# An M/G/1 G-queue with Server Breakdown, Working Vacations and Bernoulli Vacation Interruption

Jing Li, Tao Li

***Abstract*—In this paper, an M/G/1 G-queue with server breakdown, working vacations and Bernoulli vacation interruption is considered. In the normal busy period, the arrival of a negative customer not only takes away the positive customer being in service, but also causes the server break down. During the repair time, the system is out of service until the repair is completed. And during the working vacation period, the vacation will be interrupted with probability $p\,(0 \le p \le 1)$ or continues with probability $\overline{p} = 1 - p$, if there are customers in the system at a service completion instant. By applying the matrix-analytic method and the supplementary variable technique, the probability generating functions of the queue length and the server state are obtained. Finally, the sensitivity analysis and cost analysis of the model are presented.**

***Index Terms*—G-queue, working vacation, Bernoulli vacation interruption, embedded Markov chain, supplementary variable method.**

## I. Introduction

WORKING vacation queues are the extension of classical vacation queues. Servi and Finn [1] first studied an M/M/1 queue with working vacations. For the working vacation policy, the server can still work at a lower rate during the vacation. This low-speed service strategy utilizes the server effectively. Thus, the working vacation plays an important role in many practical applications such as e-commerce, production management and service industries. In recent years, many scholars have made extensive research on the queueing systems with working vacations and obtained plenteous theoretical results. Wu and Takagi [2] dealt with an M/G/1 queue with multiple working vacations and derived the distributions for the queue size in the steady state. On the basis of the classical vacation decomposition in M/G/1 queue, Li and Tian [3] analyzed the M/G/1 queue with exponential working vacation and obtained a conditional stochastic decomposition result. Recently, Gao and Yao [4], Zhang and Zhou [5], Jailaxmi et al. [6] and Kasim and Gupur [7] also considered the queueing systems with working vacation.

In order to improve the operating efficiency of the system, Li and Tian [8] introduced the working vacation interruption policy. In a working vacation period, if the system is non-empty at a service completion instant, a new normal busy period starts. If the system is still empty, on the other hand, the server continues the vacation. Subsequently, Zhang and Shi [9] proposed Bernoulli vacation interruption model. The Bernoulli vacation interruption policy is related to many service systems. After each service completion, the vacation is interrupted with probability $p$ if the system is non-empty, or continues with probability $1-p$. In addition, Gao and Liu [10] treated an M/G/1 queue with single working vacation and vacation interruption under Bernoulli schedule. They got a variety of stationary performance measures for this system and gave a conditional stochastic decomposition result. Li et al. [11] discussed an M/G/1 retrial model with Bernoulli working vacation interruption. Using the matrix analytic method, Li et al. [12] investigated a GI/M/1 queue with single working vacation and Bernoulli vacation interruption. In this system, they obtained the steady-state distributions for the queue length.

Erol et al. [13] put forward the queueing model with negative customers in 1991. The queues with negative customer arrivals are called G-queues. A negative customer will vanish automatically if it arrives to the queue when the server is idle or down or on vacation. During the normal busy period, the arrival of a negative customer either removes all customers in the system or removes only one customer from the head or the end of the system. Negative customers cannot accumulate in a queue and do not receive service. Usually, the negative customer also cause the server break down. And queueing systems with negative customers and server breakdown have many applications in telecommunication and computer network systems. For example, in computer networks, the system will be infected if the virus enters the system and it will be restored as good as new after data backing-up the infected file. In recent years, queueing models with negative customers and server breakdown have been well studied. Considered the practical problems of bank performance, Harrison et al. [14] studied reliability modeling using G-queues. Wang and Zhang [15] combined negative customers with repairable systems and studied a single-server discrete-time retrial G-queue with server breakdowns and repair. Some other results about G-queues can be found in Inoue and Takine [16] and Xu et al. [17].

Considered the model of stochastic production and inventory systems with multi-purpose production facilities, Zhang and Liu [18] studied an M/G/1 G-queue with server breakdown, working vacations and vacation interruption. Positive and negative customers represent the generation and cancellation of orders, respectively. The cancellation of orders in normal production period results in the cessation of major production and the start of optional jobs. If there are orders after the disaster caused by negative customer,

Manuscript received October 04, 2019; revised February 12, 2020.
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the production facility will perform the major production. Otherwise, it will perform the optional jobs. In addition, the production equipment returns to main production after the completion of optional operations. This motivates us to analyze the queueing model that is applied to such practical production system. In order to improve the efficiency of the main production equipment, we can selectively decide whether to start the main production of the production equipment after the optional operation is completed. Therefore, combined with Bernoulli interruption strategy, we consider an M/G/1 G-queue with server breakdown, working vacations and Bernoulli vacation interruption.

This paper is organized as follows. Section II gives a brief description of the model. In Section III, we obtain the stability conditions by the matrix-analytic method. In Section IV, we deal with the joint distribution of the serve state and the number of customers. Some performance measures of this model are also discussed. Numerical results are presented in Section V. Finally, Section VI concludes the paper.

## II. Description of the queueing system

In this paper, we consider an M/G/1 G-queue with server breakdown, working vacations and Bernoulli vacation interruption. The detailed description of this model is given as follows:

1) There are two types of customers arriving in the system, positive customers and negative customers. The inter-arrival times of positive and negative customers are exponentially distributed with parameter $\lambda^+$ and $\lambda^-$, respectively.

2) In the normal busy period, the arrival of negative customer not only takes away the positive customer being served but also breaks the server down. If the negative customer arrives when the server is in the working vacation period or under the repair period, the negative customer will disappear automatically and has no impact on the system.

3) The server takes a working vacation of random length $V$ once the system becomes empty. And vacation duration $V$ follows an exponential distribution with parameter $\theta$. Upon the completion of a service in the vacation period, if there are customers in the system, the vacation can be interrupted with probability $p\,(0 \le p \le 1)$ or continues with probability $\bar{p} = 1 - p$. Meanwhile, when a vacation ends, if the system is empty, another new vacation is taken.

4) The failed server is repaired immediately and it is assumed as good as new after repair. If the system has no customer after the repair is completed, the server takes a working vacation. Otherwise, it starts a new busy period.

5) During the normal busy period, the normal service time $S_b$ has a distribution function $S_b\,(x)$, Laplace-Stieltjes transform (LST) $\widetilde{S_b}\,(s)$ and $n$th moments $\beta_n$, $n \ge 1$. Clearly, $\beta_1 = E\,(S_b) \triangleq 1/\mu_b$. During the working vacation period, positive customers can be served at a lower rate and the service time $S_v$ has a distribution function $S_v\,(x)$, LST $\widetilde{S_v}\,(s)$ and $n$th moments $\eta_n$, $n \ge 1$. And $\eta_1 = E\,(S_v) \triangleq 1/\mu_v$. During the repair period, the repairing time $R$ has a distribution function $R(t)$, LST $\widetilde{R}\,(s)$ and $n$th moments $\gamma_n$, $n \ge 1$. Clearly, $\gamma_1 = E\,(R)$.

We assume that inter-arrival times, service times, working vacation times and repair times are mutually independent. Further, it is assumed that $\mathrm{S}_b\,(0) = 0, \mathrm{S}_b\,(\infty) = 1, \mathrm{S}_v\,(0) = 0, \mathrm{S}_v\,(\infty) = 1, R\,(0) = 0, R\,(\infty) = 1$, and $S_b\,(x)$, $S_v\,(x)$ and $R\,(x)$ are continuous at $x = 0$. The functions $\mu_b\,(x)$, $\mu_v\,(x)$ and $\alpha\,(x)$ are the conditional completion rates for normal service, lower service and repair, respectively, i.e.

$$\mu_b\,(x)\,dx = \frac{dS_b\,(x)}{1 - S_b\,(x)},$$

$$\mu_v\,(x)\,dx = \frac{dS_v\,(x)}{1 - S_v\,(x)},$$

$$\alpha\,(x)\,dx = \frac{dR\,(x)}{1 - R\,(x)}.$$

Throughout the rest of the paper, for a distribution function $F\,(x)$, we define $\bar{F}\,(x) = 1 - F\,(x)$ to be the tail of $F\,(x)$. We also denote $\tilde{F}(s) = \int_0^\infty e^{-sx} dF\,(x)$, $\bar{F}^*(s) = \int_0^\infty e^{-sx} \bar{F}(x)dx$. Clearly, we have $\bar{F}^*\,(s) = \frac{1-\tilde{F}(s)}{s}$.

Let $N\,(t)$ represent the number of positive customers in the system at time t and $I\,(t)$ denote the state of server at time t. Define

$$I(t) = \begin{cases} 0, & \textit{the server is in a working vacation} \\ & \textit{period at time } t, \\ 1, & \textit{the server is during a normal service} \\ & \textit{period at time } t, \\ 2, & \textit{the server is under the repair period} \\ & \textit{at time } t. \end{cases}$$

At time $t \ge 0$, we define the random variable $\xi\,(t)$ as follows: if $I\,(t) = 0$, $\xi\,(t)$ denotes the elapsed lower service time; if $I\,(t) = 1$, $\xi\,(t)$ represents the elapsed normal service time; if $I\,(t) = 2$, $\xi\,(t)$ stands for the elapsed repair time. Then, $X\,(t) = \{I\,(t)\,, N\,(t)\,, \xi\,(t)\,, t \ge 0\}$ is a Markov process. And the state space of the process is $\Omega = \{(0,0)\} \cup \{(2,0,x)\} \cup \{(i,n,x)\,, i = 0,1,2, n \ge 1, x \ge 0\}$. Let $\{t_n, n = 1,2,3\cdots\}$ be the sequence of epochs at which a normal service or a lower service or a repair completion or a breakdown occurs and $Y_n = \{I\,(t_n^+)\,, N\,(t_n^+)\}$. Then, the sequence of random variables $\{Y_n; n \ge 1\}$ forms an embedded Markov chain with state space $\{(0,0)\} \cup \{(2,0)\} \cup \{(i,k)\,, i = 0,1,2, k \ge 1\}$.

## III. Stable condition and stationary distribution

To develop the transition matrix of $\{Y_n, n \ge 1\}$, we introduce a few definitions:

a) Define

$$a_k = \int_0^\infty \frac{(\lambda^+ x)^k}{k!} e^{-\lambda^+ x} e^{-\lambda^- x} dS_b\,(x)\,, \quad k \ge 0.$$

Then, $\{a_k; k \ge 0\}$ is the probability that $k$ positive customers arrive during $S_b$ and no negative customer arrives. We have

$$A\,(z) \triangleq \sum_{k=0}^\infty a_k z^k = \widetilde{S_b}\left[\lambda^+ (1-z) + \lambda^-\right],$$

$$A\left(1\right)=\widetilde{S_b}\left(\lambda^-\right),\quad A'\left(1\right)=\lambda^+\int_0^\infty xe^{-\lambda^- x}dS_b\left(x\right),$$

$$A''\left(1\right)=\left(\lambda^+\right)^2\int_0^\infty x^2e^{-\lambda^- x}dS_b\left(x\right).$$

b) Define

$$b_k=\int_0^\infty\frac{\left(\lambda^+x\right)^k}{k!}e^{-\lambda^+x}\lambda^-e^{-\lambda^-x}\left[1-S_b\left(x\right)\right]dx,\quad k\geq 0.$$

Thus, $\{b_k;k\geq 0\}$ is the probability that $k$ positive customers arrive before the negative customer arrives and no service complete. We can get

$$B\left(z\right)\triangleq\sum_{k=0}^\infty b_kz^k=\frac{\lambda^-}{\lambda^+\left(1-z\right)+\lambda^-}\left[1-A\left(z\right)\right],$$

$$B\left(1\right)=1-A\left(1\right),\quad B'\left(1\right)=\frac{\lambda^+}{\lambda^-}\left[1-A\left(1\right)\right]-A'\left(1\right),$$

$$B''\left(1\right)=2\left(\frac{\lambda^+}{\lambda^-}\right)^2\left[1-A\left(1\right)\right]-2\left(\frac{\lambda^+}{\lambda^-}\right)A'\left(1\right)-A''\left(1\right).$$

c) Define

$$c_k=\int_0^\infty\frac{\left(\lambda^+x\right)^k}{k!}e^{-\lambda^+x}e^{-\theta x}dS_v\left(x\right),\quad k\geq 0.$$

Then $\{c_k;k\geq 0\}$ represents the probability that $V\geq S_v$ and $k$ positive customers arrive during $S_v$. And we have

$$C\left(z\right)\triangleq\sum_{k=0}^\infty c_kz^k=\widetilde{S_v}\left[\lambda^+\left(1-z\right)+\theta\right],$$

$$C\left(1\right)=\widetilde{S_v}\left(\theta\right),\quad C'\left(1\right)=\lambda^+\int_0^\infty xe^{-\lambda^- x}dS_v\left(x\right),$$

$$C''\left(1\right)=\left(\lambda^+\right)^2\int_0^\infty x^2e^{-\lambda^- x}dS_v\left(x\right).$$

d) Define

$$d_k=\int_0^\infty\frac{\left(\lambda^+x\right)^k}{k!}e^{-\lambda^+x}\theta e^{-\theta x}\left[1-S_v\left(x\right)\right]dx,\quad k\geq 0.$$

Hence, $\{d_k;k\geq 0\}$ explains the probability that $V\leq S_v$ and $k$ positive customers arrive during $V$. And we can get

$$D\left(z\right)\triangleq\sum_{k=0}^\infty d_kz^k=\frac{\theta}{\lambda^+\left(1-z\right)+\theta}\left[1-C\left(z\right)\right],$$

$$D\left(1\right)=1-C\left(1\right),\quad D'\left(1\right)=\frac{\lambda^+}{\theta}\left[1-C\left(1\right)\right]-C'\left(1\right),$$

$$D''\left(1\right)=2\left(\frac{\lambda^+}{\theta}\right)^2\left[1-C\left(1\right)\right]-2\left(\frac{\lambda^+}{\theta}\right)C'\left(1\right)-C''\left(1\right).$$

e) Define

$$f_k=\int_0^\infty\frac{\left(\lambda^+x\right)^k}{k!}e^{-\lambda^+x}dR\left(x\right),\quad k\geq 0.$$

Thus, $\{f_k;k\geq 0\}$ is the probability that $k$ positive customers arrive during $R$. We have

$$F\left(z\right)\triangleq\sum_{k=0}^\infty f_kz^k=\widetilde{R}\left[\lambda^+\left(1-z\right)\right],\quad F\left(1\right)=1,$$

$$F'\left(1\right)=\lambda^+E\left(R\right),\quad F''\left(1\right)=\left(\lambda^+\right)^2\int_0^\infty x^2dR\left(x\right).$$

f) Define

$$g_k=\sum_{j=0}^k d_ja_{k-j},\quad k\geq 0.$$

Hence, $\{g_k;k\geq 0\}$ represents the probability that $V\leq S_v$ and $k$ positive customers arrive during $V$ plus $S_b$. We have

$$G\left(z\right)\triangleq\sum_{k=0}^\infty g_kz^k=D\left(z\right)A\left(z\right),$$

$$G\left(1\right)=D\left(1\right)A\left(1\right),\quad G'\left(1\right)=D'\left(1\right)A\left(1\right)+D\left(1\right)A'\left(1\right),$$

$$G''\left(1\right)=D''\left(1\right)A\left(1\right)+2D'\left(1\right)A'\left(1\right)+D\left(1\right)A''\left(1\right).$$

g) Define

$$q_k=\sum_{j=0}^k d_jb_{k-j},\quad k\geq 0.$$

Then $\{q_k;k\geq 0\}$ explains the probability that $V\leq S_v$ and the new started service does not complete before the negative customer arrives, and $k$ positive customers arrive during the whole period. We can get

$$Q\left(z\right)\triangleq\sum_{k=0}^\infty q_kz^k=D\left(z\right)B\left(z\right),$$

$$Q\left(1\right)=D\left(1\right)B\left(1\right),\quad Q'\left(1\right)=D'\left(1\right)B\left(1\right)+D\left(1\right)B'\left(1\right),$$

$$Q''\left(1\right)=D''\left(1\right)B\left(1\right)+2D'\left(1\right)B'\left(1\right)+D\left(1\right)B''\left(1\right).$$

Using the lexicographical sequence for the states, we can establish the following block-Jacobi matrix as the transition probability matrix of $\{Y_n;n\geq 1\}$

$$\begin{pmatrix} W_0 & W_1 & W_2 & \cdots \\ H_0 & A_1 & A_2 & \cdots \\ & A_0 & A_1 & \cdots \\ & & \ddots & \ddots \end{pmatrix},$$

where

$$W_0=\begin{pmatrix} c_0+g_0 & q_0 \\ f_0 & 0 \end{pmatrix},\qquad H_0=\begin{pmatrix} c_0+g_0 & q_0 \\ a_0 & b_0 \\ 0 & 0 \end{pmatrix},$$

$$A_0=\begin{pmatrix} \bar{p}c_0 & pc_0+g_0 & q_0 \\ 0 & a_0 & b_0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$W_k=\begin{pmatrix} \bar{p}c_k & pc_k+g_k & q_k \\ 0 & f_k & 0 \end{pmatrix},\quad k\geq 1,$$

$$A_k=\begin{pmatrix} \bar{p}c_k & pc_k+g_k & q_k \\ 0 & a_k & b_k \\ 0 & f_{k-1} & 0 \end{pmatrix},\quad k\geq 1.$$

We can easily check that

$$W_0e_0+\sum_{k=1}^\infty W_ke=e_0,\quad H_0e+\sum_{k=1}^\infty A_ke=e,\quad \sum_{k=0}^\infty A_ke=e,$$

where $e_0=\begin{pmatrix} 1 & 1 \end{pmatrix}^T,\quad e=\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T.$

**Theorem 1.** *The embedded $Markov$ chain $\{Y_n;n\geq 1\}$ is* *ergodic if and only if*

$$\left[1-\widetilde{S_b}\left(\lambda^-\right)\right]\left[\frac{\lambda^+}{\lambda^-}+\lambda^+E\left(R\right)\right]<1.$$

*Proof:* It is not difficult to see that $\{Y_n; n\geq 1\}$ is an irreducible and aperiodic Markov chain, so we just need to prove that $\{Y_n; n\geq 1\}$ is positive recurrent if and only if

$$\left[1-\widetilde{S_b}\left(\lambda^-\right)\right]\left[\frac{\lambda^+}{\lambda^-}+\lambda^+E\left(R\right)\right]<1.$$

Since

$$A=\sum_{k=0}^{\infty}A_k=\begin{pmatrix}\overline{p}C\left(1\right) & pC\left(1\right)+G\left(1\right) & Q\left(1\right)\\ 0 & A\left(1\right) & B\left(1\right)\\ 0 & F\left(1\right) & 0\end{pmatrix}$$

is a reducible stochastic matrix.
Denote

$$A_{0(2)}=\begin{pmatrix}a_0 & b_0\\ 0 & 0\end{pmatrix},\quad A_{k(2)}=\begin{pmatrix}a_k & b_k\\ f_{k-1} & 0\end{pmatrix}, k\geq 1,$$

and

$$A_{(2)}=\sum_{k=0}^{\infty}A_{k(2)}=\begin{pmatrix}A\left(1\right) & B\left(1\right)\\ F\left(1\right) & 0\end{pmatrix},$$

then $A_{(2)}$ is the degenerative stochastic matrix and the invariant probability vector of matrix $A_{(2)}$ is $\pi=\begin{pmatrix}\pi_1 & \pi_2\end{pmatrix}$, where

$$\pi_1=\frac{1}{1+B\left(1\right)},\quad \pi_2=\frac{B\left(1\right)}{1+B\left(1\right)}.$$

The vector $\beta$ is defined by

$$\beta=\sum_{k=1}^{\infty}kA_{k(2)}e_0.$$

Thus $\beta$ is explicitly given by

$$\beta=\begin{pmatrix}\frac{\lambda^+}{\lambda^-}\left[1-\widetilde{S_b}\left(\lambda^-\right)\right], & 1+\lambda^+E\left(R\right)\end{pmatrix}^T.$$

It is obvious from chapter 2 of Neuts [19] that the embedded Markov chain $\{Y_n; n\geq 1\}$ is positive recurrent if and only if

$$\pi\beta<1\Leftrightarrow\left[1-\widetilde{S_b}\left(\lambda^-\right)\right]\left[\frac{\lambda^+}{\lambda^-}+\lambda^+E\left(R\right)\right]<1.$$

And with the Burke's theorem [20], the steady state probabilities of the Markov process $X\left(t\right)$ exist if and only if the stable condition $\left[1-\widetilde{S_b}\left(\lambda^-\right)\right]\left[\frac{\lambda^+}{\lambda^-}+\lambda^+E\left(R\right)\right]<1$ holds. ■

Now we define the limiting probability and limiting probability densities:

$$P_{0,0}=\lim_{t\to\infty}P\left(I\left(t\right)=0,N\left(t\right)=0\right),$$
$$P_{0,n}\left(x\right)dx=\lim_{t\to\infty}P(I\left(t\right)=0,N\left(t\right)=n,\ x\leq\xi\left(t\right)<x+dx),n\geq 1,$$
$$P_{1,n}\left(x\right)dx=\lim_{t\to\infty}P(I\left(t\right)=1,N\left(t\right)=n,\ x\leq\xi\left(t\right)<x+dx),n\geq 1,$$
$$P_{2,n}\left(x\right)dx=\lim_{t\to\infty}P(I\left(t\right)=2,N\left(t\right)=n,\ x\leq\xi\left(t\right)<x+dx),n\geq 0.$$

## IV. Steady state analysis

The following system of equations that govern the dynamics of the system are obtained by the method of supplementary variable technique.

$$\lambda^+P_{0,0}=\int_0^{\infty}P_{0,1}\left(x\right)\mu_v\left(x\right)dx+\int_0^{\infty}P_{1,1}\left(x\right)\mu_b\left(x\right)dx+\int_0^{\infty}P_{2,0}\left(x\right)\alpha\left(x\right)dx,\quad (1)$$

$$\frac{dP_{0,n}\left(x\right)}{dx}=-\left[\lambda^++\theta+\mu_v\left(x\right)\right]P_{0,n}\left(x\right)+\left(1-\delta_{n,1}\right)\lambda^+P_{0,n-1}\left(x\right),\quad n\geq 1,\quad (2)$$

$$\frac{dP_{1,n}\left(x\right)}{dx}=-\left[\lambda^++\lambda^-+\mu_b\left(x\right)\right]P_{1,n}\left(x\right)+\left(1-\delta_{n,1}\right)\lambda^+P_{1,n-1}\left(x\right),\quad n\geq 1,\quad (3)$$

$$\frac{dP_{2,n}\left(x\right)}{dx}=-\left[\lambda^++\alpha\left(x\right)\right]P_{2,n}\left(x\right)+\left(1-\delta_{n,0}\right)\lambda^+P_{2,n-1}\left(x\right),\quad n\geq 0,\quad (4)$$

where $\delta_{n,0}$ and $\delta_{n,1}$ are the Kronecker's symbol.
The boundary conditions are

$$P_{0,n}\left(0\right)=\int_0^{\infty}P_{0,n+1}\left(x\right)\mu_v\left(x\right)\overline{p}dx+\delta_{n,1}\lambda^+P_{0,0},\quad n\geq 1,\quad (5)$$

$$P_{1,n}\left(0\right)=\int_0^{\infty}P_{1,n+1}\left(x\right)\mu_b\left(x\right)dx+\theta\int_0^{\infty}P_{0,n}\left(x\right)dx+p\int_0^{\infty}P_{0,n+1}\left(x\right)\mu_v\left(x\right)dx+\int_0^{\infty}P_{2,n}\left(x\right)\alpha\left(x\right)dx,\quad n\geq 1,\quad (6)$$

$$P_{2,n}\left(0\right)=\lambda^-\int_0^{\infty}P_{1,n+1}\left(x\right)dx,\quad n\geq 0,\quad (7)$$

and the normalization condition is

$$P_{0,0}+\sum_{n=1}^{\infty}\int_0^{\infty}P_{0,n}\left(x\right)dx+\sum_{n=1}^{\infty}\int_0^{\infty}P_{1,n}\left(x\right)dx+\sum_{n=0}^{\infty}\int_0^{\infty}P_{2,n}\left(x\right)dx=1.\quad (8)$$

By introducing the generating functions

$$P_i\left(x,z\right)=\sum_{n=b}^{\infty}P_{i,n}\left(x\right)z^n, i=0,1,b=1; i=2,b=0,$$

from (2)-(4), we can have

$$P_0\left(x,z\right)=\left[1-S_v\left(x\right)\right]e^{-\left[\lambda^+(1-z)+\theta\right]x}P_0\left(0,z\right),\quad (9)$$
$$P_1\left(x,z\right)=\left[1-S_b\left(x\right)\right]e^{-\left[\lambda^+(1-z)+\lambda^-\right]x}P_1\left(0,z\right),\quad (10)$$
$$P_2\left(x,z\right)=\left[1-R\left(x\right)\right]e^{-\left[\lambda^+(1-z)\right]x}P_2\left(0,z\right).\quad (11)$$

By (1), (5)-(7), after some computations, we can have

$$\lambda^+P_{0,0}=C\left(0\right)P_{0,1}\left(0\right)+A\left(0\right)P_{1,1}\left(0\right)+F\left(0\right)P_{2,0}\left(0\right),\quad (12)$$

$$P_0\left(0,z\right)=\frac{z\overline{p}C\left(0\right)P_{0,1}\left(0\right)-z^2\lambda^+P_{0,0}}{f\left(z\right)},\quad (13)$$

$$P_1(0,z) = \frac{h(z)}{g(z)} P_0(0,z) + \frac{z(1-z)}{g(z)} \lambda^+ P_{0,0}, \tag{14}$$

$$P_2(0,z) = \frac{B(z)}{z} P_1(0,z), \tag{15}$$

where

$$\begin{aligned} g(z) &\triangleq F(z)B(z) + A(z) - z, \\ f(z) &\triangleq \bar{p}C(z) - z, \\ h(z) &\triangleq z - C(z) - zD(z). \end{aligned}$$

In order to obtain $P_i(0,z), i = 0,1,2$, we give the following lemma to analyze the roots of $g(z) = F(z)B(z) + A(z) - z = 0$ in the range $0 < z < 1$.

**Lemma 1.** *If* $\left[1 - \widetilde{S_b}(\lambda^-)\right]\left[\frac{\lambda^+}{\lambda^-} + \lambda^+ E(R)\right] < 1$, *the equation* $g(z) = 0$ *has no root in the interval* $(0,1)$ *and has a root* $z = 1$.

*Proof:* It is obvious that $g(1) = 0$ and $g(0) > 0$. For any $0 < z < 1$, we have

$$\begin{aligned} g'(z) &= F'(z)B(z) + F(z)B'(z) + A'(z) - 1, \\ g'(1) &= \lambda^+ E(R)[1 - A(1)] + \frac{\lambda^+}{\lambda^-}[1 - A(1)] - 1 \\ &= \left[1 - \tilde{S}_b(\lambda^-)\right]\left[\frac{\lambda^+}{\lambda^-} + \lambda^+ E(R)\right] - 1 < 0, \\ g''(z) &= F''(z)B(z) + F'(z)B'(z) + F(z)B''(z) \\ &\quad + A''(z) > 0. \end{aligned}$$

Thus, $g''(z) > 0$ indicats that $g'(z) < g'(1) < 0$. And $g'(z) < 0$ means that $g(z) > g(1) = 0$. ■

In the following, we give the second lemma.

**Lemma 2.** *If* $0 \le p < 1$, *the equation* $f(z) = 0$ *has the unique root* $z = \beta$ *in the interval* $(0,1)$.

*Proof:* Clearly,

$$f(0) = \bar{p}C(0) > 0, \quad f(1) = \bar{p}C(1) - 1 < 0.$$

For any $0 < z < 1$, we have

$$f'(z) = \bar{p}C'(z) - 1, \quad f''(z) = \bar{p}C''(z) > 0,$$

which means $f(z)$ is a convex function in the interval $(0,1)$. Thus $f(0) > 0$ and $f(1) < 0$ indicate that the $f(z) = 0$ has the unique root $z = \beta$ in the interval $(0,1)$. ■

By Lemma 2, taking $z = \beta$ in (13) leads to

$$\beta\bar{p}C(0)P_{0,1}(0) = \beta^2 \lambda^+ P_{0,0}.$$

Thus, we have

$$P_0(0,z) = \frac{z\beta^2 - z^2}{f(z)} \lambda^+ P_{0,0}. \tag{16}$$

Taking (16) into (14), we can obtain

$$P_1(0,z) = \frac{h(z)(z\beta - z^2)}{g(z)f(z)} \lambda^+ P_{0,0} + \frac{z(1-z)}{g(z)} \lambda^+ P_{0,0}. \tag{17}$$

Then we get

$$P_2(0,z) = \frac{h(z)(\beta - z)B(z)}{g(z)f(z)} \lambda^+ P_{0,0} + \frac{(1-z)B(z)}{g(z)} \lambda^+ P_{0,0}. \tag{18}$$

**Remark 1.** *If* $p = 1$, *the model reduces to an M/G/1 G-queue with working vacations and vacation interruption. Hence* $f(z) = -z$. *And takes* $f(z) = -z$ *into (13)-(15), we get*

$$\begin{aligned} P_0(0,z) &= z\lambda^+ P_{0,0}, \\ P_1(0,z) &= \frac{h(z) + (1-z)}{g(z)} z\lambda^+ P_{0,0} \\ &= \frac{z(\frac{\lambda^+}{\theta} + 1)(1-z)D(z)}{g(z)} \lambda^+ P_{0,0}, \\ P_1(0,z) &= \frac{h(z) + (1-z)}{g(z)} B(z)\lambda^+ P_{0,0} \\ &= \frac{(\frac{\lambda^+}{\theta} + 1)(1-z)D(z)B(z)}{g(z)} \lambda^+ P_{0,0}. \end{aligned}$$

Obviously, $f(z) = -z = 0$ has the root $z = 0$, i.e., $\beta = 0$, which means (16)-(18) also hold for $p = 1$.

Next we need to introduce the following lemma before find $P_{0,0}$, and we omit the proof.

**Lemma 3.**

$$\begin{aligned} & f(1) = \bar{p}C(1) - 1, \quad f'(1) = \bar{p}C'(1) - 1, \\ & f''(1) = \bar{p}C''(1), \quad g(1) = 0, \quad h(1) = 0, \\ & g'(1) = F'(1)B(1) + F(1)B'(1) + A'(1) - 1, \\ & g''(1) = F''(1)B(1) + 2F'(1)B'(1) + F(1)B''(1) + A''(1), \\ & h'(1) = 1 - C'(1) - D(1) - D'(1), \\ & h''(1) = -C''(1) - 2D'(1) - D''(1), \\ & S_b^*[\lambda^+(1-z) + \lambda^-] = \frac{B(z)}{\lambda^-}, \\ & R^*[\lambda^+(1-z)] = \frac{1 - F(z)}{\lambda^+(1-z)}, \\ & S_v^*[\lambda^+(1-z) + \theta] = \frac{D(z)}{\theta}. \end{aligned}$$

Define the marginal generating functions $\Phi_i(z) = \int_0^\infty P_i(x,z)dx$, $i = 0,1,2$. Substituting (16)-(18) into (9)-(11), respectively. Then the following theorem is obtained by calculation.

**Theorem 2.**

$$\begin{aligned} \Phi_0(z) &= \int_0^\infty P_0(x,z)dx \\ &= \frac{z\beta - z^2}{f(z)} S_v^*[\lambda^+(1-z) + \theta]\lambda^+ P_{0,0}, \\ \Phi_1(z) &= \int_0^\infty P_1(x,z)dx \\ &= \left[\frac{h(z)(z\beta - z^2)}{g(z)f(z)} + \frac{z(1-z)}{g(z)}\right] S_b^*[\lambda^+(1-z) + \lambda^-] \\ &\quad \cdot \lambda^+ P_{0,0}, \\ \Phi_2(z) &= \int_0^\infty P_2(x,z)dx \\ &= \left[\frac{h(z)(\beta - z)}{g(z)f(z)} + \frac{1-z}{g(z)}\right] B(z)R^*[\lambda^+(1-z)] \\ &\quad \cdot \lambda^+ P_{0,0}. \end{aligned}$$

*Based on the normalization condition*

$$P_{0,0} + \Phi_0(1) + \Phi_1(1) + \Phi_2(1) = 1,$$

$P_{0,0}$ *is calculated by*

$$P_{0,0} = (\lambda^+ \Delta)^{-1},$$

*where*

$$\Delta = 1 + \frac{(\beta-1)S_v^*(\theta)}{f(1)} + \frac{[h'(1)(\beta-1) - f(1)]S_b^*(\lambda^-)}{g'(1)f(1)} + \frac{B(1)R^*(0)[h'(1)(\beta-1) - f(1)]}{g'(1)f(1)}.$$

Apparently, the probability generating function of the number of customers in the system is given by

$$\Phi(z) = P_{0,0} + \Phi_0(z) + \Phi_1(z) + \Phi_2(z).$$

The probability that the server is in a working vacation period is given by

$$P_w = P_{0,0} + \Phi_0(1) = \frac{f(1) + \lambda^+(\beta-1)S_v^*(\theta)}{f(1)} P_{0,0}.$$

The probability that the server is in a normal period is given by

$$P_b = \Phi_1(1) = \frac{h'(1)(\beta-1) - f(1)}{g'(1)f(1)} S_b^*(\lambda^-)\lambda^+ P_{0,0}.$$

The probability that the server is in a repair period is given by

$$P_r = \Phi_2(1) = \frac{h'(1)(\beta-1) - f(1)}{g'(1)f(1)} B(1)R^*(0)\lambda^+ P_{0,0}.$$

Let $E[L_i]$ denote the average number of customers in the system when the server's state is $i$, $i = 0, 1, 2$ and from Theorem 2, after the calculations we can get

$$E[L_0] = \lim_{z\to 1} \Phi_0'(z) = \triangle_0 \lambda^+ P_{0,0},$$
$$E[L_1] = \lim_{z\to 1} \Phi_1'(z) = \triangle_1 \lambda^+ P_{0,0},$$
$$E[L_2] = \lim_{z\to 1} \Phi_2'(z) = \triangle_2 \lambda^+ P_{0,0},$$

where

$$\triangle_0 = \frac{[(\beta-2)S_v^*(\theta) + (\beta-1)S_v^{*'}(\theta)]f(1)}{f^2(1)} - \frac{f'(1)(\beta-1)S_v^*(\theta)}{f^2(1)},$$

$$\triangle_1 = \left[\frac{[h''(1)(\beta-1) - 2h'(1)]}{2g'(1)f(1)} - \frac{g''(1)}{2g'(1)^2}\right] S_b^*(\lambda^-) - \frac{(\beta-1)\left[\frac{g''(1)h'(1)}{g'(1)} + \frac{2f'(1)h'(1)}{f(1)}\right]}{2g'(1)f(1)} S_b^*(\lambda^-) + \left[\frac{h'(1)(\beta-1)}{g'(1)f(1)} - \frac{1}{g'(1)}\right]\left[S_b^*(\lambda^-) + S_b^{*'}(\lambda^-)\right],$$

$$\triangle_2 = \left[\frac{h''(1)(\beta-1) - 2h'(1)}{2g'(1)f(1)} - \frac{g''(1)}{2g'(1)^2}\right] B(1)R^*(0) - \frac{(\beta-1)\left[\frac{g''(1)h'(1)}{g'(1)} + \frac{2f'(1)h'(1)}{f(1)}\right]}{2g'(1)f(1)} B(1)R^*(0) + \left[\frac{h'(1)(\beta-1)}{g'(1)f(1)} - \frac{1}{g'(1)}\right] B'(1)R^*(0) + \left[\frac{h'(1)(\beta-1)}{g'(1)f(1)} - \frac{1}{g'(1)}\right] B(1)R^{*'}(0).$$

Hence, the mean system length ($E[L]$) is given by

$$E[L] = \lim_{z\to 1} \Phi'(z) = E[L_0] + E[L_1] + E[L_2].$$

Let $E[W]$ be the expected sojourn time of a customer in the system, using Little's formula, we can have $E[W] = \frac{E[L]}{\lambda^+}$.

## V. Numerical results

In this section, we present some numerical examples to illustrate the effect of the varying parameters on the mean system length. It is assumed that $S_v(x)$, $S_b(x)$ and $R(x)$ are exponential distribution functions with rates $\mu_v$, $\mu_b$ and $\alpha$, respectively.

Under the stable condition $\lambda^+ + \frac{\lambda^- \lambda^+}{\alpha} < \lambda^- + \mu_b$, we set the value of some parameters in the model to be $\lambda^+ = 1.5$, $\lambda^- = 1$, $\theta = 1$, $p = 0.5$, $\alpha = 1.8$, $\mu_b = 8$, $\mu_v = 1$ unless they are selected as independent variables in the numerical analysis.

### A. Sensitivity Analysis

In the first situation, the effect of negative customer arrival rate $\lambda^-$ on the mean system length $E[L]$ are showed in Fig.1. It is clear that $E[L]$ increases with the increase of $\lambda^-$. The reason is that the arrival of a negative customer causes the server break down when the system is in a normal period, and the server does not provide service to the arriving customers during the repair period. Thus, the probability that the server is under the repair period increases as the value of $\lambda^-$ is increasing, and $E[L]$ also increases. Fig.1 also reflects that the increase of $\alpha$ reduces the value of $E[L]$, this is because that the expected repair time is $1/\alpha$.

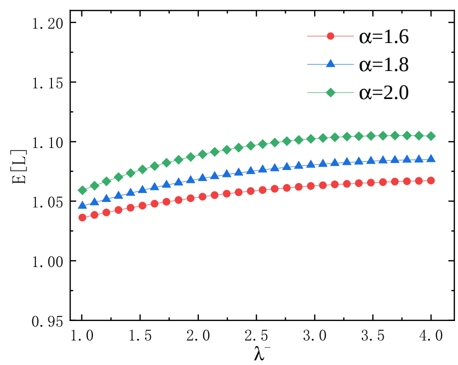


Fig. 1: The effect of $\lambda^-$ on $E[L]$ for different values of $\alpha$.

In this paper, we consider the Bernoulli vacation interruption strategy, where the vacation can be interrupted with probability $p$ if the system is non-empty at s service completion. As shown in Fig.2, the mean system length $E[L]$ decreased as the vacation interruption probability $p$ increase. This is due to the fact that the probability of the server in the normal working state increase with the increasing value of $p$, and the service rate $\mu_v$ is smaller than service rate $\mu_b$. As expected, it can also be observed from Fig.2 that three curves are plotted in decreasing order which correspond to $\mu_v = 0.8, 1, 1.2$.

Finally, the effect of $\mu_v$ on $E[L]$ is presented in Fig.3. As praxis proves, the bigger the value of $\mu_v$ is, the smaller the mean system length $E[L]$ is. In our system, the server takes a working vacation once the system becomes empty, and the expected vacation time is $1/\theta$. Therefore, from Fig.3, the increase of $\theta$ causes the value of $E[L]$ to decrease. At the same time, the effect of $\theta$ on $E[L]$ is more obvious when $\mu_v$ is relatively small. With the increase of $\mu_v$, the influence of $\theta$ on $E[L]$ has a tendency to weaken.

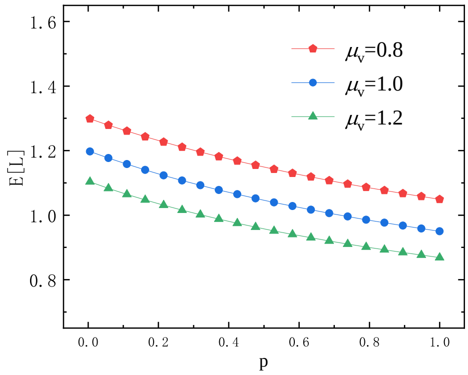


Fig. 2: The effect of $p$ on $E[L]$ for different values of $\mu_v$.

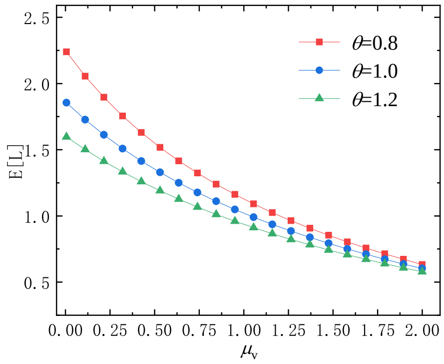


Fig. 3: The effect of $\mu_v$ on $E[L]$ for different values of $\theta$.

### B. Cost Analysis

In this subsection, in order to minimize the expected operating cost function per unit time, a cost function is imposed to search for the optimal service rate $\mu_v$.

Define the following cost elements:

$C_L$ =cost per unit time for each customer present in the system;

$C_{\mu_b}$ =cost per unit time for service during a normal service period;

$C_{\mu_v}$ =cost per unit time for service during a lower service period;

$C_\theta$ =cost per unit time during a vacation period;

$C_\alpha$ =cost per unit time during a repair period;

An expected operating cost function per unit time is established as

$$\min_{\mu_v} : f(\mu_v) = C_L E[L] + C_{\mu_b}\mu_b + C_{\mu_v}\mu_v + C_\theta\theta + C_\alpha\alpha.$$

The operating cost function per unit time is highly non-linear and complex, so we use parabolic method to solve the optimization problem. Based on the polynomial approximation theory, the unique optimum of the quadratic function agreeing with $f(x)$ at 3-point pattern $\{x_0, x_1, x_2\}$ occurs at

$$\bar{x} = \frac{1}{2}\frac{f(x_0)(x_1^2-x_2^2)+f(x_1)(x_2^2-x_0^2)+f(x_2)(x_0^2-x_1^2)}{f(x_0)(x_1-x_2)+f(x_1)(x_2-x_0)+f(x_2)(x_0-x_1)}.$$

We assume $C_L = 35$, $C_{\mu_b} = 40$, $C_{\mu_v} = 30$, $C_\theta = 25$, $C_\alpha = 20$, and use the parabolic method to find the optimum value of $\mu_v$, called $\mu_v^*$. Besides, the specific steps of the parabolic method are summarized in [21].

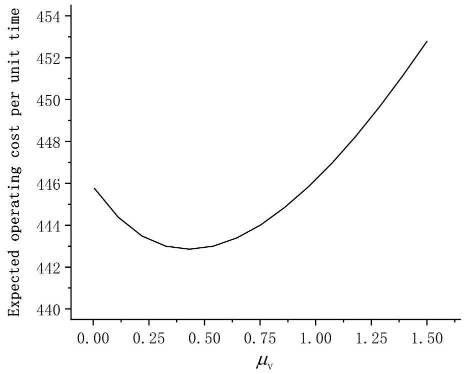


Fig. 4: The effect of $\mu_v$ on the expected operating cost per unit time.

TABLE I: Parabolic Method in Searching the Optimum Solution.

| iterations | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $x_0$ | 0.250000 | 0.250000 | 0.301880 | 0.301880 |
| $x_1$ | 0.400000 | 0.301880 | 0.304597 | 0.303444 |
| $x_2$ | 0.600000 | 0.400000 | 0.400000 | 0.304597 |
| $f(x_0)$ | 442.364440 | 442.364440 | 442.322712 | 442.322712 |
| $f(x_1)$ | 442.450342 | 442.322712 | 442.322697 | 442.322677 |
| $f(x_2)$ | 443.431831 | 442.450342 | 442.450342 | 442.322697 |
| $\bar{x}$ | 0.301880 | 0.304597 | 0.303444 | 0.303436 |
| $f(\bar{x})$ | 442.322712 | 442.322697 | 442.322677 | 442.322677 |
| tolerance | 0.098120 | 0.003717 | 0.001153 | 0.000008 |

Fig.4 shows that there is an optimal service rate $\mu_v$ to minimize the cost. After three iterations, the results in Table.1 are obtained by the parabolic method. And the error is controlled by $\varepsilon = 10^{-5}$. Obviously, the minimum expected operating cost per unit time converges to the solution $\mu_v^* = 0.303436$ with a value $f(\mu_v^*) = 442.322677$.

## VI. Conclusion

This paper analyzes an M/G/1 G-queues with serve breakdown, working vacations and Bernoulli vacation interruption. We derive the embedded Markov chain by the method of supplementary variable. And the transition probability matrix of the embedder Markov chain is established. Then we obtain the condition of stability. The generating functions of the server state and the number of customers in the system are also analyzed. In addition, we discuss various representative rerformance measures of the model. Finally, the numerical examples are presented and the effect of the parameters on the mean system length are reflected. At the same time, the cost minimization problem is also studied. For future study, one can extend this model to an $M^X/G/1$ queue, or consider the similar model but with the working breakdown policy.

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