Cyclic Brunn-Minkowski Inequalities for the General L_p-Mixed Brightness Integrals and General L_p-Dual Mixed Brightness Integrals

Linmei Yu, Weidong Wang* and Yuanyuan Zhang

Abstract—**Based on the general L_p-projection bodies, Yan and Wang introduced the notion of general L_p-mixed brightness integrals. Combining with general L_p-intersection bodies, Zhang and Wang gave the general L_p-dual mixed brightness integrals. In this paper, we establish the new cyclic Brunn-Minkowski inequalities for general L_p-mixed brightness integrals and general L_p-dual mixed brightness integrals. Our results unify the relevant cyclic inequalities and Brunn-Minkowski inequalities.**

Index Terms—**cyclic inequality; Brunn-Minkowski inequality; L_p-mixed brightness integral; L_p-dual mixed brightness integral.**

I. Introduction

THE setting for this paper is the Euclidean n-space $\mathbf{R}^n$. If K is nonempty compact convex set in $\mathbf{R}^n$, the support function, $h_K = h(K,\cdot) : \mathbf{R}^n \to \mathbf{R}$, of $K \in \mathcal{K}^n$ is defined by (see [6])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \qquad x \in \mathbf{R}^n$$

where $x \cdot y$ denotes the standard inner product of x and y. If K is compact convex set with nonempty interiors in $\mathbf{R}^n$, then K is called a convex body. Let $\mathcal{K}_o^n$ denote the set of convex bodies that containing the origin in $\mathbf{R}^n$.

For a compact set K in $\mathbf{R}^n$ which is star shaped with respect to the origin, the radial function, $\rho_K = \rho(K,\cdot) : \mathbf{R}^n \backslash \{0\} \to [0,\infty)$, was defined by (see [6])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \qquad x \in \mathbf{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, then K will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denote the subset of star bodies containing the origin in $\mathbf{R}^n$. Two star bodies K and L are dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$, where S^{n-1} denotes the unit sphere in $\mathbf{R}^n$.

Asymmetric L_p-Brunn-Minkowski theory has its origins in the work of Ludwig, Haberl and Schuster (see [10], [11], [12], [14], [13], [21], [22]). Based on Lutwak, Yang and Zhang's L_p-projection body (see [24]), Ludwig ([21]) discovered general L_p-projection bodies: For $K \in \mathcal{K}_o^n$, $p \geq 1$, the function $\varphi_\tau : \mathbf{R} \to [0,\infty)$ given by $\varphi_\tau(t) = |t| + \tau t$, where $\tau \in [-1,1]$, the general L_p-projection body, $\Pi_p^\tau K \in \mathcal{K}_o^n$, of K is defined by

$$h^p_{\Pi_p^\tau K}(u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot \nu)^p dS_p(K,\nu), \qquad (1.1)$$

for all $u \in S^{n-1}$. Here $S_p(K,\cdot)$ denotes the L_p-surface area measure of K, and

$$\alpha_{n,p}(\tau) = \frac{\alpha_{n,p}}{(1+\tau)^p + (1-\tau)^p}.$$

The normalization is chosen such that $\Pi_p^\tau B = B$ for every $\tau \in [-1,1]$. Obviously, if $\tau = 0$, then $\Pi_p^\tau K = \Pi_p K$ which is just Lutwak, Yang and Zhang's L_p-projection body (see [24]). On this basis, let $p = 1$, the convex body $\Pi_1 K$ is a dilate of the classical projection body ΠK of K and $\Pi_1 B = B$.

In 2015, based on the general L_p-projection bodies, Yan and Wang ([42]) defined general L_p-mixed brightness integrals: for $K_1,...,K_n \in \mathcal{K}_o^n$, $p \geq 1$ and $\tau \in [-1,1]$, the general L_p-mixed brightness integrals, $D_p^{(\tau)}(K_1,...,K_n)$, of $K_1,...,K_n$ is defined by

$$D_p^{(\tau)}(K_1,...,K_n) = \frac{1}{n}\int_{S^{n-1}} \delta_p^{(\tau)}(K_1,u)\cdots\delta_p^{(\tau)}(K_n,u)dS(u), \qquad (1.2)$$

where $\delta_p^{(\tau)}(K,u) = \frac{1}{2}h(\Pi_p^\tau K,u)$ denotes the half general L_p-brightness of $K \in \mathcal{K}_o^n$ in the direction u. If there exist constants $\lambda_1,...,\lambda_n > 0$ such that for all $u \in S^{n-1}$

$$\lambda_1\delta_p^{(\tau)}(K_1,u) = \lambda_2\delta_p^{(\tau)}(K_2,u) = \cdots = \lambda_n\delta_p^{(\tau)}(K_n,u),$$

then we call convex bodies $K_1,...,K_n$ have similar general L_p-brightness.

For $\tau = 0$ in (1.2), we write $\delta_p^{(0)}(K,u) = \delta_p(K,u) = \frac{1}{2}h(\Pi_p K,u)$ and

$$D_p(K_1,...,K_n) = D_p^{(0)}(K_1,\cdots,K_n) = \frac{1}{n}\int_{S^{n-1}} \delta_p(K_1,u)\cdots\delta_p(K_n,u)dS(u). \qquad (1.3)$$

Here $D_p(K_1,...,K_n)$ is called the mixed L_p-brightness integrals of $K_1,...,K_n \in \mathcal{K}_o^n$. If $p = 1$ in (1.3), then it is just the mixed brightness integrals given by Li and Zhu (see [17]).

In (1.2), let $K_1 = ... = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, we write

$$D_{p,i}^{(\tau)}(K,L)$$

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$$= \frac{1}{n}\int_{S^{n-1}} \delta_p^{(\tau)}(K,u)^{n-i}\delta_p^{(\tau)}(L,u)^i dS(u), \qquad (1.4)$$

if allow i is any real number, then $D_{p,i}^{(\tau)}(K,L)$ is called the i-th general L_p-mixed brightness integrals of K and L.

Let $L = B$ in (1.4), we write $D_{p,i}^{(\tau)}(K,B) = \frac{1}{2^i}D_{p,i}^{(\tau)}(K)$ and notice that $\delta_p^{(\tau)}(B,u) = \frac{1}{2}h(\Pi_p^\tau B,u) = \frac{1}{2}$ for all $u \in S^{n-1}$, then by (1.4) we have

$$D_{p,i}^{(\tau)}(K) = \frac{1}{2^i \cdot n}\int_{S^{n-1}} \delta_p^{(\tau)}(K,u)^{n-i} dS(u),$$

where the $D_{p,i}^{(\tau)}(K)$ is called the i-th general L_p-mixed brightness integrals of K.

In [42], Yan and Wang established the following Brunn-Minkowski type inequality and cyclic inequality for i-th general L_p-mixed brightness integrals.

Theorem 1.A. *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, $i \in \mathbf{R}$ and $i \neq n$, then for $i < n-p$,*

$$D_{p,i}^{(\tau)}(K \oplus_p L)^{\frac{p}{n-i}} \leq D_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + D_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}; \quad (1.5)$$

for $n-p < i < n$ or $i > n$, we have

$$D_{p,i}^{(\tau)}(K \oplus_p L)^{\frac{p}{n-i}} \geq D_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + D_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}. \quad (1.6)$$

In each case, equality holds if and only if K and L have similar general L_p-brightness. For $i = n-p$, equality always holds in (1.5) and (1.6). Here the "$\oplus_p$" denotes the L_p-Blaschke sum.

Theorem 1.B. *Let $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$ and $i, j, k \in \mathbf{R}$. If $i < j < k$, then*

$$D_{p,j}^{(\tau)}(K)^{k-i} \leq D_{p,i}^{(\tau)}(K)^{k-j} D_{p,k}^{(\tau)}(K)^{j-i},$$

with equality if and only if K and L have similar general L_p-brightness, i.e., K has constant general L_p-brightness.

In 2006, Haberl and Ludwig ([11]) introduced the notion of L_p-intersection body, it is an important concept in L_p-Brunn-Minkowski theory. Recently, Wang and Li ([35], [36]) using the function $\varphi_\tau' : \mathbf{R} \to [0,+\infty)$ which is given by

$$\varphi_\tau'(t) = |t| - \tau t, \quad \tau \in [-1,1],$$

to define the general L_p-intersection body with parameter τ as follows: for $K \in \mathcal{S}_o^n$, $0 < p < 1$, and $\tau \in [-1,1]$, the general L_p-intersection body, $I_p^\tau K \in \mathcal{S}_o^n$, of K is defined by (see [35], [36])

$$\begin{aligned} &\rho(I_p^\tau K,u)^p \\ &= i(\tau)\int_K \varphi_\tau'(u\cdot x)^{-p}dx \\ &= i(\tau)\int_K [|u\cdot x| - \tau(u\cdot x)]^{-p}dx \\ &= \frac{i(\tau)}{n-p}\int_{S^{n-1}} [|u\cdot v| - \tau(u\cdot v)]^{-p}\rho(K,v)^{n-p}dv, \quad (1.7) \end{aligned}$$

for any $u \in S^{n-1}$, where

$$i(\tau) = \frac{(1+\tau)^p(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \qquad (1.8)$$

Motivated by the general L_p-mixed brightness integrals and based on the general L_p-intersection bodies, Zhang and Wang ([43]) defined the general L_p-dual mixed brightness integrals as follows: For $K_1, ..., K_n \in \mathcal{S}_o^n$, $0 < p < 1$ and $\tau \in [-1,1]$, the general L_p-dual mixed brightness integrals, $\widetilde{D}_p^{(\tau)}(K_1,...,K_n)$, of $K_1,...,K_n$ is defined by

$$\begin{aligned} &\widetilde{D}_p^{(\tau)}(K_1,...,K_n) \\ &= \frac{1}{n}\int_{S^{n-1}} \widetilde{\delta}_p^{(\tau)}(K_1,u)\cdots\widetilde{\delta}_p^{(\tau)}(K_n,u), \qquad (1.9) \end{aligned}$$

where $\widetilde{\delta}_p^{(\tau)}(K,u) = \frac{1}{2}\rho(I_p^\tau K,u)$ denotes the half general L_p-dual brightness of $K \in \mathcal{S}_o^n$ in direction $u \in S^{n-1}$.

If $\tau = 0$, by (1.7) and (1.8), we have $I_p^0 K = I_p K$, the $I_p K$ called the L_p-intersection body which was given by Haberl and Ludwig (see [11]). So we write $\widetilde{\delta}_p^0(K,u) = \frac{1}{2}\rho(I_p K,u)$ and $\widetilde{D}_p^{(0)}(K_1,...,K_n) = \widetilde{D}_p(K_1,...,K_n)$. $\widetilde{D}_p(K_1,...,K_n)$ was called the L_p-dual mixed brightness integrals of $K_1,...,K_n \in S_o^n$, this was introduced by Zhang and Wang (see [43])

Let $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$ $(i = 0,1,...,n)$ in (1.9), that is $\widetilde{D}_{p,i}^{(\tau)}(K,L) = \widetilde{D}_p^{(\tau)}(K,...,K,L,...,L)$. If i is any real number, $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1,1]$, then the i-th general L_p-dual mixed brightness integrals, $\widetilde{D}_{p,i}^{(\tau)}(K,L)$, of K and L is defined by (see [43])

$$\begin{aligned} &\widetilde{D}_{p,i}^{(\tau)}(K,L) \\ &= \frac{1}{n}\int_{S^{n-1}} \widetilde{\delta}_p^{(\tau)}(K,u)^{n-i}\widetilde{\delta}_p^{(\tau)}(L,u)^i dS(u) \\ &= \frac{1}{2^n\cdot n}\int_{S^{n-1}} \rho(I_p^\tau K,u)^{n-i}\rho(I_p^\tau L,u)^i dS(u). \quad (1.10) \end{aligned}$$

In (1.10), let $L = B$ and notice that the $\widetilde{\delta}_p^{(\tau)}(B,u) = \frac{1}{2}\rho(I_p^\tau B,u) = \frac{1}{2}$ for all $u \in S^{n-1}$, so we write $\widetilde{D}_{p,i}^{(\tau)}(K,B) = \frac{1}{2^i}\widetilde{D}_{p,i}^{(\tau)}(K)$, which together with (1.10) yields

$$\widetilde{D}_{p,i}^{(\tau)}(K) = \frac{1}{2^i\cdot n}\int_{S^{n-1}} \widetilde{\delta}_p^{(\tau)}(K,u)^{n-i}dS(u).$$

Here we call $\widetilde{D}_{p,i}^{(\tau)}(K)$ the i-th general L_p-dual mixed brightness integrals of K.

For the i-th general L_p-dual mixed brightness integrals, Zhang and Wang ([43]) gave related Brunn-Minkowski type inequality and cyclic inequality as follows:

Theorem 1.C. *Let $K, K', L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1,1]$, $i \in \mathbf{R}$ and $i \neq n$. If $i < n-p$, then*

$$\begin{aligned} &\widetilde{D}_{p,i}^{(\tau)}(K \widehat{+}_p K', L)^{\frac{p}{n-i}} \\ &\leq \widetilde{D}_{p,i}^{(\tau)}(K,L)^{\frac{p}{n-i}} + \widetilde{D}_{p,i}^{(\tau)}(K',L)^{\frac{p}{n-i}}, \qquad (1.11) \end{aligned}$$

equality holds if and only if $I_p^\tau K$ and $I_p^\tau K'$ are dilates. If $i > n-p$, then inequality (1.11) is reversed. For $i = n-p$, (1.11) becomes an equality. Here "$\widehat{+}_p$" denotes L_p radial Blaschke sum.

Theorem 1.D. *Let $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1,1]$, $i, j, k \in \mathbf{R}$. If $\frac{k-i}{k-j} > 1$, then*

$$\widetilde{D}_{p,i}^{(\tau)}(K,L)^{k-j}\widetilde{D}_{p,k}^{(\tau)}(K,L)^{j-i} \geq \widetilde{D}_{p,j}^{(\tau)}(K,L)^{k-i}, \quad (1.12)$$

equality holds if and only if $I_p^\tau K$ and $I_p^\tau L$ are dilates. If $0 < \frac{k-i}{k-j} < 1$, the inequality (1.12) is reversed.

In this paper, associated with the Brunn-Minkowski type inequality (Theorem 1.A or Theorem 1.C) and cyclic inequality (Theorem 1.B or Theorem 1.D), we establish two cyclic Brunn-Minkowski inequalities for i-th general L_p-mixed brightness integrals and i-th general L_p-dual mixed brightness integrals, respectively. For more cyclic Brunn-Minkowski inequality, please refer to references ([44], [45]).

Theorem 1.1. *Let $K, K', L \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, $i, j, k \in \mathbf{R}$. If $j < n-p$ and $i \leq j \leq k$ $(i \neq k)$, then*

$$D_{p,j}^{(\tau)}(K \oplus_p K', L)^{\frac{p}{n-j}} \leq D_{p,i}^{(\tau)}(K, L)^{\frac{p(k-j)}{(k-i)(n-j)}} D_{p,k}^{(\tau)}(K, L)^{\frac{p(j-i)}{(k-i)(n-j)}} + D_{p,i}^{(\tau)}(K', L)^{\frac{p(k-j)}{(k-i)(n-j)}} D_{p,k}^{(\tau)}(K', L)^{\frac{p(j-i)}{(k-i)(n-j)}}; \quad (1.13)$$

if $n-p < j < n$ and $j \leq i \leq k$ or $j > n$ and $i \leq j \leq k$ $(i \neq k)$, then

$$D_{p,j}^{(\tau)}(K \oplus_p K', L)^{\frac{p}{n-j}} \geq D_{p,i}^{(\tau)}(K, L)^{\frac{p(k-j)}{(k-i)(n-j)}} D_{p,k}^{(\tau)}(K, L)^{\frac{p(j-i)}{(k-i)(n-j)}} + D_{p,i}^{(\tau)}(K', L)^{\frac{p(k-j)}{(k-i)(n-j)}} D_{p,k}^{(\tau)}(K', L)^{\frac{p(j-i)}{(k-i)(n-j)}}. \quad (1.14)$$

In each case, with equality if and only if K and K' have similar general L_p-brightness.

Theorem 1.2. *Let $K, K', L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1,1]$, $i, j, k \in \mathbf{R}$. If $j < n-p$ and $i \leq j \leq k$ $(i \neq k)$, then*

$$\widetilde{D}_{p,j}^{(\tau)}(K \widehat{+}_p K', L)^{\frac{p}{n-j}} \leq \widetilde{D}_{p,i}^{(\tau)}(K, L)^{\frac{p(k-j)}{(k-i)(n-j)}} \widetilde{D}_{p,k}^{(\tau)}(K, L)^{\frac{p(j-i)}{(k-i)(n-j)}} + \widetilde{D}_{p,i}^{(\tau)}(K', L)^{\frac{p(k-j)}{(k-i)(n-j)}} \widetilde{D}_{p,k}^{(\tau)}(K', L)^{\frac{p(j-i)}{(k-i)(n-j)}}; \quad (1.15)$$

if $n-p < j < n$ and $j \leq i \leq k$ or $j > n$ and $i \leq j \leq k$ $(i \neq k)$, then

$$\widetilde{D}_{p,j}^{(\tau)}(K \widehat{+}_p K', L)^{\frac{p}{n-j}} \geq \widetilde{D}_{p,i}^{(\tau)}(K, L)^{\frac{p(k-j)}{(k-i)(n-j)}} \widetilde{D}_{p,k}^{(\tau)}(K, L)^{\frac{p(j-i)}{(k-i)(n-j)}} + \widetilde{D}_{p,i}^{(\tau)}(K', L)^{\frac{p(k-j)}{(k-i)(n-j)}} \widetilde{D}_{p,k}^{(\tau)}(K', L)^{\frac{p(j-i)}{(k-i)(n-j)}}. \quad (1.16)$$

In each case, with equality if and only if $I_p^\tau K$ and $I_p^\tau K'$ are dilates.

Remark 1.1. Let $j = i$ in Theorem 1.1, we can get the following Brunn-Minkowski type inequality for i-th general L_p-mixed brightness integrals.

Corollary 1.1. *Let $K, K', L \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, $i \in \mathbf{R}$ and $i \neq n$. If $i < n-p$, then*

$$D_{p,i}^{(\tau)}(K \oplus_p K', L)^{\frac{p}{n-i}} \leq D_{p,i}^{(\tau)}(K, L)^{\frac{p}{n-i}} + D_{p,i}^{(\tau)}(K', L)^{\frac{p}{n-i}};$$

if $i > n-p$, then

$$D_{p,i}^{(\tau)}(K \oplus_p K', L)^{\frac{p}{n-i}} \geq D_{p,i}^{(\tau)}(K, L)^{\frac{p}{n-i}} + D_{p,i}^{\tau}(K', L)^{\frac{p}{n-i}}.$$

In each case, with equality if and only if K and K' have similar general L_p-brightness.

If $L = B$, then Corollary 1.1 yields Theorem 1.A.

Remark 1.2. Let $K' = \{o\}$ in (1.13), then we may obtain the following result which was given by Yan and Wang (see [42]).

Corollary 1.2. *Let $K, L \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$ and $i, j, k \in \mathbf{R}$, if $i < j < k$, then*

$$D_{p,j}^{\tau}(K, L)^{k-i} \leq D_{p,i}^{\tau}(K, L)^{k-j} D_{p,k}^{\tau}(K, L)^{j-i},$$

with equality if and only if K and L have similar general L_p-brightness.

In particular, let $L = B$ in Corollary 1.2, this is just Theorem 1.B.

Remark 1.3. Let $j = i$ in Theorem 1.2, we immediately obtain Theorem 1.C. If $K' = \{o\}$ in Theorem 1.2, then Theorem 1.D can be given.

Our results unify the relevant cyclic inequalities and Brunn-Minkowski inequalities for the general L_p-mixed brightness integrals and general L_p-dual mixed brightness integrals. Our work belongs to a new and rapidly evolving asymmetric L_p Brunn-Minkowski theory, the further researches for this theory, readers can refer to papers [2], [3], [4], [10], [11], [12], [14], [18], [19], [20], [25], [26], [27], [30], [31], [38], [39], [29], [33], [34], [35], [36], [37], [40], [32], [41], [42], [43], [46], [47].

II. Preliminaries

A. L_p-Blaschke combination

According to the existence's theorem of L_p-Minkowski problem (see Theorem 9.2.3 in [28]), the author defined the L_p-Blaschke combinations of convex bodies as follows: For $K, L \in \mathcal{K}_o^n$, $n \neq p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-Blaschke combination of K and L is defined by (see [28]):

$$S_p(\lambda \circ K \oplus_p \mu \circ L, \cdot) = \lambda S_p(K, \cdot) + \mu S_p(L, \cdot), \quad (2.1)$$

where " $\oplus_p$ " denotes L_p-Blaschke addition, " $\lambda \circ K$ " denotes L_p-Blaschke scalar multiplication. When K and L both are origin-symmetric convex bodies, the (2.1) was given by Lutwak ([23]). For more information on these and other binary operations between convex and star bodies, see ([7], [8], [9], [16]).

By (1.1) and (2.1), we have, for all $u \in S^{n-1}$,

$$h(\Pi_p^\tau(\lambda \circ K \oplus_p \mu \circ L), u) = \lambda h(\Pi_p^\tau K, u)^p + \mu h(\Pi_p^\tau L, u)^p, \quad (2.2)$$

i.e.,

$$\Pi_p^\tau(\lambda \circ K \oplus_p \mu \circ L) = \lambda \cdot \Pi_p^\tau K +_p \mu \cdot \Pi_p^\tau L.$$

B. L_p-radial combination and L_p-radial Blaschke combination

For $K, L \in \mathcal{S}_o^n$, $p \neq 0$ and $\lambda, \mu \geq 0$ (note both zero), the L_p-radial combination, $\lambda * K \widetilde{+}_p \mu * L \in \mathcal{S}_o^n$, of K and L was defined by (see [5], [28])

$$\rho(\lambda * K \widetilde{+}_p \mu * L, \cdot)^p = \lambda \rho(K, \cdot)^p + \lambda \rho(L, \cdot)^p, \quad (2.3)$$

where "$\widetilde{+}_p$ " and "$\lambda * K$ " denote L_p radial addition and L_p radial scalar multiplication, respectively.

If $n > p > 0$, then according to (2.3), we defined the L_p-radial Blaschke combination, $\lambda \odot K \widehat{+}_p \mu \odot L \in \mathcal{S}_o^n$, of K and L by (see [32])

$$\lambda \odot K \widehat{+}_p \mu \odot L = \lambda * K \widetilde{+}_{n-p} \mu * L. \quad (2.4)$$

For $K, L \in \mathcal{S}_o^n$, $0 < p < 1$, $\tau \in [-1,1]$, from (1.8), (2.3) and (2.4), we can obtain for any $u \in S^{n-1}$,

$$\rho^p(I_p^\tau(K \widehat{+}_p L), u)$$

$$= \frac{i(\tau)}{n-p}\int_{S^{n-1}}[|u\cdot v|-\tau(u\cdot v)]^{-p}\rho((K\widehat{+}_pL),v)^{n-p}dv$$
$$= \frac{i(\tau)}{n-p}\int_{S^{n-1}}[|u\cdot v|-\tau(u\cdot v)]^{-p}\rho(K,v)^{n-p}dv$$
$$+\frac{i(\tau)}{n-p}\int_{S^{n-1}}[|u\cdot v|-\tau(u\cdot v)]^{-p}\rho(L,v)^{n-p}dv$$
$$= \rho(I_p^\tau K,u)^p+\rho(I_p^\tau L,u)^p,$$

i.e.,

$$I_p^\tau(K\widehat{+}_pL)=I_p^\tau K\widetilde{+}_pI_p^\tau L. \tag{2.5}$$

III. Results and Proofs

In this part, we will give the proofs of Theorem 1.1 and Theorem 1.2. First, we give the following lemmas.

Lemma 3.1 ([1]). *Let $f\in L^p(E), g\in L^q(E)$, real number $p,q\neq 0$ and $\frac{1}{p}+\frac{1}{q}=1$, if $p>1$, then*

$$\left(\int_E|f(x)|^pdx\right)^{\frac{1}{p}}\left(\int_E|g(x)|^qdx\right)^{\frac{1}{q}} \geq \int_E|f(x)g(x)|dx; \tag{3.1}$$

if $p<0$ or $0<p<1$, then

$$\left(\int_E|f(x)|^pdx\right)^{\frac{1}{p}}\left(\int_E|g(x)|^qdx\right)^{\frac{1}{q}} \leq \int_E|f(x)g(x)|dx. \tag{3.2}$$

In each case, with equality if and only if there exists constant c_1 and c_2, such that $c_1|f(x)|^p=c_2|g(x)|^q$. Here $L^p(E)$ denotes all function sets defined on measurable set E in L_p spaces.

Lemma 3.2 ([15]). *Let $f\in L^p(E)$ and $g\in L^p(E)$, if real number $p\neq 0$ and $p>1$, then*

$$\left(\int_E|f(x)|^pdx\right)^{\frac{1}{p}}+\left(\int_E|g(x)|^pdx\right)^{\frac{1}{p}} \geq \left(\int_E|f(x)+g(x)|^p\right)^{\frac{1}{p}}dx; \tag{3.3}$$

if $p<0$ or $0<p<1$, then

$$\left(\int_E|f(x)|^pdx\right)^{\frac{1}{p}}+\left(\int_E|g(x)|^pdx\right)^{\frac{1}{p}} \leq \left(\int_E|f(x)+g(x)|^p\right)^{\frac{1}{p}}dx. \tag{3.4}$$

In each case, with equality if and only if there exists constant c_1 and c_2, such that $c_1|f(x)|=c_2|g(x)|$.

Proof of Theorem 1.1. Since $p\geq 1$ and $j<n-p$, thus $\frac{n-j}{p}>1$. From this, by (1.4), (2.2) and (3.3), we get

$$D_{p,j}^{(\tau)}(K\oplus_pK^{'},L)^{\frac{p}{n-j}}$$
$$=\left[\frac{1}{2^i\cdot n}\int_{S^{n-1}}\delta_p^{(\tau)}(K\oplus_pK^{'},u)^{n-j}\delta_p^{(\tau)}(L,u)^jdS(u)\right]^{\frac{p}{n-j}}$$
$$=\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left(h(\Pi_p^\tau(K\oplus_pK^{'}),u)^ph(\Pi_p^\tau L,u)^{\frac{jp}{n-j}}\right)^{\frac{n-j}{p}}dS(u)\right]^{\frac{p}{n-j}}$$
$$=\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left((h(\Pi_p^\tau K,u)^p+h(\Pi_p^\tau K^{'},u)^p)h(\Pi_p^\tau L,u)^{\frac{jp}{n-j}}\right)^{\frac{n-j}{p}}dS(u)\right]^{\frac{p}{n-j}}$$
$$=\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left(h(\Pi_p^\tau K,u)^{\frac{p(k-j)(n-i)}{(k-i)(n-j)}}h(\Pi_p^\tau K,u)^{\frac{p(j-i)(n-k)}{(k-i)(n-j)}}h(\Pi_p^\tau L,u)^{\frac{jp}{n-j}}+h(\Pi_p^\tau K^{'},u)^{\frac{p(k-j)(n-i)}{(k-i)(n-j)}}h(\Pi_p^\tau K^{'},u)^{\frac{p(j-i)(n-k)}{(k-i)(n-j)}}h(\Pi_p^\tau L,u)^{\frac{jp}{n-j}}\right)^{\frac{n-j}{p}}dS(u)\right]^{\frac{p}{n-j}}$$
$$\leq\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}h(\Pi_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}h(\Pi_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}h(\Pi_p^\tau L,u)^jdS(u)\right]^{\frac{p}{n-j}}$$
$$+\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}h(\Pi_p^\tau K^{'},u)^{\frac{(k-j)(n-i)}{k-i}}h(\Pi_p^\tau K^{'},u)^{\frac{(j-i)(n-k)}{k-i}}h(\Pi_p^\tau L,u)^jdS(u)\right]^{\frac{p}{n-j}}. \tag{3.5}$$

In the first item of right hand of inequality (3.5), notice that $i\leq j\leq k$ $(i\neq k)$ implies $\frac{k-i}{k-j}>1$, by (3.1) we know

$$\frac{1}{2^n\cdot n}\int_{S^{n-1}}h(\Pi_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}h(\Pi_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}h(\Pi_p^\tau L,u)^jdS(u)$$
$$=\frac{1}{2^n\cdot n}\int_{S^{n-1}}h(\Pi_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}h(\Pi_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}h(\Pi_p^\tau L,u)^{\frac{i(k-j)}{k-i}}h(\Pi_p^\tau L,u)^{\frac{k(j-i)}{k-i}}dS(u)$$
$$\leq\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left(h(\Pi_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}h(\Pi_p^\tau L,u)^{\frac{i(k-j)}{k-i}}\right)^{\frac{k-i}{k-j}}dS(u)\right]^{\frac{k-j}{k-i}}$$
$$\times\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left(h(\Pi_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}h(\Pi_p^\tau L,u)^{\frac{k(j-i)}{k-i}}\right)^{\frac{k-i}{j-i}}dS(u)\right]^{\frac{j-i}{k-i}}$$
$$=D_{p,i}^{(\tau)}(K,L)^{\frac{k-j}{k-i}}D_{p,k}^{(\tau)}(K,L)^{\frac{j-i}{k-i}}. \tag{3.6}$$

Combining $\frac{p}{n-j}>0$ and (3.6), we can obtain the following inequality

$$\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}h(\Pi_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}h(\Pi_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}h(\Pi_p^\tau L,u)^jdS(u)\right]^{\frac{p}{n-j}}$$
$$\leq D_{p,i}^{(\tau)}(K,L)^{\frac{p(k-j)}{(k-i)(n-j)}}D_{p,k}^{(\tau)}(K,L)^{\frac{p(j-i)}{(k-i)(n-j)}}. \tag{3.7}$$

Similarly, we have

$$\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}} h(\Pi_p^\tau K',u)^{\frac{(k-j)(n-i)}{k-i}} h(\Pi_p^\tau K',u)^{\frac{(j-i)(n-k)}{k-i}} h(\Pi_p^\tau L,u)^j dS(u)\right]^{\frac{p}{n-j}} \leq D_{p,i}^{(\tau)}(K',L)^{\frac{p(k-j)}{(k-i)(n-j)}} D_{p,k}^{(\tau)}(K',L)^{\frac{p(j-i)}{(k-i)(n-j)}}. \quad (3.8)$$

Hence, by inequalities (3.5), (3.7) and (3.8), we get

$$D_{p,j}^{(\tau)}(K\oplus_p K',L)^{\frac{p}{n-j}} \leq D_{p,i}^{(\tau)}(K,L)^{\frac{p(k-j)}{(k-i)(n-j)}} D_{p,k}^{(\tau)}(K,L)^{\frac{p(j-i)}{(k-i)(n-j)}} + D_{p,i}^{(\tau)}(K',L)^{\frac{p(k-j)}{(k-i)(n-j)}} D_{p,k}^{(\tau)}(K',L)^{\frac{p(j-i)}{(k-i)(n-j)}}.$$

This yields (1.13).

Similarly, from $n-p<j<n$ and $k\geq i\geq j$ $(i\neq k)$, and notice that $p\geq 1$, we know that $0<\frac{n-j}{p}<1$ and $0<\frac{k-i}{k-j}<1$, by (3.2) and (3.4), the (3.5) and (3.6) are reversed. Since $\frac{p}{n-j}>0$, so the (3.7) and (3.8) are reversed. For $j>n$, the $\frac{n-j}{p}<0$, by (3.4), the (3.5) is reversed, and using the $\frac{p}{n-j}<0$ and (3.6), the inequality (3.7) is reversed. Similarly, the inequality (3.8) is also reversed. In summary, we may obtain inequality (1.14).

By the equality conditions of Lemma 3.1 and Lemma 3.2, there exists equalities in (1.13) and (1.14) if and only if K and L have similar general L_p-brightness.

Proof of Theorem 1.2. From $0<p<1$ and $j<n-p$, we know $\frac{n-j}{p}>1$. Hence, according to (1.10), (2.3), (2.5) and (3.3), we get

$$\begin{aligned}
&\widetilde{D}_{p,j}^{(\tau)}(K\widehat{+}_pK',L)^{\frac{p}{n-j}}\\
&=\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\rho(I_p^\tau(K\widehat{+}_pK',u))^{n-j}\rho(I_p^\tau L,u)^j dS(u)\right]^{\frac{p}{n-j}}\\
&=\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left(\rho(I_p^\tau(K\widehat{+}_pK'),u)^p\rho(I_p^\tau L,u)^{\frac{jp}{n-j}}\right)^{\frac{n-j}{p}} dS(u)\right]^{\frac{p}{n-j}}\\
&=\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left((\rho(I_p^\tau K,u)^p+\rho(I_p^\tau K',u)^p)\rho(I_p^\tau L,u)^{\frac{jp}{n-j}}\right)^{\frac{n-j}{p}} dS(u)\right]^{\frac{p}{n-j}}\\
&=\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left(\rho(I_p^\tau K,u)^{\frac{p(k-j)(n-i)}{(k-i)(n-j)}}\rho(I_p^\tau K,u)^{\frac{p(j-i)(n-k)}{(k-i)(n-j)}}\rho(I_p^\tau L,u)^{\frac{jp}{n-j}}\right.\right.\\
&\quad\left.\left.+\rho(I_p^\tau K',u)^{\frac{p(k-j)(n-i)}{(k-i)(n-j)}}\rho(I_p^\tau K',u)^{\frac{p(j-i)(n-k)}{(k-i)(n-j)}}\rho(I_p^\tau L,u)^{\frac{jp}{n-j}}\right)^{\frac{n-j}{p}} dS(u)\right]^{\frac{p}{n-j}}\\
&\leq\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\rho(I_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}\rho(I_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}\rho(I_p^\tau L,u)^j dS(u)\right]^{\frac{p}{n-j}}\\
&\quad+\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\rho(I_p^\tau K',u)^{\frac{(k-j)(n-i)}{k-i}}\rho(I_p^\tau K',u)^{\frac{(j-i)(n-k)}{k-i}}\rho(I_p^\tau L,u)^j dS(u)\right]^{\frac{p}{n-j}}. \quad (3.9)
\end{aligned}$$

In the first item of right side of the above inequality (3.9), since $i\leq j\leq k$ $(i\neq k)$, thus $\frac{k-i}{k-j}>1$. Therefore, by (3.1) we get

$$\begin{aligned}
&\frac{1}{2^n\cdot n}\int_{S^{n-1}}\rho(I_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}\rho(I_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}\rho(I_p^\tau L,u)^j dS(u)\\
&=\frac{1}{2^n\cdot n}\int_{S^{n-1}}\rho(I_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}\rho(I_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}\rho(I_p^\tau L,u)^{\frac{i(k-j)}{k-i}}\rho(I_p^\tau L,u)^{\frac{k(j-i)}{k-i}} dS(u)\\
&\leq\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left(\rho(I_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}\rho(I_p^\tau L,u)^{\frac{i(k-j)}{k-i}}\right)^{\frac{k-i}{k-j}} dS(u)\right]^{\frac{k-j}{k-i}}\\
&\quad\times\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\left(\rho(I_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}\rho(I_p^\tau L,u)^{\frac{k(j-i)}{k-i}}\right)^{\frac{k-i}{j-i}} dS(u)\right]^{\frac{j-i}{k-i}}\\
&=\widetilde{D}_{p,i}^{(\tau)}(K,L)^{\frac{k-j}{k-i}}\widetilde{D}_{p,k}^{(\tau)}(K,L)^{\frac{j-i}{k-i}}. \quad (3.10)
\end{aligned}$$

So, using (3.10), we get

$$\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\rho(I_p^\tau K,u)^{\frac{(k-j)(n-i)}{k-i}}\rho(I_p^\tau K,u)^{\frac{(j-i)(n-k)}{k-i}}\rho(I_p^\tau L,u)^j dS(u)\right]^{\frac{p}{n-j}} \leq \widetilde{D}_{p,i}^{(\tau)}(K,L)^{\frac{p(k-j)}{(k-i)(n-j)}}\widetilde{D}_{p,k}^{(\tau)}(K,L)^{\frac{p(j-i)}{(k-i)(n-j)}}. \quad (3.11)$$

Similarly to the above mothod, we also have

$$\left[\frac{1}{2^n\cdot n}\int_{S^{n-1}}\rho(I_p^\tau K',u)^{\frac{(k-j)(n-i)}{k-i}}\rho(I_p^\tau K',u)^{\frac{(j-i)(n-k)}{k-i}}\rho(I_p^\tau L,u)^j dS(u)\right]^{\frac{p}{n-j}} \leq \widetilde{D}_{p,i}^{(\tau)}(K',L)^{\frac{p(k-j)}{(k-i)(n-j)}}\widetilde{D}_{p,k}^{(\tau)}(K',L)^{\frac{p(j-i)}{(k-i)(n-j)}}. \quad (3.12)$$

Hence, by (3.9), (3.11) and (3.12), we have

$$\widetilde{D}_{p,j}^{(\tau)}(K\widehat{+}_pK',L)^{\frac{p}{n-j}} \leq \widetilde{D}_{p,i}^{(\tau)}(K,L)^{\frac{p(k-j)}{(k-i)(n-j)}}\widetilde{D}_{p,k}^{(\tau)}(K,L)^{\frac{p(j-i)}{(k-i)(n-j)}} + \widetilde{D}_{p,i}^{(\tau)}(K',L)^{\frac{p(k-j)}{(k-i)(n-j)}}\widetilde{D}_{p,k}^{(\tau)}(K',L)^{\frac{p(j-i)}{(k-i)(n-j)}}.$$

This completes the proof of (1.15).

When $n-p<j<n$ and $k\geq i\geq j$ $(i\neq k)$, we see that $0<\frac{n-j}{p}<1$ and $0<\frac{k-i}{k-j}<1$, by (3.2) and (3.4), the (3.9) and (3.10) are reversed. Since $\frac{p}{n-j}>0$, so the (3.11) and (3.12) are reversed. For $j>n$, by (3.4), we can get the (3.9) is reversed. Using $j>n$ and (3.10), we can get the (3.11) is reversed. Similarly, the (3.12) is also reversed. So, when $n-p<j<n$ and $j\leq i\leq k$ or $j>n$ and $i\leq j\leq k$ $(i\neq k)$, we can get (1.16).

From the equality conditions of the Lemma 3.1 and Lemma 3.2, we see that equalities hold in (1.15) and (1.16) if and only if $I_p^{\tau}K$ and $I_p^{\tau}K^{'}$ are dilates.

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