CTE Method and Exact Solutions for a High-order Boussinesq-Burgers Equation

Jinming Zuo

Abstract—**In this work, A consistent tanh expand (CTE) method is developed for a high-order Boussinesq-Burgers (HB-B) equation. Via the CTE method, we first obtain two kinds of branches to the HBB equation. The main branches are consistent while the auxiliary branches are not consistent. From the main consistent branches, we can obtain many exact significant solutions including soliton-resonant solutions, soliton-periodic wave interactions and soliton-rational wave interactions. From the inconsistent branches, many special solutions can be found. The CTE related nonlocal symmetries are also proposed. The nonlocal symmetries can be localized to find finite Bäcklund transformations (BT) by prolonging the model to an enlarged one.**

Index Terms—**consistent tanh expand (CTE) method, high-order Boussinesq-Burgers (HBB) equation, interaction solutions, nonlocal symmetries.**

I. Introduction

nonlinear evolution equations (NLEEs) play an important role in describing nonlinear scientific phenomena, such as marine engineering, fluid dynamics, plasma physics, chemistry, physics and other fields. In order to understand the mechanisms of their physical phenomena, it is necessary to explore their explicit solutions. Some elegant methods have been constructed for finding exact solutions of NLEEs, such as Hirota's bilinear method [1], Bäcklund transformation (BT) [2], Darboux transformation (DT) [3], Painlevé analysis [4,5], Homogeneous balance method (HB) [6,7], Compact difference schemes [8], Hybrid series method [9] and so on. Recently, Lou and his group [10-13] propose the consistent Riccati expansion (CRE) and consistent tanh expansion (CTE) method through the nonlocal symmetry to find interaction solutions of NLEEs including soliton-resonant solutions, soliton-periodic wave interactions and so on. On account of this, there are a lot of paper here to study this problem [14-23].

In this work, we will discuss the following high-order Boussinesq-Burgers (HBB) equation [24-28]

$$u_t - 3\sigma u^2 u_x + \tfrac{3}{2}\sigma(uv)_x - \tfrac{1}{4}\sigma u_{xxx} = 0, \tag{1a}$$

$$\begin{aligned} &v_t + \tfrac{3}{2}\sigma v v_x - 3\sigma(u^2 v)_x + 3\sigma u_x u_{xx} \\ &+\tfrac{3}{2}\sigma u u_{xxx} - \tfrac{1}{4}\sigma v_{xxx} = 0. \end{aligned} \tag{1b}$$

where σ is a non-zero arbitrary constant.

Zuo and Zhang [24] first applied the simplified Hirota's method to derive multiple kink solutions. Guo et al. [25] applied the homogeneous balance method to find multiple-soliton (kink) solutions of the HBB equation (1). Jaradat et al. [26] and Wazwaz [27] used function expansion methods to investigate soliton and periodic solutions. Zuo [28] gave out Painlevé analysis, Lax pairs and a Darboux transformation of the HBB equation (1). The aim of this work to use the CTE method to seek the interaction solutions. The CTE related nonlocal symmetries are also proposed. The nonlocal symmetries can be localized to find finite BT by prolonging the model to an enlarged one.

Manuscript received November 25, 2019; revised February 13, 2020. The work was supported by the Natural Science Foundation of Shandong Province of China (ZR2017MA021).

Jinming Zuo is with School of Mathematics and Statistics, Shandong University of Technology, Zibo, Shandong, 255049 P. R. China; e-mail: zuojinming@sdut.edu.cn.

II. CTE Solvability for the HBB equation

For a given derivative nonlinear polynomial system

$$P(\boldsymbol{x}, t, u) = 0, \quad \boldsymbol{x} = \{x_1, x_2, \cdots, x_n\}, \tag{2}$$

we look for the following possible truncated expansion solution

$$u = \sum_{i=0}^{n} u_i \tanh^i(w), \tag{3}$$

where w is an undetermined function of space-time $\boldsymbol{x}$ and t, n should be determined from the leading order analysis of (2), and all the expansion coefficient functions u_i should be determined by vanishing the coefficients of powers $\tanh(w)$ after substituting (3) into (2).

If the system for $u_i (i = 0, 1, \cdots, n)$ and w is obtained by vanishing all the coefficients of powers $\tanh(w)$ after substituting (3) into (2) is consistent (or not overdetermined), we call the expansion (3) is a CTE and the nonlinear system (2) is CTE solvable.

By using leading order analysis (balance the highest order of $\tanh(w)$) for the HBB equation (1), we can take the following truncated $\tanh$ function expansions

$$u = u_1 \tanh(w) + u_0 + \tfrac{1}{2} w_x, \tag{4a}$$

$$v = v_2 \tanh^2(w) + v_1 \tanh(w) + v_0. \tag{4b}$$

In the expansion (4a), we have written u_0 as $u_0 + \frac{1}{2}w_x$ for convenience later. Substituting (4) into (1), and assuming all coefficients of $\tanh(w)$ vanish independently, we get the following two kids of results.

Case 1 Principle branchs. We have

$$v_2 = \tfrac{1}{2} w_x^2, u_1 = \pm\tfrac{1}{2} w_x, \tag{5}$$

$$v_1 = -\tfrac{1}{2} w_{xx}, v_0 = \mp u_{0x} - \tfrac{1}{2} w_x^2 \mp \tfrac{1}{2} w_{xx}, \tag{6}$$

while w and u_0 are determined by the following two equations

$$\text{STO}: \ u_{0t} - \sigma(\tfrac{1}{4} u_{0xx} \pm \tfrac{3}{2} u_0 u_{0x} + u_0^3)_x = 0, \tag{7a}$$

$$\begin{aligned} \text{PSTO}: \ & w_t - \sigma(\tfrac{1}{4} w_{xx} \pm \tfrac{3}{4} w_x^2 \pm \tfrac{3}{2} u_0 w_x)_x - \sigma w_x^3 \\ & -3\sigma u_0 w_x (u_0 + w_x) = 0. \end{aligned} \tag{7b}$$

Thus, according to the definition of the CTE, we have the following CTE solvable theorem for the HBB equation (1).

Theorem 1 The HBB equation (1) is CTE solvable with the CTE

$$u = \tfrac{1}{2}w_x[\pm \tanh(w) + 1] + u_0, \tag{8a}$$

$$\begin{aligned} v = {} & \tfrac{1}{2}w_x^2[\tanh^2(w) - 1] \\ & -\tfrac{1}{2}w_{xx}[\tanh(w) \pm 1] \mp u_{0x}. \end{aligned} \tag{8b}$$

and the consistent system (7).

In the CTE (7a), the w_x term has been separated out from u_0 such as the u_0 equation becomes w independent. In order to prove the CTE solvability of the HBB equation (1), it is sufficient to take u_0 as zero. However, nonzero u_0 leads to more general solutions of the HBB equation (1). The u_0 equation (7a) is just the well known linearizable Sharma-Tasso-Olver (STO) equation [29,30]. The w equation (7b) can also be linearized because it is a potential form of the variable coefficient STO equation.

Case 2 Auxiliary branchs. We have

$$v_2 = w_x^2, u_1 = \pm w_x, \tag{9a}$$

$$v_1 = -w_{xx}, u_0 = -\tfrac{1}{2}w_x \mp \tfrac{1}{2}\tfrac{w_{xx}}{w_x}, \tag{9b}$$

$$v_0 = -\tfrac{2w_t}{3\sigma w_x} + \tfrac{w_{xxx}}{6w_x} - \tfrac{1}{3}w_x^2, \tag{9c}$$

while w is determined by the following over-determined systems

$$w_t - \sigma w_x^3 + \tfrac{1}{2}\sigma w_x xx - \tfrac{3}{4}\sigma\tfrac{w_{xx}^2}{w_x} = 0, \tag{10a}$$

$$(\partial_x^2 - 4w_x^2)\left(\tfrac{w_{xxx}}{w_x} - \tfrac{3}{2}\tfrac{w_{xx}^2}{w_x^2}\right) - 10w_{xx}^2 = 0. \tag{10b}$$

Though the auxiliary branches are not a CTE, They are still useful to get a special type of exact solutions of the HBB equation (1) by solving the over-determined systems (10)

$$u = \pm\left[w_x\tanh(w) - \tfrac{1}{2}\tfrac{w_{xx}}{w_x}\right], \tag{11a}$$

$$\begin{aligned} v = {} & w_x^2[\tanh^2(w) - 1] - w_{xx}\tanh(w) \\ & +\tfrac{1}{2}\tfrac{w_{xxx}}{w_x} - \tfrac{1}{2}\tfrac{w_{xx}^2}{w_x^2}. \end{aligned} \tag{11b}$$

III. Explicit solutions of the HBB equation

According to the CTE solvability Theorem 1, by solving the w and u_0 equation (7), we can get various interaction solutions among different types of nonlinear excitations.

From Eq. (7), we have known that for any fixed solution of the STO equation (7a), the solution of the w equation (7b) can be obtained by solving a variable coefficient potential STO (PSTO) eqaution. Therefore, the corresponding solution of the HBB equation (1) can be obtained from the CTE (8).

In this work, we only restrict the trivial STO solution

$$u_0 = c. \tag{12}$$

where c is an arbitrary constant. In this case the w equation (7b) is simplified to a constant coefficient PSTO equation

$$\begin{aligned} w_t - \sigma(\tfrac{1}{4}w_{xx} \pm \tfrac{3}{4}w_x^2 \pm \tfrac{3}{2}cw_x)_x & \\ -\sigma w_x^3 - 3\sigma c w_x(c + w_x) = 0. & \end{aligned} \tag{13}$$

A. Single soliton solutions

Eq. (13) has the following trivial solution

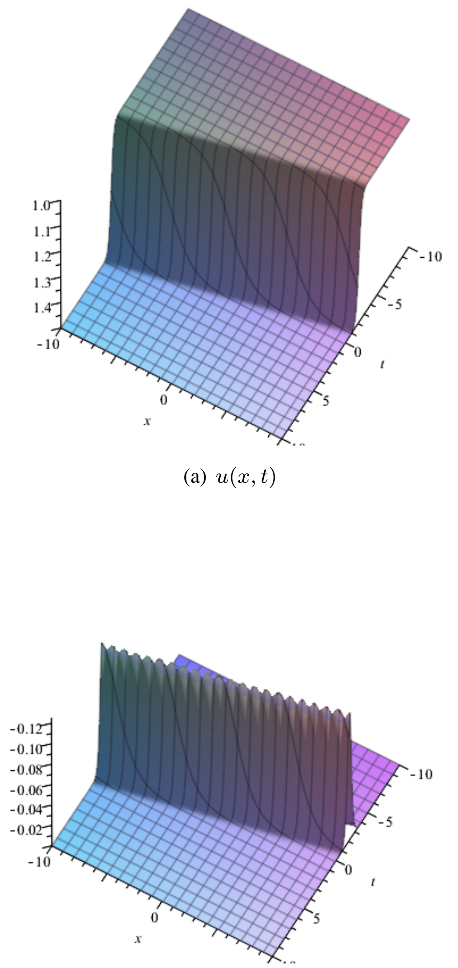
$$w = kx + \omega t, \quad \omega = \sigma k(k^2 + 3ck + 3c^2), \tag{14}$$

which leads to the single soliton solutions of the HBB equation (1)

$$\begin{aligned} u = {} & \pm\tfrac{1}{2}k\tanh[kx + \sigma(k^3 + 3ck^2 + 3c^2k)t] \\ & +c + \tfrac{1}{2}k, \end{aligned} \tag{15a}$$

$$v = -\tfrac{1}{2}k^2 sech^2[kx + \sigma(k^3 + 3ck^2 + 3c^2k)t]. \tag{15b}$$

Taking $k = \frac{1}{2}, c = 1, \sigma = 1$ in (15), we can show the single soliton solutions of the HBB equation (1) in Fig 1.



(a) $u(x,t)$

(b) $v(x,t)$

Fig. 1. Single soliton solutions of the HBB equation.

B. Solitons and the any Potential STO wave solutions

In order to obtain the interaction solutions of Eqs. (1), we consider w in the form

$$w = kx + \omega t + g, \tag{16}$$

on account of which, Eq. (13) leads to a potential STO wave

$$\begin{aligned} g_t - \tfrac{\sigma}{4}(g_{xx} \pm 6c_1 g_x + 12c_1^2 g \pm 3g_x^2)_x & \\ -\sigma g_x^3 - 3\sigma c_1 g_x^2 + \omega_0 = 0, & \end{aligned} \tag{17}$$

where c_1 and ω_0 are related to k, c and ω by

$$c_1 = c + k, \ \omega_0 = \omega - \sigma k(k^2 + 3ck + 3c^2), \tag{18}$$

Substituting Eq. (16) along with Eq. (17) into Eq. (8), we get the interaction solutions for the original HBB equation (1)

$$u = \frac{1}{2}(k+g_x)[\pm\tanh(w)+1]+c, \tag{19a}$$

$$\begin{aligned} v = &\frac{1}{2}(k+g_x)^2[\tanh^2(w)-1] \\ &-\frac{1}{2}g_{xx}(\tanh(w)\pm 1). \end{aligned} \tag{19b}$$

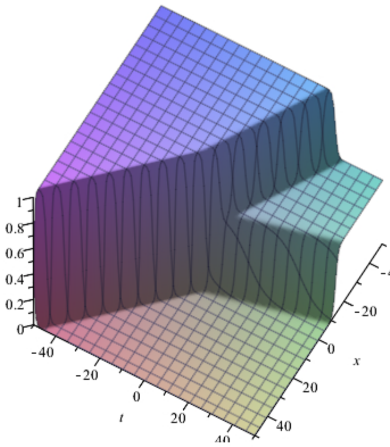
It is well known that the potential STO equation has many types of known exact solutions, such as resonant soliton solutions and so on. Thus, we can use those known solutions to construct the interaction solutions between a soliton and those PSTO waves.

1) Multiple resonant soliton solutions: Eq. (17) possesses the following multiple wave solutions

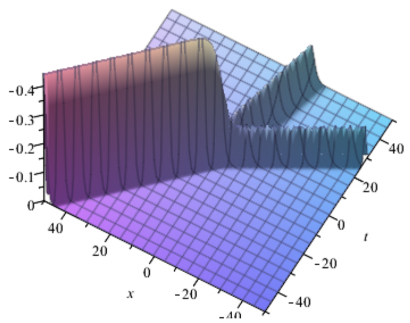
$$g = \pm\frac{1}{2}\ln\Big[\sum_{i=1}^{n} l_i e^{(k_i x+\omega_i t)}\Big] \tag{20}$$

with $\omega_i = \mp 2\omega_0 + \sigma k_i(\frac{1}{4}k_i^2 \pm \frac{3}{2}c_1 k_i + 3c_1^2)$, and k_i, l_i are arbitrary constants. Substituting (20) into the CTE (19), we obtain $(n+1)$ resonant soliton solutions of the HBB equation (1), which displays soliton fission and fusions.

Taking $n = 2, k = \frac{1}{2}, c = 1, \sigma = 1, \omega = 1, l_1 = 1, k_1 = 2, l_2 = 2, k_2 = 3$ in (20), we can show multiple resonant soliton solutions of the HBB equation (1) in Figure 2.



(a) $u(x,t)$



(b) $v(x,t)$

Fig. 2. Multiple resonant soliton solutions of the HBB equation.

2) Multiple interactions with periodic waves: It is not difficult to find soliton interactions with sine-cosine periodic waves because the PSTO (17) possess the following exact solutions

$$g = \pm\frac{1}{2}\ln\sum_{i=1}^{n}\Big\{d_i\sin\big[l_i(x+a_i t)\big]e^{(k_i x+b_i t)}\Big\} \tag{21}$$

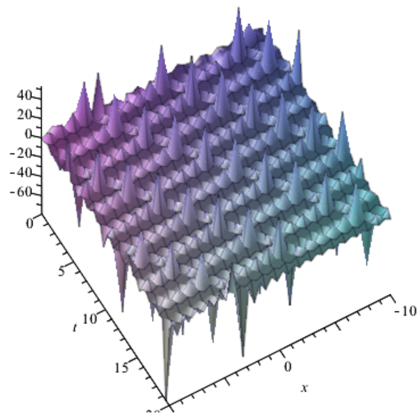
and

$$g = \pm\frac{1}{2}\ln\sum_{i=1}^{n}\Big\{d_i\cos\big[l_i(x+a_i t)\big]e^{(k_i x+b_i t)}\Big\} \tag{22}$$

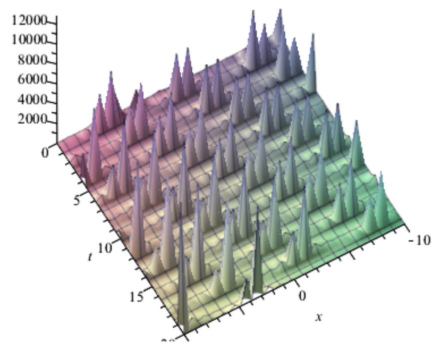
where $a_i = 3\sigma(c_1 \pm \frac{k_i^2}{2}) - \frac{1}{4}\sigma l_i^2, b_i = \mp 2\omega_0 + \sigma[3k_i c_1^2 \mp \frac{3}{2}c_1(l_i^2 - k_i^2) + \frac{1}{4}k_i^2 - \frac{3}{4}k_i l_i^2]$, and $d_i, k_i, l_i (i = 1, 2, \cdots, n)$ are arbitrary constants.

In other words, the CTE (21) or (22) with (19) exhibits the interaction solutions of multiple solitons and multiple periodic waves.

Taking $n = 2, k = 1, c = 2, \sigma = 1, \omega = 1, l_1 = 2, k_1 = 1, d_1 = 1, l_2 = 3, k_2 = 2, d_2 = 2$ in (22), we can show multiple interactions with periodic waves of the HBB equation (1) in Figure 3.



(a) $u(x,t)$



(b) $v(x,t)$

Fig. 3. Multiple interactions with periodic waves of the HBB equation.

3) Multiple interactions with rational waves: In order to obtain more solutions of Eq. (17), we consider ω in the following result

$$\omega = \sigma k(k^2 + 3ck + 3c^2). \tag{23}$$

Thus, the PSTO wave (17) is simplified to

$$g_t - \frac{\sigma}{4}(g_{xx} \pm 6c_1 g_x + 12c_1^2 g \pm 3g_x^2)_x \\ \sigma g_x^3 - 3\sigma c_1 g_x^2 = 0, \tag{24}$$

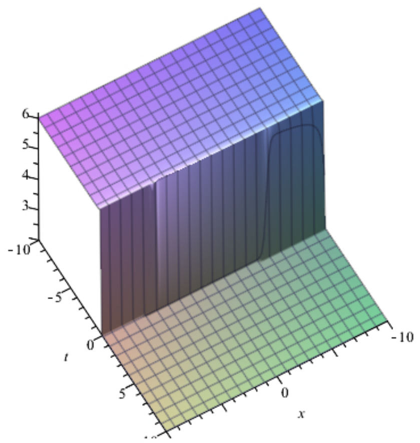
It is not difficult to verify that Eq. (24) possesses the following solution

$$g = \pm\frac{1}{2}\ln\left[a_1 x^2 + a_2 x + 6\sigma a_1 c_1^2 xt \\ +9\sigma^2 a_1 c_1^4 t^2 + 3\sigma c_1(a_2 c_1 \pm a_1)t\right], \tag{25}$$

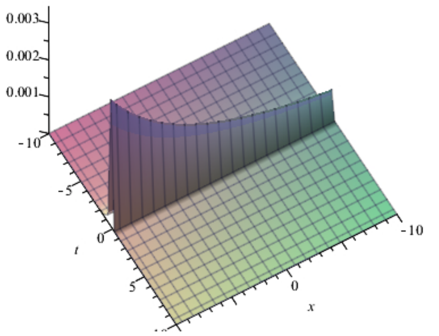
which means the CTE solutions (19) along with (25) become an interaction solution between a soliton and a rational wave.

So, starting from any solution of the STO u_0 equation (7a) and the variable coefficient PSTO system (7b) we can find the corresponding interaction solutions of the HBB equation (1) via the CTE (8).

Taking $k = 4, c = 2, \sigma = 1, a_1 = 1, a_2 = 2$ in (25), we can show multiple interactions with rational waves of the HBB equation (1) in Figure 4.



(a) $u(x,t)$



(b) $v(x,t)$

Fig. 4. Multiple interactions with rational waves of the HBB equation.

Now, we should return back to the non-CTE (11) case. Though the expansions (11) are not consistent in our definition, it still can be used to find some possible exact special solutions. For instance, it's not difficult to find that the over-determined system (10) possesses the solutions

$$w = \pm\frac{1}{2}k(x + \sigma k^2 t) - \frac{1}{2}\ln sech[k(x + \sigma k^2 t)], \tag{26}$$

where k is an arbitrary constant. Substituting the special solutions (26) into the expansion (11) yield a kids of single soliton solutions.

IV. Nonlocal Symmetries of the HBB equation Related to CTE

To find nonlocal symmetries related to the CTE, we write down a non-auto Bäcklund (BT) theorem for the HBB equation.

Theorem 2 If $\{u_0, w\}$ is a solution of the coupled STO and PSTO system (7), then

$$u = w_x + u_0, v = \mp(w_{xx} + u_{0x}). \tag{27}$$

are solutions of the HBB equation (1).

Now it is ready to study the nonlocal symmetries of the HBB equation (1) related to the consistent system (7) and the non-auto BT (27), A symmetry of the HBB equation is defined as a solution of its linearized system

$$\sigma_t^u - \sigma(3u^2\sigma^u - \frac{3}{2}v\sigma^u - \frac{3}{2}u\sigma^v + \frac{1}{4}\sigma_{xx}^u)_x = 0, \tag{28a}$$

$$\sigma_t^v - \sigma[3u^2\sigma^v + 6uv\sigma^u - \frac{3}{2}v\sigma^v - \frac{3}{2}(u\sigma^u)_{xx} \\ +\frac{3}{2}u_x\sigma_x^u + \frac{1}{4}\sigma_{xx}^v]_x = 0. \tag{28b}$$

which means the HBB equation (1) is form invariant under the transformation

$$\begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v \end{pmatrix} + \epsilon \begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} \tag{29}$$

with the infinitesimal parameter ϵ. Thus, the HBB equation (1) has the following nonlocal symmetry theorem.

Theorem 3 If $\{u, v\}$ is related to $\{u_0, w\}$ by (27), and $\{u_0, w\}$ is a solution of (7), then

$$\begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} = \begin{pmatrix} w_x \mathrm{e}^{\mp 2w} \\ (\mp w_{xx} + 2w_x^2)\mathrm{e}^{\mp 2w} \end{pmatrix} \tag{30}$$

is a nonlocal symmetry of the HBB equation (1).

Recently, it is found that the nonlocal symmetries can be localized by introducing an enlarged system. Thus, we have the following localization theorem for the enlarged system

$$\begin{aligned} & u_t - 3\sigma u^2 u_x + \frac{3}{2}\sigma(uv)_x - \frac{1}{4}\sigma u_{xxx} = 0, \\ & v_t + \frac{3}{2}\sigma v v_x - 3\sigma(u^2 v)_x + 3\sigma u_x u_{xx} \\ & \quad +\frac{3}{2}\sigma u u_{xxx} - \frac{1}{4}\sigma v_{xxx} = 0, \\ & u = w_x + u_0, \\ & v = \mp(w_{xx} + u_{0x}), \\ & w_1 = w_x, \\ & w_2 = \mp w_{1x}, \\ & u_{0t} - \sigma(\frac{1}{4}u_{0xx} \pm \frac{3}{2}u_0 u_{0x} + u_0^3)_x = 0, \\ & w_t - \sigma(\frac{1}{4}w_{xx} \pm \frac{3}{4}w_x^2 \pm \frac{3}{2}u_0 w_x)_x \\ & \quad -\sigma w_x^3 - 3\sigma u_0 w_x(u_0 + w_x) = 0. \end{aligned} \tag{31}$$

Theorem 4 The HBB equation (1) possesses a Lie point symmetry

$$\begin{pmatrix} \sigma^u \\ \sigma^v \\ \sigma^w \\ \sigma^{u_0} \\ \sigma^{w_1} \\ \sigma^{w_2} \end{pmatrix} = \begin{pmatrix} w_1 \mathrm{e}^{\mp 2w} \\ (w_2 + 2w_1^2)\mathrm{e}^{\mp 2w} \\ \mp\frac{1}{2}\mathrm{e}^{\mp 2w} \\ 0 \\ w_1 \mathrm{e}^{\mp 2w} \\ (w_2 + 2w_1^2)\mathrm{e}^{\mp 2w} \end{pmatrix}, \tag{32}$$

which is a localization of the nonlocal symmetry for the original HBB equation (1). When a nonlocal symmetry is localized, it can be used to find its finite transformations and the related symmetry reductions. Thus, we have the following finite transformation theorem.

Theorem 5 if $\{u, v, w, u_0, w_1, w_2\}$ is a solution of the prolonged HBB equation (31), so $\{u', v', w', u'_0, w'_1, w'_2\}$ is with

$$
\begin{aligned}
u' &= u + \frac{\epsilon w_1}{-\epsilon + \mathrm{e}^{\pm 2w}}, \\
v' &= v + \frac{\epsilon w_2}{-\epsilon + \mathrm{e}^{\pm 2w}} + \frac{2\epsilon w_1^2 \mathrm{e}^{\pm 2w}}{(-\epsilon + \mathrm{e}^{\pm 2w})^2}, \\
w' &= \pm\frac{1}{2}\ln(-\epsilon + \mathrm{e}^{\pm 2w}), \qquad (33)\\
v'_0 &= v_0,\ w'_1 = \frac{w_1 \mathrm{e}^{\pm 2w}}{-\epsilon + \mathrm{e}^{\pm 2w}}, \\
w'_2 &= \frac{w_2 \mathrm{e}^{\pm 2w}}{-\epsilon + \mathrm{e}^{\pm 2w}} + \frac{2\epsilon w_1^2 \mathrm{e}^{\pm 2w}}{(-\epsilon + \mathrm{e}^{\pm 2w})^2}.
\end{aligned}
$$

From the finite BT transformation theorem 5, we can obtain new solutions of the HBB equation (1) from any seed solutions. For instance, starting form the trivial seed solution $u = v = u_0 = 0$ and $w = kx + \omega t$, we can re-obtain the single soliton solution (15) of the HBB equation (1).

V. Conclusions

In the work, we have used the CTE method to solve the HBB equation (1). It is found that the HBB equation (1) is not only integrable under some traditional meaning [28] but also CTE solvable. With the help of the CTE method, abundant interaction solutions among solitons and other types of nonlinear waves especially any STO/PSTO waves such as the multiple resonant solitons and periodic waves are obtained.

Furthermore, the nonlocal symmetries of the HBB equation (1) related to the CTE are obtained. The nonlocal symmetries related to the CTE can also be localized by introducing suitable prolonged system. After finishing the localization procedure, the finite transformation of the nonlocal system can be obtained by solving the standard Lie's initial value problem. These results are important and may have significant impact on future research. It is also worth noting that this method can be applied to other nonlinear evolution equations.

Acknowledgement

I would like to express my sincere thanks to referees for their valuable suggestions and comments.

References

[1] R. Hirota, "Exact N-soliton solutions of the wave equation of long waves in shallow-water and in nonlinear lattices," *J. Math. Phys.*, vol. 14, pp. 810-814, 1973.

[2] Y. Shang, "Bäcklund transformation, Lax pairs and explicit exact solutions for the shallow water waves equation," *Appl. Math. Comput.*, vol. 187, pp. 1286-1297, 2007.

[3] E.G. Fan, "Darboux transformation and soliton-like solutions for the Gerdjikov-Ivanov equation," *J. Phys. A: Math. Gen.*, vol. 33, pp. 6925-6933, 2000.

[4] J. Weiss, M. Tabor and G. Carnevale, "The Painlevé property for partial differential equations," *J. Math. Phys.*, vol. 24, pp. 522-526, 1983.

[5] S. Zhang and J.H. Li, "On Nonisospectral AKNS System with Infinite Number of Terms and its Exact Solutions, " *IAENG International Journal of Applied Mathematics*, vol. 47, no. 1, pp. 89-96, 2017.

[6] M.L. Wang, "Solitary wave solution for variant Boussinesq equations," *Phys. Lett. A*, vol. 199, pp.169-172, 1995.

[7] E.G. Fan and H.Q. Zhang, "New exact solutions to a system of coupled KdV equations," *Phys. Lett. A*, vol. 245, pp. 389-392, 1998.

[8] Q.H. Feng, "Compact Difference Schemes for a Class of Space-time Fractional Differential Equations, " *Engineering Letters*, vol. 27, no. 2, pp. 269-277, 2019.

[9] J.K. Zhang, L.Y. Yin, L.N. Li, and Q.D. Huang, "Analytical Approximate Solutions for a Class of Non-homogeneous Time-fractional Partial Differential Equations by a Hybrid Series Method, " *Engineering Letters*, vol. 27, no. 3, pp. 601-612, 2019.

[10] S.Y. Lou, X. Hu and Y. Chen, "Nonlocal symmetries related to Bäcklund transformation and their applications," *J. Phys. A: Math. Theor.*, vol. 45, no. 15, 155209, 2012.

[11] X.R. Hu, S.Y. Lou and Y. Chen, "Explicit solutions from eigenfunction symmetry of the Korteweg-de Vries equation," *Phys. Rev. E*, vol. 85, 056607, 2012.

[12] C.L. Chen and S.Y. Lou, "CTE solvability, nonlocal symmetries and exact solutions of dispersive water wave system," *Commun. Theor. Phys.*, vol. 61, pp. 545-550, 2014.

[13] S.Y. Lou, "Consistent Riccati Expansion for Integrable Systems," *Stud. Appl. Math.*, vol. 134, pp. 372-402, 2015.

[14] J.C. Chen and Y. Chen, "Nonlocal symmetry constraints and exact interaction solutions of the (2+1)-dimensional modified generalized long dispersive wave equation," *J. Nonlinear Math. Phys.*, vol. 21, pp. 454-472, 2014.

[15] Y.H. Wang, "CTE method to the interaction solutions of Boussinesq-Burgers equations," *Appl. Math. Lett.*, vol. 38, pp. 100-105, 2014.

[16] S.F. Tian, Y.F. Zhang, B.L. Feng and H.Q. Zhang, "On the Lie algebras, generalized symmetries and Darboux transformations of the fifth-order evolution equations in shallow water," *Chin. Ann. Math. B*, vol. 36, no. 4, pp. 543-560, 2015.

[17] W.G. Cheng, B. Li and Y. Chen, "Construction of Soliton-Cnoidal wave interaction solution for the (2+1)-dimensional breaking soliton equation," *Commun. Theor. Phys.*, vol. 63, pp. 549-553, 2015.

[18] B. Ren and J. Lin, "Interaction behaviours between soliton and cnoidal periodic waves for the cubic generalised Kadomtsev-Petviashvili equation," *Z. Naturforsch. A*, vol. 70, pp. 539-544, 2015.

[19] X.P. Xin and X.Q. Liu, "Interaction solutions for (1+1)-dimensional higher-order Broer-Kaup system," *Commun. Theor. Phys.*, vol. 66, pp. 479-482, 2016.

[20] S.F. Tian, Y.F. Zhang, B.L. Feng and H.Q. Zhang, "Nonlocal symmetries, consistent Riccati expansion, and analytical solutions of the variant Boussinesq system," *Z. Naturforsch. A*, vol. 72, pp. 655-663, 2017.

[21] L.L. Feng, S.F. Tian and T.T Zhang, "Nonlocal symmetries and consistent Riccati expansions of the (2+1)-dimensional dispersive long wave equation," *Z. Naturforsch. A*, vol. 72, pp. 425-431, 2017.

[22] X.W. Yan, S.F. Tian, M.J. Dong, X.B. Wang and T.T. Zhang, "Nonlocal symmetries, conservation laws and interaction solutions of the generalised dispersive modified Benjamin-Bona-Mahony equation," *Z. Naturforsch. A*, vol. 73, pp. 399-405, 2018.

[23] M.J. Dong, S.F. Tian, X.W. Yan and T.T. Zhang, "Nonlocal symmetries, conservation laws and interaction solutions for the classical Boussinesq-Burgers equation," *Nonlinear Dyn.*, vol. 95, pp. 273-291, 2019.

[24] J.M. Zuo and Y.M. Zhang, "The Hirota bilinear method for the coupled Burgers equation and the high-order Boussinesq-Burgers equation," *Chin. Phys. B*, vol. 20, Article ID 010205, 2011.

[25] P. Guo, X. Wu and L. Wang, "New multiple-soliton (kink) solutions for the high-order Boussinesq-Burgers equation," *Waves Random Complex Media*, vol. 26, pp. 383-396, 2016.

[26] A. Jaradat, M.S. Noorani, M. Alquran and H.M. Jaradat, "Construction and solitary wave solutions of two-mode higher-order Boussinesq-Burger system," *Adv. Differ. Equ.*, vol. 2017, pp. 376-385, 2017.

[27] A.M. Wazwaz, "A Variety of Soliton Solutions for the Boussinesq-Burgers Equation and the Higher-Order Boussinesq-Burgers Equation, " *Filomat* , vol. 31, pp. 831-840, 2017.

[28] J.M. Zuo, "Painlevé analysis, Lax Pairs and new analytic solutions for a high-order Boussinesq-Burgers equation, " *IAENG International Journal of Applied Mathematics*, vol. 48, no. 3, pp. 337-341, 2018.

[29] P.J. Olver, "Evolution equations possessing infinitely many symmetries, " *J. Math. Phys.*, vol. 18, pp. 1212-1215, 1977.

[30] S. Wang, X.Y. Tang and S.Y. Lou, "Soliton fission and fusion: Burgers equation and Sharma-Tasso-Olver equation, " *Chaos, Solitons & Fractals*, vol. 21, pp. 231-239, 2004.