1(t)x(t)], t\ne t_j, j\in N,\\ x(t_j^+)=(1-h_{1j})x(t_j). \end{cases} \tag{14}$$

Theorem 2. *Under the assumptions of Theorem 1. Let $x(t)=(x_1(t),x_2(t))^T$ be any positive solution of system (1) with initial condition (2), then the species x_2 will be driven to extinction, that is, $x_2(t)\to 0$ as $t\to+\infty$, and $x_1(t)\to x^*(t)$ as $t\to+\infty$, where $x^*(t)$ is any positive solution of equation (13).*

Proof: Let $x(t)=(x_1(t),x_2(t))^T$ be a solution of system (1) with $x_i(0)>0, i=1,2$. From Lemma 1, $x_1(t)$ is bounded above by positive constants on $[0,+\infty)$. To finish the proof of Theorem 2, it is enough to show that $x_1(t)\to x^*(t)$ as $t\to+\infty$, where $x^*(t)$ is any positive solution of equation (13).

From system (1), we have

$$\begin{cases} x_1'(t) < x_1(t)\Big[r_1(t) -a_1(t)\int_0^{+\infty}k_1(s)x_1(t-s)ds\Big], \\ \qquad t\ne t_j, j\in N,\\ x_1(t_j^+) = (1-h_{1j})x_1(t_j), \end{cases} \tag{15}$$

then $x_1(t)<x^*(t)$ for all $t>0$, where $x^*(t)$ is any positive solution of equation (13) with $x(0)=x_1(0)$. Clearly, $x^*(t)$ is bounded above and below by positive constants on $[0,+\infty)$.

Define a function $V(t)$ on $[0,+\infty)$ as

$$V(t)=-(\ln x_1(t)-\ln x^*(t)). \tag{16}$$

For $t\ne t_j, j\in N$, calculating the derivative of $V(t)$ along the solution $x_1(t)$ and $x^*(t)$, it follows that

$$\begin{aligned}V'(t) &= -\left(\frac{x_1'(t)}{x_1(t)}-\frac{x^{*\prime}(t)}{x^*(t)}\right)\\ &= -a_1(t)\int_0^{+\infty}k_1(s)[x^*(t-s)-x_1(t-s)]ds\\ &\quad+\frac{b_2(t)x_2(t-\tau)}{c(t)+x_2(t-\tau)}.\end{aligned}$$

Since $\lim\limits_{t\to+\infty}x_2(t)=0$, there exists a positive constant $T_2>0$ such that for all $t>T_2+\tau$,

$$x_2(t)<\varepsilon.$$

Hence,

$$\begin{aligned}V'(t) &< -a_1^l m_1\int_0^{+\infty}k_1(s)[x^*(t-s)-x_1(t-s)]ds\\ &\quad+\frac{b_2^u\varepsilon}{c(t)+x_2(t-\tau)}.\end{aligned}$$

Let $\varepsilon\to 0$, then

$$V'(t)\le -a_1^l m_1\int_0^{+\infty}k_1(s)[x^*(t-s)-x_1(t-s)]ds.$$

The above inequality implies that

$$\frac{d}{dt}\left(-V(t)-a_1^l m_1\int_{T_3}^{t}\int_0^{+\infty}k_1(s) \times[x^*(\theta-s)-x_1(\theta-s)]dsd\theta\right)\ge 0. \tag{17}$$

where $T_3=T_2+\tau$.

For $t=t_j, j\in N$, we can easily check that

$$V(t_j^+)=V(t_j).$$

Integrating both sides of (17) on the interval $[T_3,t)$, we have

$$\begin{aligned}&-V(t)-a_1^l m_1\int_{T_3}^{t}\int_0^{+\infty}k_1(s)\\ &\times[x^*(\theta-s)-x_1(\theta-s)]dsd\theta\ge -V(T_3),\end{aligned}$$

that is,

$$\begin{aligned}&a_1^l m_1\int_{T_3}^{t}\int_0^{+\infty}k_1(s)[x^*(\theta-s)-x_1(\theta-s)]dsd\theta\\ &\le V(T_3)-V(t).\end{aligned}$$

Since $V(t)$ is bounded, let $t\to+\infty$, then

$$0<\int_{T_3}^{+\infty}\int_0^{+\infty}k_1(s)[x^*(\theta-s)-x_1(\theta-s)]dsd\theta<+\infty.$$

On the other hand, $x^*(t)-x_1(t)$ is a nonnegative, bounded and differential function such that $x^{*\prime}(t)-x_1'(t)$ is bounded on $[T_3,+\infty)$. Hence, by the mean valued theorem, $x^{*\prime}(t) - x_1'(t)$ is uniformly continuous on $[T_3,+\infty)$. Thus by Barbalat's Lemma, one can conclude that

$$\lim_{t\to+\infty}(x^*(t)-x_1(t))=0.$$

This completes the proof.

Theorem 3. *Under the assumption of Theorem 1, let $x(t)=(x_1(t),x_2(t))^T$ be any positive solution of system (1) with initial condition (2), then the species x_2 will be driven to extinction, that is, $x_2(t)\to 0$ as $t\to+\infty$, and $x_1(t)\to$*

$x^*(t)$ as $t \to +\infty$, where $x^*(t)$ is any positive solution of equation (14).

Proof: The logistic equation (14)

$$\begin{cases} x'(t) = x(t)[r_1(t) - a_1(t)x(t)], t \neq t_j, j \in N, \\ x(t_j^+) = (1 - h_{1j})x(t_j). \end{cases}$$

can be written as

$$\begin{cases} x'(t) &= x(t)\Big[r_1(t) - a_1(t)\int_0^{+\infty} k_1(s)x(t)ds\Big], \\ & \quad t \neq t_j, j \in N, \\ x(t_j^+) &= (1 - h_{1j})x(t_j). \end{cases}$$

The following proof is similar to that of Theorem 2, we omit it here. This completes the proof.

IV. Extinction of x_1 and stability of x_2

In this section, we present the extinction of the species x_1.

Theorem 4. *Assume that the inequality*

$$\limsup_{t\to+\infty} \frac{\tilde{r}_1(t)}{\tilde{r}_2(t)} < \liminf_{t\to+\infty} \left\{ \frac{a_1(t)c(t)}{b_1(t)}, \frac{b_2(t)}{a_2(t)(c(t)+x_2^*)} \right\} \quad (18)$$

or

$$\liminf_{t\to+\infty} \frac{\tilde{r}_2(t)}{\tilde{r}_1(t)} > \limsup_{t\to+\infty} \left\{ \frac{b_1(t)}{a_1(t)c(t)}, \frac{a_2(t)(c(t)+x_2^*)}{b_2(t)} \right\} \quad (19)$$

holds, where $\tilde{r}_1(t)$ and $\tilde{r}_2(t)$ have been defined in (5) and (6), respectively. Then the species x_1 will be driven to extinction, that is, for any positive solution $(x_1(t), x_2(t))^T$ of system (1), $x_1(t) \to 0$ exponentially as $t \to +\infty$.

Remark 2. *The proof of Theorem 4 is similar to the proof of Theorem 1. So we omit here.*

Consider the following impulsive logistic equation

$$\begin{cases} x'(t) = x(t)(r_2(t) - a_2(t)x(t-\tau)), t \neq t_j, j \in N, \\ x(t_j^+) = (1 - h_{2j})x(t_j). \end{cases} \quad (20)$$

Next, we study the global attractivity of the species x_2 of system (1).

Theorem 5. *Under the assumptions of Theorem 4. Further assume that*

$$\delta = \limsup_{t\to+\infty} x_2^* \int_{t-\tau}^{t} [a_2(z+\tau) + a_2(z+2\tau)]dz < 2, \quad (21)$$

where $x_2^* = \frac{M_2 r_2^u}{m_2 a_2^l}\exp\{r_2^u\tau\}$. Let $\tilde{x}(t) = (x_1(t), x_2(t))^T$ be any positive solution of system (1), then the species x_1 will be driven to extinction, that is, $x_1(t) \to 0$ as $t \to +\infty$, and $x_2(t) \to x(t)$ as $t \to +\infty$, where $x(t)$ is any positive solution of system (20).

Proof: Let $\tilde{x}(t) = (x_1(t), x_2(t))^T$ is a solution of system (1) with initial conditions (2), $x(t)$ is a solution of system (20).

Set

$$x_2(t) = x(t)\exp\{\eta(t)\}. \quad (22)$$

For $t \neq t_j, j \in N$, taking derivative on both sides of (22), we have

$$\begin{aligned} \frac{d\eta(t)}{dt} &= \frac{x_2'(t)}{x_2(t)} - \frac{x'(t)}{x(t)} \\ &= -a_2(t)x_2(t-\tau) - [-a_2(t)x(t-\tau)] \\ &\quad - \frac{b_1(t)\int_0^{+\infty} k_1(s)x_1(t-s)ds}{c(t) + \int_0^{+\infty} k_1(s)x_1(t-s)ds} \\ &= -a_2(t)x(t-\tau)\exp\{\eta(t-\tau)\} \\ &\quad -[-a_2(t)x(t-\tau)] - f(t) \\ &\triangleq F(t,\eta) - F(t,0) - f(t), \end{aligned} \quad (23)$$

where $f(t) = \frac{b_1(t)\int_0^{+\infty} k_1(s)x_1(t-s)ds}{c(t)+\int_0^{+\infty} k_1(s)x_1(t-s)ds}$. From Theorem 4 and Remark 1,

$$\lim_{t\to+\infty} f(t) = 0. \quad (24)$$

By using the mean value theorem of differential calculus, it follows from (23) that

$$\frac{d\eta(t)}{dt} = -J(t)\eta(t-\tau) - f(t),$$

where

$$J(t) = -F_\eta(t,\zeta) = a_2(t)x(t-\tau)\exp\{\zeta(t)\}, \quad (25)$$

and $\zeta(t)$ lies between 0 and $\eta(t-\tau)$, then

$$\begin{aligned} &\min\{x_2(t-\tau), x(t-\tau)\} \le x(t-\tau)\exp\{\zeta(t)\} \\ &\le \max\{x_2(t-\tau), x(t-\tau)\}. \end{aligned} \quad (26)$$

Consider the following Lyapunov function

$$\begin{aligned} V(t) &= [\eta(t) - \int_{t-\tau}^{t} J(z+\tau)\eta(z)dz]^2 \\ &\quad + \int_{t-\tau}^{t} J(z+2\tau)\int_{s}^{t} J(z+\tau)\eta^2(\theta)d\theta dz. \end{aligned}$$

For $t \neq t_j, j \in N$, calculating the upper right derivative of $V(t)$, we have

$$\begin{aligned} &D^+V(t) \\ &= 2[\eta(t) - \int_{t-\tau}^{t} J(z+\tau)\eta(z)dz][-J(t+\tau)\eta(t) - f(t)] \\ &\quad + \int_{t-\tau}^{t} J(z+2\tau)dz J(t+\tau)\eta^2(t) \\ &\quad - J(t+\tau)\int_{t-\tau}^{t} J(\theta+\tau)\eta^2(\theta)d\theta \\ &< 2[\eta(t) - \int_{t-\tau}^{t} J(z+\tau)\eta(z)dz][-J(t+\tau)\eta(t)] \\ &\quad + \int_{t-\tau}^{t} J(z+2\tau)dz J(t+\tau)\eta^2(t) \\ &\quad + 2f(t)|\eta(t) - \int_{t-\tau}^{t} J(z+\tau)\eta(z)dz|. \end{aligned}$$

Noting the fact that $2\eta(t)\eta(z) \le \eta^2(t) + \eta^2(z)$, then

$$\begin{aligned} D^+V(t) &< -\eta^2(t)J(t+\tau) \\ &\quad \times[2 - \int_{t-\tau}^{t} (J(z+\tau) + J(z+2\tau))dz] \\ &\quad + 2f(t)|\eta(t) - \int_{t-\tau}^{t} J(z+\tau)\eta(z)dz|. \end{aligned}$$

From (25)-(26), it is easy to see that

$$0 < \min_{t\geq 0} J(t+\tau) < x_2^* a_2^u.$$

Further more, from (21) and (24), there exists a positive constant $\varepsilon(0<\varepsilon<\frac{2-\delta}{2})$ small sufficiently and a positive constant T_4 such that

$$2f(t)|\eta(t) - \int_{t-\tau}^{t} J(z+\tau)\eta(z)dz| < \varepsilon,$$
$$\int_{t-\tau}^{t} (J(z+\tau)+J(z+2\tau))dz < \delta+\varepsilon$$

for all $t > T_4$, and then

$$\begin{aligned} D^+V(t) &< -\eta^2(t)[(2-\delta-\varepsilon)J(t+\tau)]+\varepsilon \\ &< -\eta^2(t)[(2-\delta-\varepsilon)\min_{t\geq 0} J(t+\tau)]+\varepsilon, t > T_4. \end{aligned}$$

Let $\varepsilon \to 0$, then

$$D^+V(t) \leq -\eta^2(t)[(2-\delta)\min_{t\geq 0} J(t+\tau)] < 0, t > T_4. \quad (27)$$

For $t = t_j, j \in N$, we can easily check that

$$\eta(t_j^+) = \eta(t_j), V(t_j^+) = V(t_j).$$

Integrating both sides of (27) on the interval $[T_4, t)$, we have

$$V(t) + (2-\delta)\min_{t\geq 0} J(t+\tau)\int_{T_4}^{t} \eta^2(\theta)d\theta \leq V(T_4) < +\infty.$$

Therefore, $V(t)$ is bounded on $[T_4, +\infty)$ and there is $\int_{T_4}^{+\infty} \eta^2(t)dt < +\infty$. Then we claim that

$$\lim_{t\to+\infty} \eta(t) = 0. \quad (28)$$

Otherwise, for any given $\varepsilon > 0$, there are two cases:

(i) For any $T_4 > 0$, when $t > T_4$, $|\eta(t)| \geq \varepsilon$;

(ii) For any $T_4 > 0$, when $t > T_4$, $|\eta(t)|$ is oscillatory about ε.

For case (i), we have $\int_{T_4}^{+\infty} \eta^2(t)dt \geq \int_{T_4}^{+\infty} \varepsilon^2 dt \to +\infty$, which is a contradiction.

For case (ii), we can choose two sequences $\rho(n)$ and $\rho^*(n)$ satisfying $T_4 < \rho_1 < \rho_1^* < \rho_2 < \rho_2^* < \cdots$, and $\lim_{t\to+\infty} \rho_n = \lim_{t\to+\infty} \rho_n^* = +\infty$ such that

$$|\eta(\rho_n)| \geq \varepsilon; |\eta(\rho_n^+)| \leq \varepsilon; |\eta(\rho_n^*)| \leq \varepsilon; |\eta(\rho_n^{*+})| \geq \varepsilon;$$
$$|\eta(t)| \leq \varepsilon, \forall t \in (\rho_n, \rho_n^*); |\eta(t)| \geq \varepsilon, \forall t \in (\rho_n^*, \rho_{n+1});$$

and then

$$\begin{aligned} \int_{T_4}^{+\infty} \eta^2(t)dt &= \int_{T_4}^{\rho_1} \eta^2(t)dt + \sum_{n=1}^{+\infty}\int_{\rho_n}^{\rho_n^*} \eta^2(t)dt \\ &\quad + \sum_{n=1}^{+\infty}\int_{\rho_n^*}^{\rho_{n+1}} \eta^2(t)dt \\ &\geq \sum_{n=1}^{+\infty}\int_{\rho_n^*}^{\rho_{n+1}} \varepsilon^2 dt \to +\infty, \end{aligned}$$

which is also a contradiction.

Combine (22) and (28), we have

$$\lim_{t\to+\infty} x_2(t) = x(t).$$

This completes the proof.

V. An example

In this section, we give an example to illustrate the feasibility of our results,

$$\begin{aligned} x_1'(t) &= x_1(t)\Big[r_1(t) - a_1(t)\int_0^{+\infty} k_1(s)x_1(t-s)ds \\ &\qquad - \frac{b_2(t)x_2(t-\tau)}{c(t)+x_2(t-\tau)}\Big], \\ x_2'(t) &= x_2(t)\Big[r_2(t) - a_2(t)x_2(t-\tau) \\ &\qquad - \frac{b_1(t)\int_0^{+\infty} k_1(s)x_1(t-s)ds}{c(t)+\int_0^{+\infty} k_1(s)x_1(t-s)ds}\Big], \\ &\qquad t \neq t_j, j \in N, \\ x_1(t_j^+) &= (1-h_{1j})x_1(t_j), \\ x_2(t_j^+) &= (1-h_{2j})x_2(t_j). \end{aligned} \quad (29)$$

Let

$$p(t) = \int_0^{+\infty} k_1(s)x_1(t-s)ds, k_1(s) = \mu_1 e^{-\mu_1 s},$$

then (29) can be written as

$$\begin{aligned} x_1'(t) &= x_1(t)\Big[r_1(t) - a_1(t)p(t) - \frac{b_2(t)x_2(t-\tau)}{c(t)+x_2(t-\tau)}\Big], \\ x_2'(t) &= x_2(t)\Big[r_2(t) - a_2(t)x_2(t-\tau) - \frac{b_1(t)p(t)}{c(t)+p(t)}\Big], \\ p'(t) &= \mu_1(x_1(t)-p(t)), t \neq t_j, j \in N, \\ x_1(t_j^+) &= (1-h_{1j})x_1(t_j), \\ x_2(t_j^+) &= (1-h_{2j})x_2(t_j). \end{aligned}$$

Choose the coefficients

$$\begin{aligned} &r_1(t) = 1.6 - 0.2\cos(t), a_1(t) = 1.5 - 0.2\sin(t), \\ &r_2(t) = 0.2 - 0.1\sin(t), a_2(t) = 1.2 + 0.3\cos(t), \\ &b_1(t) = 2 + 0.5\cos(t), b_2(t) = 1 - 0.5\sin(t), \\ &c(t) = 1, \mu_1 = 0.5, \tau = 0.1, h_{1j} = 1 - \exp\{\frac{1}{3}\}, \\ &h_{2j} = 1 - \exp\{\frac{1}{2}\}. \end{aligned}$$

Let $\omega = 2\pi, t_j = j\pi/2$, then $q = 4$. By a direct calculation, we can get

$$\begin{aligned} &\liminf_{t\to+\infty} \frac{\tilde{r}_1(t)}{\tilde{r}_2(t)} = 2.8281 > 2.7461 \\ &= \limsup_{t\to+\infty} \left\{\frac{a_1(t)(c(t)+x_1^*)}{b_1(t)}, \frac{b_2(t)}{a_2(t)c(t)}\right\}; \\ &\tilde{r}_1(t) = r_1(t) + \frac{1}{\omega}\sum_{j=1}^{q} \ln(1-h_{1j}) > 0; \\ &\tilde{r}_2(t) = r_2(t) + \frac{1}{\omega}\sum_{j=1}^{q} \ln(1-h_{2j}) > 0; \end{aligned}$$

that is the conditions of Theorems 2 and 3 hold, and so the species x_2 will be driven to extinction while the species x_1 is asymptotically to any positive solution of

$$\begin{cases} x'(t) &= x(t)[r_1(t) - a_1(t)\int_0^{+\infty} k_1(s) \\ &\qquad \times x_1(t-s)ds], t \neq t_j, j \in N, \\ x(t_j^+) &= (1-h_{1j})x(t_j), \end{cases} \quad (30)$$

and

$$\begin{cases} x'(t) = x(t)[r_1(t) - a_1(t)x_1(t)], t \neq t_j, j \in N, \\ x(t_j^+) = (1 - h_{1j})x(t_j). \end{cases} \quad (31)$$

The solutions of systems (29), (30) and (31) corresponding to initial values are displayed in Figures 1 and 2.

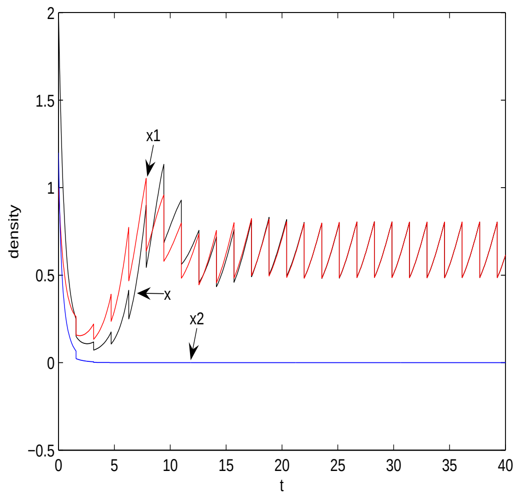

Fig. 1. Dynamic behaviors of x_1 and x_2 in system (29) with initial values $(x_1(-0.1), x_2(-0.1)) = (1, 1.2)$; x is a solution of system (30).

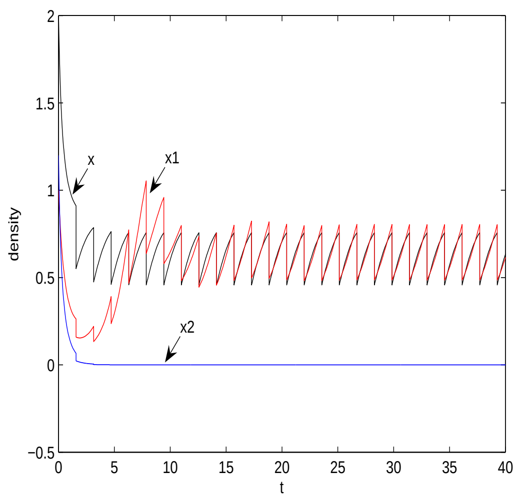

Fig. 2. Dynamic behaviors of x_1 and x_2 in system (29) with initial values $(x_1(-0.1), x_2(-0.1)) = (1, 1.2)$; x is a solution of system (31).

VI. Conclusion

This paper is concerned with an impulsive nonautonomous Lotka-Volterra competitive system with discrete delay and infinite delay, sufficient conditions which guarantee the permanence, extinction of the prey species and the predator species are obtained, respectively.

This paper provided an effective method for the further study on permanence and extinction of population dynamic systems with delays and impulses. In fact, our techniques in this paper are applicable to an impulsive competitive system with pure discrete delays or pure infinite delays.

As we know, system (1) is a basic model, based on system (1), we can establish different types of Lotka-Volterra competitive systems according to the ecological significance, such as plankton allelopathy systems, functional response systems and so on, by using the same methods and analytical techniques, and similar results can be obtained. From the obtained results, we not only can reveal the inherent law of the system, and predict the development of the population, but also can control or adjust the ecological development of the population in a better way. Future work includes the study, analysis, and modeling, one may see [16-18].

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