Oscillatory Behavior of a Class Impulsive Fractional Partial Differential Equation

Qian Feng, Anping Liu

***Abstract*—In this paper, we investigate the oscillatory behavior of a class of impulsive fractional partial differential equations with Neumann and Dirichlet boundary conditions by using the definitions and the related properties of the modified Riemann-Liouville fractional order and the differential inequality method. Some sufficient conditions for the oscillation of the solutions of the impulsive fractional differential equation are obtained. As an application, we included an example to illustrate the main result.**

***Index Terms*—impulsive; fractional partial differential equations; modified Riemann-Liouville fractional partial derivative; oscillation.**

I. Introduction

FRACTIONAL differential equations have gained increasing attention due to their various applications in science and engineering such as rheology, dynamical processes in self-similar and porous structures, heat conduction, control theory, electroanalytical chemistry, chemical physics, and economics, etc. The growing interest is caused both by the intensive development of the theory of fractional calculus itself and by the applications.

The oscillation theory as a part of the qualitative theory of differential equations has been developed rapidly in the last decades, and there has been plenty of works on the oscillatory behavior of integer order partial differential equations $[1-3]$. Some new developments in the oscillatory behavior of solutions of fractional differential equations with damping terms have been reported by authors $[4-9]$. In [5], P. Prakash has studied the oscillatory behavior of solutions of the nonlinear fractional partial differential equation with damping and forced term subject to Robin boundary condition by using the differential inequality method as well as the integral average method.

Recently, the oscillatory behavior of various classes of fractional differential equations has been investigated by many authors $[10-15]$. In [15], A. Raheem established some sufficient conditions for oscillation of solutions of a class of impulsive partial fractional differential equations with forcing term subject to Robin and Dirichlet boundary conditions by using differential inequality method. For more related references, please refer to $[19-21]$.

The main purpose of this paper is to give several oscillation theorems for the fractional impulsive differential equation:

$$\begin{cases} D_{+,t}^{\alpha}\left(r(t)D_{+,t}^{\alpha}u(x,t)\right)+p(t)D_{+,t}^{\alpha}u(x,t) \\ \quad =a(t)h(u)\Delta u(x,t) \\ \qquad +\displaystyle\sum_{i=1}^{j}a_i(t)h_i\left(u\left(x,t-\tau_i\right)\right)\Delta u\left(x,t-\tau_i\right) \\ \qquad -m(x,t,u(x,t))-q(x,t)F(u(x,t)),\ t\neq t_k \\ D_{+,t}^{\alpha}u\left(x,t_k^{+}\right)=\sigma\left(x,t_k,D_{+,t}^{\alpha}u\left(x,t_k\right)\right), \\ \quad k=1,2,3,\cdots,\quad (x,t)\in\Omega\times R_{+}\equiv E \\ u\left(x,t_k^{+}\right)=\delta\left(x,t_k,u\left(x,t_k\right)\right), \\ \quad k=1,2,3,\cdots,\quad (x,t)\in\Omega\times R_{+}\equiv E \end{cases} \tag{1}$$

with two kinds of boundary conditions

$$\frac{\partial u(x,t)}{\partial N}=g(x,t,u(x,t)),\ (x,t)\in\partial\Omega\times R_{+},\ t\neq t_k \tag{2}$$

and

$$u(x,t)=0,\ (x,t)\in\partial\Omega\times R_{+},\ t\neq t_k \tag{3}$$

where $0<\alpha<1$, Δ is the Laplacian in R^n, Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$ and $\overline{\Omega}=\Omega\cup\partial\Omega$, N is the unit out normal vector to $\partial\Omega$, $g(x,t,u(x,t))\in C(\partial\Omega\times R_{+}\times R;R_{+})$. And $0<t_1<\cdots<t_i<\cdots$, and $\lim\limits_{i\to\infty}t_i=+\infty$.

Throughout this paper, we assume that the following conditions hold:

(H1) $r(t)\in C^{\alpha}\left(R_{+};R_{+}\right)$, $p(t), a(t)\in C\left(R_{+};R_{+}\right)$, $a_i(t)\in C\left(R_{+};R_{+}\right)$; $h(u), h_i(u)\in C(R;R)$ and $\tau_i\geq 0$ are constants, $i\in I_j=\{1,2\cdots,j\}$; $u\neq 0$, $g(x,t,u(x,t))$ is a piecewise continuous function, such that $ug(x,t,u(x,t))h(u)<0$, $ug(x,t,u(x,t))h_i(u)<0$, $uh'(u)\geq 0$, $uh_i'(u)\geq 0$.

(H2) $q(x,t)\in C\left(\overline{E};R_{+}\right)$, $q(t)=\min\limits_{x\in\overline{\Omega}}q(x,t)$; $F(u)\in C(R;R)$, for $x\neq 0$, there exist positive constant c, such that $\dfrac{F(x)}{x}\geq c>0$.

(H3) $m\in C(\overline{E}\times R;R)$, and

$$m(x,t,\eta)=\begin{cases}\geq 0 & \eta\in(0,+\infty),\\ \leq 0 & \eta\in(-\infty,0);\end{cases}$$

(H4) $\sigma\left(x,t_k,D_{+,t}^{\alpha}u\left(x,t_k\right)\right),\ \delta\left(x,t_k,u\left(x,t_k\right)\right):\overline{E}\times R_{+}\times R\to R$, they are both piecewise continuous with discontinuities of first kind only at $t=t_k$, and left continuous at $t=t_k$, $k=1,2,3,\ldots$, and there exist positive constants $\overline{\alpha}_k$, $\underline{\alpha}_k$, $\overline{\beta}_k$, $\underline{\beta}_k$, such that $\xi\neq 0$, $\zeta\neq 0$,

$$\underline{\alpha}_k\leq\frac{\sigma\left(x,t_k,\xi\right)}{\xi}\leq\overline{\alpha}_k,$$

$$\underline{\beta}_k\leq\frac{\delta\left(x,t_k,\zeta\right)}{\zeta}\leq\overline{\beta}_k,$$

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where $\overline{\beta}_k \leq \underline{\alpha}_k$.

II. PRELIMINARIES

Definition 2.1(*[16]*) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : R_+ \to R$ on the half-axis R_+ is given by

$$(I_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-v)^{\alpha-1} f(v)dv \tag{4}$$

provided that the right side is pointwise defined on R_+, where Γ is the gamma function.

Definition 2.2(*[17]*) The modified Riemann-Liouville fractional partial derivative of order $\alpha > 0$ is defined as

$$\begin{cases} D_t^{\alpha} f(t) = \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \displaystyle\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, \\ \qquad\qquad\qquad\qquad \text{if } 0<\alpha<1, \\ D_t^{\alpha} f(t) = (f^n(t))^{\alpha-n}, \qquad \text{if } 1 \leq n \leq \alpha < n+1. \end{cases} \tag{5}$$

Definition 2.3 The solution $u(x,t)$ of problems (1), (2)((3)) is called nonoscillatory in the domain E, if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

The following is a list of some calculation formulas related to modified Riemann-Liouville derivative

$$D_t^{\alpha} t^r = \frac{\Gamma(1+\alpha)}{\Gamma(1+r-\alpha)} t^{r-\alpha},$$

$$D_t^{\alpha}(f(t)h(t)) = h(t) D_t^{\alpha} f(t) + f(t) D_t^{\alpha} h(t),$$

$$D_t^{\alpha} f[h(t)] = f_h' D_t^{\alpha} h(t) = D_h^{\alpha} f[h(t)](h'(t))^{\alpha}.$$

For convenience, we denote:

$$R(t) = I_+^{\alpha}\left(\frac{p(t)}{r(t)}\right), \quad \xi = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad \widetilde{r}(\xi) = r(t),$$

$$\widetilde{p}(\xi) = p(t),\ \widetilde{q}(\xi) = q(t),\ \widetilde{\psi}(\xi) = \psi(t),\ \widetilde{R}(\xi) = R(t).$$

III. MAIN RESULTS

Theorem 3.1 If impulsive fractional differential inequality

$$\begin{cases} D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} U(t)\right] + p(t) D_{+,t}^{\alpha} U(t) \leq -cq(t)U(t), \\ \underline{\alpha}_k \leq \dfrac{D_{+,t}^{\alpha} U\left(t_k^+\right)}{D_{+,t}^{\alpha} U\left(t_k\right)} \leq \overline{\alpha}_k, \quad k=1,2,3,\cdots, \\ \underline{\beta}_k \leq \dfrac{U\left(t_k^+\right)}{U\left(t_k\right)} \leq \overline{\beta}_k, \qquad k=1,2,3,\cdots, \end{cases} \tag{6}$$

has no eventually positive solutions and the impulsive fractional differential inequality

$$\begin{cases} D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} U(t)\right] + p(t) D_{+,t}^{\alpha} U(t) \geq -cq(t)U(t), \\ \underline{\alpha}_k \leq \dfrac{D_{+,t}^{\alpha} U\left(t_k^+\right)}{D_{+,t}^{\alpha} U\left(t_k\right)} \leq \overline{\alpha}_k, \quad k=1,2,3,\cdots, \\ \underline{\beta}_k \leq \dfrac{U\left(t_k^+\right)}{U\left(t_k\right)} \leq \overline{\beta}_k, \qquad k=1,2,3,\cdots, \end{cases} \tag{7}$$

has no eventually negative solutions, then every nontrivial solution $u(x,t)$ of the problem (1) and (2) is oscillatory in E.

Proof Suppose that $u(x,t)$ is a nonoscillatory solution of the problem (1) and (2). Without loss of generality, we may assume that $u(x,t)$ is an eventually positive solution of the problem (1) and (2) in the domain E, then there exists a $t_0 \geq 0$, such that $u(x,t) > 0$ and $u(x,t-\tau_i) > 0$ for $(x,t) \in \Omega \times [t_0, +\infty)$.

Case 1: $t \neq t_k$. Integrating the first equation of Eq.(1) with respect to x over the domain Ω, we have

$$\begin{aligned} & D_{+,t}^{\alpha} \int_{\Omega} (r(t) D_{+,t}^{\alpha} u(x,t))dx + p(t) \int_{\Omega} D_{+,t}^{\alpha} u(x,t)dx \\ = {} & a(t) \int_{\Omega} h(u) \Delta u(x,t)dx \\ & + \sum_{i=1}^{j} a_i(t) \int_{\Omega} h_i(u(x,t-\tau_i)) \Delta u(x,t-\tau_i)dx \\ & - \int_{\Omega} m(x,t,u(x,t))dx - \int_{\Omega} q(x,t) F(u(x,t))dx. \end{aligned} \tag{8}$$

By using Green's formula and Eq.(2), we obtain

$$\begin{aligned} & \int_{\Omega} h(u) \Delta u(x,t)dx \\ = {} & \int_{\partial\Omega} h(u) \frac{\partial u(x,t)}{\partial N} dS - \int_{\Omega} h'(u) |\text{grad } u|^2 dx \\ = {} & -\int_{\partial\Omega} h(u) g(x,t,u(x,t))dS - \int_{\Omega} h'(u) |\text{grad } u|^2 dx \\ \leq {} & 0, \quad t \geq t_0, \end{aligned} \tag{9}$$

$$\begin{aligned} & \int_{\Omega} h_i(u(x,t-\tau_i)) \Delta u(x,t-\tau_i)dx \\ = {} & -\int_{\partial\Omega} g(x,t-\tau_i,u(x,t-\tau_i)) h_i(u(x,t-\tau_i))dS \\ & - \int_{\Omega} h_i'(u(x,t-\tau_i)) |\text{grad } u(x,t-\tau_i)|^2 dx \\ \leq {} & 0, \qquad t \geq t_0, \end{aligned} \tag{10}$$

where dS is an area element of $d\Omega$.

It is obvious that

$$\int_{\Omega} m(x,t,u(x,t))dx \geq 0, \quad t \geq t_0, \tag{11}$$

$$\begin{aligned} \int_{\Omega} q(x,t) F(u(x,t))dx \geq {} & \int_{\Omega} cq(t) u(x,t)dx \\ \geq {} & cq(t)U(t),\ t \geq t_0. \end{aligned} \tag{12}$$

From (8)-(12), we obtain

$$D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} U(t)\right] + p(t) D_{+,t}^{\alpha} U(t) \leq -cq(t)U(t),\ t \geq t_0, \tag{13}$$

where

$$U(t) = \int_{\Omega} u(x,t)dx.$$

Case 2: $t = t_k$. From the second and third equations of Eq.(1), together with the assumption (H4) can deduce that

$$\underline{\alpha}_k \leq \frac{D_{+,t}^{\alpha} u\left(x,t_k^+\right)}{D_{+,t}^{\alpha} u\left(x,t_k\right)} = \frac{\sigma\left(x,t_k,D_{+,t}^{\alpha} u\left(x,t_k\right)\right)}{D_{+,t}^{\alpha} u\left(x,t_k\right)} \leq \overline{\alpha}_k, \quad k=1,2,3,\cdots.$$

$$\underline{\beta}_k \leq \frac{u\left(x,t_k^+\right)}{u\left(x,t_k\right)} = \frac{\delta\left(x,t_k,u\left(x,t_k\right)\right)}{u\left(x,t_k\right)} \leq \overline{\beta}_k,\ k=1,2,3,\cdots.$$

then integrating the two inequalities above with respect to x over the domain Ω, we derive

$$\underline{\alpha}_k \leq \frac{D_{+,t}^{\alpha} U\left(t_k^{+}\right)}{D_{+,t}^{\alpha} U\left(t_k\right)} = \frac{\int_\Omega D_{+,t}^{\alpha} u\left(x, t_k^{+}\right) dx}{\int_\Omega D_{+,t}^{\alpha} u\left(x, t_k^{+}\right) dx} \leq \overline{\alpha}_k, \quad k = 1, 2, 3, \cdots . \tag{14}$$

$$\underline{\beta}_k \leq \frac{U\left(t_k^{+}\right)}{U\left(t_k\right)} = \frac{\int_\Omega u\left(x, t_k^{+}\right) dx}{\int_\Omega u\left(x, t_k\right) dx} \leq \overline{\beta}_k, \quad k = 1, 2, 3, \cdots . \tag{15}$$

Thus (13)-(15) imply that the function $U(t) = \int_\Omega u(x,t)dx$ is an eventually positive solution of the fractional impulsive differential inequality (6) which contradicts the conditions of theorem.

Secondly, if $u(x,t)$ is an eventually negative solution of the problem (1) and (2) in the domain E, then using above procedure, we can easily show that $U(t) = \int_\Omega u(x,t)dx$ is an eventually negative solution of the fractional impulsive differential inequality (7) which again contradicts the conditions of theorem. This completes the proof.

Lemma 3.2(*[15]*) The smallest eigenvalue λ_0 of the Dirichlet problem

$$\begin{cases} \Delta w(x) + \lambda w(x) = 0, & in\ \Omega \\ w(x) = 0, & on\ \partial\Omega \end{cases} \tag{16}$$

is positive and the corresponding eigenfunction $\phi(x)$ is positive in Ω.

Theorem 3.3 If impulsive fractional differential inequality

$$\begin{cases} D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} V(t)\right] + p(t) D_{+,t}^{\alpha} V(t) \leq -cq(t) V(t) \\ \underline{\alpha}_k \leq \dfrac{D_{+,t}^{\alpha} V\left(t_k^{+}\right)}{D_{+,t}^{\alpha} V\left(t_k\right)} \leq \overline{\alpha}_k, \ k = 1, 2, 3, \cdots \\ \underline{\beta}_k \leq \dfrac{V\left(t_k^{+}\right)}{V\left(t_k\right)} \leq \overline{\beta}_k, \ k = 1, 2, 3, \cdots \end{cases} \tag{17}$$

has no eventually positive solutions and the impulsive fractional differential inequality

$$\begin{cases} D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} V(t)\right] + p(t) D_{+,t}^{\alpha} V(t) \geq -cq(t) V(t) \\ \underline{\alpha}_k \leq \dfrac{D_{+,t}^{\alpha} V\left(t_k^{+}\right)}{D_{+,t}^{\alpha} V\left(t_k\right)} \leq \overline{\alpha}_k, \ k = 1, 2, 3, \cdots \\ \underline{\beta}_k \leq \dfrac{V\left(t_k^{+}\right)}{V\left(t_k\right)} \leq \overline{\beta}_k, \ k = 1, 2, 3, \cdots \end{cases} \tag{18}$$

has no eventually negative solutions, then every nontrivial solution $u(x,t)$ of the problem (1), (3) is oscillatory in E.

Proof To obtain a proof by contradiction, let $u(x,t)$ be a nonoscillatory solution of the problem (1) and (3). Then, $u(x,t)$ is either eventually positive or eventually negative in E. If $u(x,t)$ is an eventually positive solution of the problem (1) and (3) in the domain E, then there exists $t_1 \geq 0$, such that $u(x,t) > 0$ and $u(x, t-\tau_i) > 0$ for $(x,t) \in \Omega \times [t_1, +\infty)$.

Case 1: $t \neq t_k$. Multiplying the first equation of Eq.(1) by $\phi(x)$ and integrating with respect to x over the domain Ω, we have

$$\begin{aligned} & D_{+,t}^{\alpha} \int_\Omega \left(r(t) D_{+,t}^{\alpha} u(x,t)\right) \phi(x) dx \\ & + p(t) \int_\Omega D_{+,t}^{\alpha} u(x,t) \phi(x) dx \\ = & a(t) \int_\Omega h(u) \Delta u(x,t) \phi(x) dx \\ & + \sum_{i=1}^{j} a_i(t) \int_\Omega h_i\left(u\left(x, t-\tau_i\right)\right) \Delta u\left(x, t-\tau_i\right) \phi(x) dx \\ & - \int_\Omega m(x,t,u(x,t)) \phi(x) dx - \int_\Omega q(x,t) F(u(x,t)) \phi(x) dx \end{aligned} \tag{19}$$

By using Green's formula and lemma 3.2, we obtain

$$\begin{aligned} \int_\Omega \Delta u(x,t) \phi(x) dx = & \int_\Omega u(x,t) \Delta\phi(x) dx \\ = & -\lambda_0 \int_\Omega u(x,t) \phi(x) dx \\ \leq & \ 0, \quad t \geq t_1, \end{aligned} \tag{20}$$

$$\begin{aligned} & \int_\Omega \Delta u(x, t-\tau_i) \phi(x) dx \\ = & -\lambda_0 \int_\Omega u(x, t-\tau_i) \phi(x) dx \leq 0, \quad t \geq t_1, \end{aligned} \tag{21}$$

and

$$\int_\Omega m(x,t,u(x,t)) \phi(x) dx \geq 0, \ t \geq t_1, \tag{22}$$

$$\begin{aligned} \int_\Omega q(x,t) F(u(x,t)) \phi(x) dx \geq & \int_\Omega c q(t) u(x,t) \phi(x) dx \\ \geq & \ c q(t) V(t), \ t \geq t_1. \end{aligned} \tag{23}$$

From (19)-(23), we get

$$D_{+,t}^{\alpha}\left[r(t) D_{+,t}^{\alpha} V(t)\right] + p(t) D_{+,t}^{\alpha} V(t) \leq -cq(t) V(t), \ t \geq t_1, \tag{24}$$

where

$$V(t) = \int_\Omega u(x,t) \phi(x) dx.$$

Case 2: $t = t_k$. From the second and third equations of Eq.(1), together with the assumption (H4) can deduce that

$$\underline{\alpha}_k \leq \frac{D_{+,t}^{\alpha} u\left(x, t_k^{+}\right)}{D_{+,t}^{\alpha} u\left(x, t_k\right)} = \frac{\sigma\left(x, t_k, D_{+,t}^{\alpha} u\left(x, t_k\right)\right)}{D_{+,t}^{\alpha} u\left(x, t_k\right)} \leq \bar{\alpha}_k, \quad k = 1, 2, 3, \cdots .$$

$$\underline{\beta}_k \leq \frac{u\left(x, t_k^{+}\right)}{u\left(x, t_k\right)} = \frac{\delta\left(x, t_k, u\left(x, t_k\right)\right)}{u\left(x, t_k\right)} \leq \bar{\beta}_k, \ k = 1, 2, 3, \cdots .$$

then multiplying the above two inequalities by $\phi(x)$ respectively and integrating with respect to x over the domain Ω, we derive

$$\underline{\alpha}_k \leq \frac{D_{+,t}^{\alpha} V\left(t_k^{+}\right)}{D_{+,t}^{\alpha} V\left(t_k\right)} = \frac{\int_\Omega D_{+,t}^{\alpha} u\left(x, t_k^{+}\right) \phi(x) dx}{\int_\Omega D_{+,t}^{\alpha} u\left(x, t_k\right) \phi(x) dx} \leq \overline{\alpha}_k, \quad k = 1, 2, 3, \cdots . \tag{25}$$

$$\underline{\beta}_k \leq \frac{V\left(t_k^{+}\right)}{V\left(t_k\right)} = \frac{\int_\Omega u\left(x, t_k^{+}\right) \phi(x) dx}{\int_\Omega u\left(x, t_k\right) \phi(x) dx} \leq \overline{\beta}_k, \quad k = 1, 2, 3, \cdots . \tag{26}$$

Thus (24)-(26) imply that the function $V(t) = \int_\Omega u(x,t)\phi(x)dx$ is an eventually positive solution of the fractional impulsive differential inequality (17) which contradicts the conditions of the theorem.

Secondly, if $u(x,t)$ is an eventually negative solution of the problem (1) and (3) in the domain E, then using above procedure, we can easily get a contradiction to the conditions of theorem. This completes the proof.

Using above results, next we shall establish some more oscillation criteria for the impulsive fractional differential equations.

For this, we need the following lemma.

Lemma 3.4(*[15]*) Let

$$G(t) = \int_0^t (t-v)^{-\alpha} f(v)dv, \quad \alpha \in (0,1),\ t>0.$$

Then

$$G'(t) = \Gamma(1-\alpha)(D_+^\alpha f)(t), \quad \alpha \in (0,1),\ t>0.$$

Lemma 3.5(*[18]*) Assume that

$$w'(t) \le h(t), \quad t \neq t_k, \quad t \ge t_0,$$
$$w\left(t_k^+\right) \le (1+d_k)\, w(t_k), \quad k = 1,2,3,\cdots,$$

where $0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = +\infty$; $w \in PC^1[R_+, R]$, $h \in C[R_+, R]$ and $d_k > 0$ are constants. Then

$$w(t) \le w(t_0) \prod_{t_0 < t_k < t} (1+d_k) + \int_{t_0}^t \prod_{s<t_k<t} (1+d_k)\, h(s)ds, \quad t \ge t_0.$$

Lemma 3.6 If $0<\alpha<1$, and then

$$\left(D_+^\alpha I_{0+}^\alpha f\right)(x) = f(x)$$

Proof By the definition 2.1 and 2.2, we get

$$\begin{aligned}
&\left(D_+^\alpha I_{0+}^\alpha f\right)(x) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{dt}{(x-t)^\alpha} \times \\
&\qquad \left[\frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)ds}{(t-s)^{1-\alpha}} - I_{0+}^\alpha f(0)\right] \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{dt}{(x-t)^\alpha} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x f(s)ds \int_s^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt.
\end{aligned}$$

Letting $t = s + \mu(x-s)$, using the definition of the Beta function

$$\begin{aligned}
\int_s^x (x-t)^{-\alpha}(t-s)^{\alpha-1}dt &= \int_0^1 \mu^{\alpha-1}(1-\mu)^{-\alpha}d\mu \\
&= B(\alpha, 1-\alpha),
\end{aligned}$$

$$\left(D_+^\alpha I_{0+}^\alpha f\right)(x) = \frac{B(\alpha,1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x f(s)ds = f(x).$$

This completes the proof.

Theorem 3.7 Suppose that for some $t^* \ge 0$,

$$\int_{t^*}^\infty \frac{1}{r(s)e^{R(s)}} ds = \infty \tag{27}$$

and

$$\lim_{\xi\to\infty} \int_{t^*}^\xi \prod_{t^*<\xi_k<s} \frac{\beta_k}{\overline{\alpha}_k} \widetilde{\psi}(s)ds = +\infty \tag{28}$$

where $\widetilde{\psi}(s) = ce^{\widetilde{R}(s)}\widetilde{q}(s)$. Then every nontrival solution of problem (1)-(2) is oscillatory in the domain E.

Proof To prove the theorem, it is sufficient to prove that the impulsive fractional differential inequality (1) and (2) admit no eventually positive solutions and no eventually negative solutions, respectively.

Again, we argue by contradiction. If the impulsive fractional differential inequality (6) has an eventually positive solution $U(t)$, then there exists $t^* \ge 0$ satisfying $U(t) > 0$, $U(t-\tau_i) > 0$, $G(t) > 0$ for $t \ge t^*$, such that

$$\begin{aligned}
&D_+^\alpha\left[e^{R(t)} r(t) D_+^\alpha U(t)\right] \\
&= e^{R(t)} D_+^\alpha\left[r(t) D_+^\alpha U(t)\right] + r(t) D_+^\alpha U(t) D_+^\alpha e^{R(t)} \\
&= e^{R(t)} D_+^\alpha\left[r(t) D_+^\alpha U(t)\right] + r(t) D_+^\alpha U(t) e^{R(t)} D_+^\alpha I_+^\alpha \left(\frac{p(t)}{r(t)}\right) \\
&= e^{R(t)} D_+^\alpha\left[r(t) D_+^\alpha U(t)\right] + e^{R(t)} p(t) D_+^\alpha U(t) \\
&= e^{R(t)} \left[D_+^\alpha[r(t) D_+^\alpha U(t)] + p(t) D_+^\alpha U(t)\right] \\
&< -ce^{R(t)} q(t) U(t) \\
&< 0.
\end{aligned} \tag{29}$$

Thus $e^{R(t)} r(t) D_+^\alpha U(t)$ is strictly decreasing for $t \ge t^*$ and $D_+^\alpha U(t)$ is eventually of constant sign. We claim $D_+^\alpha U(t) > 0$ on $t \in [t^*, \infty)$. Otherwise, assume that there exists a sufficiently large $T \in [t^*, \infty)$ such that $D_+^\alpha U(T) < 0$. Then it is obvious that

$$e^{R(t)} r(t) D_+^\alpha U(t) \le e^{R(T)} r(T) D_+^\alpha U(T) = c_1 < 0,$$

where c_1 is a constant for $t \in [T, \infty)$.

Lemma 3.4 yields that

$$\frac{G'(t)}{\Gamma(1-\alpha)} = D_+^\alpha U(t) \le \frac{c_1}{e^{R(t)} r(t)}, \tag{30}$$

Integrating the above inequality from T to t enables us to get

$$\int_T^t \frac{G'(s)}{\Gamma(1-\alpha)} ds \le \int_T^t \frac{c_1}{e^{R(s)} r(s)} ds, \tag{31}$$

$$G(t) \le G(T) + \Gamma(1-\alpha)c_1 \int_T^t \frac{1}{e^{R(s)} r(s)} ds. \tag{32}$$

As $t \to \infty$, $\lim_{t\to\infty} G(t) \le -\infty$, which contradicts the fact that $G(t) > 0$. Hence $D_+^\alpha U(t) > 0$ for $t > T$.

Let

$$w(t) = e^{R(t)} \frac{r(t) D_+^\alpha U(t)}{U(t)}.$$

Then we have $w(t) > 0$, and it follows from (6) that

$$\begin{aligned}
&D_+^\alpha w(t) \\
&= e^{R(t)} D_+^\alpha\left[\frac{r(t) D_+^\alpha U(t)}{U(t)}\right] + \frac{r(t) D_+^\alpha U(t)}{U(t)} D_+^\alpha e^{R(t)} \\
&= e^{R(t)} \frac{D_+^\alpha[r(t) D_+^\alpha U(t)]}{U(t)} - \frac{e^{R(t)} r(t) D_+^\alpha U(t) D_+^\alpha U(t)}{U^2(t)} \\
&\quad + \frac{p(t)}{r(t)} w(t)
\end{aligned}$$

$$
\begin{aligned}
&= e^{R(t)} \frac{-p(t) D_{+,t}^{\alpha} U(t) - cq(t)U(t)}{U(t)} - \frac{w^2(t)}{e^{R(t)} r(t)} \\
&\quad + \frac{p(t)}{r(t)} w(t) \\
&= -ce^{R(t)} q(t) - \frac{p(t)}{r(t)} w(t) - \frac{w^2(t)}{e^{R(t)} r(t)} + \frac{p(t)}{r(t)} w(t) \\
&\leq -ce^{R(t)} q(t). \qquad (33)
\end{aligned}
$$

i.e.,

$$D_{+}^{\alpha} w(t) \leq -\psi(t), \qquad (34)$$

where $\psi(t) = ce^{R(t)} q(t)$.

From (H1) and inequality (6), it is easy to see that

$$w\left(t_k^{+}\right) \leq \frac{\bar{\alpha}_k}{\beta_k} w\left(t_k\right), \quad k = 1, 2, 3, \cdots \qquad (35)$$

Let $\xi = \dfrac{t^{\alpha}}{\Gamma(1+\alpha)}$, $\widetilde{w}(\xi) = w(t)$, $\widetilde{\psi}(\xi) = \psi(t)$, we derive

$$D_{+}^{\alpha} w(t) = D_{+}^{\alpha} \widetilde{w}(\xi) = \widetilde{w'}(\xi) D_{+}^{\alpha}(\xi) = \widetilde{w'}(\xi), \qquad (36)$$

Therefore, we have

$$\widetilde{w'}(\xi) \leq -\widetilde{\psi}(\xi). \qquad (37)$$

From (35)-(37), we get

$$
\begin{cases}
\widetilde{w'}(\xi) \leq -\widetilde{\psi}(\xi), & \xi \neq \xi_k, \quad \xi \geq t^{*} \\
\widetilde{w}\left(\xi_k^{+}\right) \leq \dfrac{\overline{\alpha}_k}{\underline{\beta}_k} \widetilde{w}\left(\xi_k\right), & k = 1, 2, 3, \cdots
\end{cases} \qquad (38)
$$

Using Lemma 3.5 and (28), we obtain

$$
\begin{aligned}
&\widetilde{w}(\xi) \\
&\leq \widetilde{w}(t^{*}) \prod_{t^{*} < \xi_k < \xi} \frac{\overline{\alpha}_k}{\underline{\beta}_k} - \int_{t^{*}}^{\xi} \prod_{s < \xi_k < \xi} \frac{\overline{\alpha}_k}{\underline{\beta}_k} \widetilde{\psi}(s) ds \\
&= \prod_{t^{*} < \xi_k < \xi} \frac{\overline{\alpha}_k}{\underline{\beta}_k} \left[\widetilde{w}(t^{*}) - \int_{t^{*}}^{\xi} \prod_{t^{*} < \xi_k < s} \frac{\underline{\beta}_k}{\overline{\alpha}_k} \widetilde{\psi}(s) ds \right] \\
&< 0, \qquad (39)
\end{aligned}
$$

which contradicts the fact that $\widetilde{w}(\xi) > 0$.

Secondly, suppose that $U(t)$ is an eventually negative solution of the fractional differential inequality (7) and there exists $G(t) < 0,\ t \in [t^{*}, \infty)$. Then by using above process it can be easily shown that $D_{+}^{\alpha} U(t) < 0$ for $t > t^{*}$.

Let $w(t) = -e^{R(t)} \dfrac{r(t) D_{+}^{\alpha} U(t)}{U(t)}$, thus $w < 0$. And from inequality (7), we get $D_{+}^{\alpha} w(t) \geq ce^{R(t)} q(t) = \psi(t)$ and $\dfrac{\overline{\alpha}_k}{\underline{\beta}_k} w\left(t_k\right) \leq w\left(t_k^{+}\right) \leq \dfrac{\underline{\alpha}_k}{\overline{\beta}_k} w\left(t_k\right), \quad k = 1, 2, 3, \cdots.$

that is

$$
\begin{cases}
\widetilde{w'}(\xi) \geq \widetilde{\psi}(\xi), \quad \xi \neq \xi_k, \quad \xi \geq t^{*} \\
\dfrac{\overline{\alpha}_k}{\underline{\beta}_k} \widetilde{w}\left(\xi_k\right) \leq \widetilde{w}\left(\xi_k^{+}\right) \leq \dfrac{\underline{\alpha}_k}{\overline{\beta}_k} \widetilde{w}\left(\xi_k\right), \ k = 1, 2, 3, \cdots
\end{cases} \qquad (40)
$$

If $\widetilde{w}(\xi) = -\widetilde{v}(\xi)$

$$
\begin{cases}
\widetilde{v}'(\xi) \leq -\widetilde{\psi}(\xi), \quad \xi \neq \xi_k, \quad \xi \geq t^{*} \\
\widetilde{v}\left(\xi_k^{+}\right) \leq \dfrac{\overline{\alpha}_k}{\underline{\beta}_k} \widetilde{v}\left(\xi_k\right), \ k = 1, 2, 3, \cdots
\end{cases} \qquad (41)
$$

Using lemma 3.5

$$\widetilde{v}(\xi) \leq \widetilde{v}(t^{*}) \prod_{t^{*} < \xi_k < \xi} \frac{\overline{\alpha}_k}{\underline{\beta}_k} - \int_{t^{*}}^{\xi} \prod_{s < \xi_k < \xi} \frac{\overline{\alpha}_k}{\underline{\beta}_k} \widetilde{\psi}(s) ds, \qquad (42)$$

Therefore,

$$
\begin{aligned}
&\widetilde{w}(\xi) \\
&\geq \widetilde{w}(t^{*}) \prod_{t^{*} < \xi_k < \xi} \frac{\overline{\alpha}_k}{\underline{\beta}_k} + \int_{t^{*}}^{\xi} \prod_{s < \xi_k < \xi} \frac{\overline{\alpha}_k}{\underline{\beta}_k} \widetilde{\psi}(s) ds \\
&= \prod_{t^{*} < \xi_k < \xi} \frac{\overline{\alpha}_k}{\underline{\beta}_k} \left[\widetilde{w}(t^{*}) + \int_{t^{*}}^{\xi} \prod_{t^{*} < \xi_k < s} \frac{\underline{\beta}_k}{\overline{\alpha}_k} \widetilde{\psi}(s) ds \right] \qquad (43) \\
&> 0,
\end{aligned}
$$

which contradicts the fact that $\widetilde{w}(\xi) < 0$. The proof is complete.

Theorem 3.8 If all the conditions of Theorem 3.7 hold. Then every solution of problem (1) and (3) oscillates in E.

Proof Suppose that $V(t)$ is a nonoscillatory solution of (17). Without loss of generality, the proof that (17) is oscillatory is similar to that of Theorems 3.7, therefore, we omit it.

IV. APPLICATION

Example 4.1 Consider the fractional differential equation

$$
\begin{cases}
D_{+,t}^{\frac{1}{2}} \left[e^{-t - \frac{2}{3\sqrt{\pi}} t^{\frac{3}{2}}} D_{+,t}^{\frac{1}{2}} u(x,t) \right] + \frac{1}{2} t e^{-t - \frac{2}{3\sqrt{\pi}} t^{\frac{3}{2}}} D_{+,t}^{\frac{1}{2}} u(x,t) \\
\quad = e^{t} u^2 \Delta u(x,t) + t^{\frac{2}{3}} \Delta u(x, t - \frac{\pi}{3}) \\
\qquad - \dfrac{u^3(x,t)}{1 + t^2 + x^2} - (x^2 + t^2) u(x,t), \quad t \neq t_k \\
D_{+,t}^{\frac{1}{2}} u(x, t_k^{+}) = 3 D_{+,t}^{\frac{1}{2}} u(x, t_k), \\
\qquad k = 1, 2, 3, \cdots, \quad (x,t) \in (0, \pi) \times R_{+} \equiv E \\
u(x, t_k^{+}) = u(x, t_k), \\
\qquad k = 1, 2, 3, \cdots, \quad (x,t) \in (0, \pi) \times R_{+} \equiv E
\end{cases} \qquad (44)
$$

with the boundary condition

$$u(0,t) = u(\pi, t) = 0, \qquad t > 0 \qquad (45)$$

Note here that

$$
\begin{aligned}
&r(t) = e^{-t - \frac{2}{3\sqrt{\pi}} t^{\frac{3}{2}}}, \quad p(t) = \tfrac{1}{2} t e^{-t - \frac{2}{3\sqrt{\pi}} t^{\frac{3}{2}}}, \\
&a(t) = e^{t}, \ a_1(t) = t^{\frac{2}{3}}, \ m(x,t,u) = \tfrac{u^3(x,t)}{1 + t^2 + x^2}, \\
&h(u) = u^2, \quad h_1(u) = 1, \\
&q(x,t) = x^2 + t^2, \quad q(t) = \min_{x \in \overline{\Omega}} (x^2 + t^2) = t^2, \\
&\sigma\left(x, t_k, D_{+,t}^{\frac{1}{2}} u\left(x, t_k\right)\right) = 3 D_{+,t}^{\frac{1}{2}} u(x, t_k), \\
&\delta\left(x, t_k, u(x, t_k)\right) = u(x, t_k), \\
&F(u) = u, \quad \frac{F(u)}{u} \geq c = 1.
\end{aligned}
$$

And take $\Omega = (0, \pi)$, $j = 1$, $\underline{\beta}_k = 1$, $\overline{\alpha}_k = 3$.

We can easily check the conditions of Theorem 3.8, as follows

$$R(t) = I_{0+}^{\frac{1}{2}} \left(\frac{p(t)}{r(t)} \right) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t - v)^{\frac{1}{2} - 1} \cdot \frac{v}{2} dv$$

$$= \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (-v) d(t-v)^{-\frac{1}{2}}$$
$$= \tfrac{2}{3\sqrt{\pi}} t^{\frac{3}{2}},$$

$$\int_{t^*}^{\infty} \frac{1}{r(s)e^{R(s)}} ds = \int_{t^*}^{\infty} \left(e^{-t-\frac{2}{3\sqrt{\pi}} t^{\frac{3}{2}}} e^{\frac{2}{3\sqrt{\pi}} t^{\frac{3}{2}}} \right)^{-1} ds$$
$$= \int_{t^*}^{\infty} e^s ds = +\infty,$$

Therefore

$$\lim_{\xi\to\infty} \int_{t^*}^{\xi} e^{\widetilde{R(s)}} \widetilde{q(s)} ds$$
$$= \lim_{\xi\to\infty} \int_{t^*}^{\xi} e^{\frac{2}{3\sqrt{\pi}} s^{\frac{3}{2}}} s^2 ds$$
$$= \lim_{\xi\to\infty} \left[\sqrt{\pi} s^{\frac{3}{2}} e^{\frac{2}{3\sqrt{\pi}} s^{\frac{3}{2}}} |_{t^*}^{\xi} - \int_{t^*}^{\xi} e^{\frac{2}{3\sqrt{\pi}} s^{\frac{3}{2}}} \sqrt{\pi} ds^{\frac{3}{2}} \right]$$
$$= \lim_{\xi\to\infty} \left[\sqrt{\pi} s^{\frac{3}{2}} e^{\frac{2}{3\sqrt{\pi}} s^{\frac{3}{2}}} |_{t^*}^{\xi} - \int_{t^*}^{\xi} \frac{3\sqrt{\pi}}{2} e^{\frac{2}{3\sqrt{\pi}} s^{\frac{3}{2}}} s^{\frac{1}{2}} ds \right]$$
$$= \lim_{\xi\to\infty} \left[\sqrt{\pi} s^{\frac{3}{2}} e^{\frac{2}{3\sqrt{\pi}} s^{\frac{3}{2}}} |_{t^*}^{\xi} - \frac{3\pi}{2} \int_{t^*}^{\xi} de^{\frac{2}{3\sqrt{\pi}} s^{\frac{3}{2}}} \right]$$
$$= \lim_{\xi\to\infty} \left[\sqrt{\pi} \xi^{\frac{3}{2}} e^{\frac{2}{3\sqrt{\pi}} \xi^{\frac{3}{2}}} - \frac{3\pi}{2} e^{\frac{2}{3\sqrt{\pi}} \xi^{\frac{3}{2}}} \right.$$
$$\left. - \sqrt{\pi} (t^*)^{\frac{3}{2}} e^{\frac{2}{3\sqrt{\pi}} (t^*)^{\frac{3}{2}}} - \frac{3\pi}{2} e^{\frac{2}{3\sqrt{\pi}} (t^*)^{\frac{3}{2}}} \right]$$
$$= +\infty,$$

$$\lim_{\xi\to\infty} \int_{t^*}^{\xi} \prod_{t^*<\xi_k<s} \frac{\beta_k}{\overline{\alpha}_k} \widetilde{\psi}(s) ds = +\infty.$$

Therefore, every solution of the problem (44) and (45) oscillates.

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