Positive Solutions for Fractional Nonlocal Boundary Value Problems with Dependence on the First Order Derivative

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Abstract—This research work is dedicated to an investigation of the existence of solutions for a class of fractional nonlocal boundary value problems of the type

\[
D_{0+}^\alpha u(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3,
\]

\[
u(0) = u'(0) = 0, \quad D_{0+}^\alpha u(1) = \int_0^\eta a(t)D_{0+}^\beta u(t)dt,
\]

where \(D_{0+}^\alpha\) is the standard Riemann-Liouville fractional derivative. A full analysis of existence of positive solutions is proved by using the monotone iterative technique. The interesting point is the nonlinear term \(f\) is involved with the first order derivative explicitly. The case \(f = f(t, u)\) existence results are proved via Schauder and a classical Krasnosel’skii fixed point theorems.

Index Terms—Positive solution; Boundary value problem; Fractional differential equation; Fixed point theorem.

I. INTRODUCTION

In this paper, we consider the existence results of positive solutions to the fractional nonlocal boundary value problems

\[
D_{0+}^\alpha u(t) + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3,
\]

\[
u(0) = u'(0) = 0, \quad D_{0+}^\alpha u(1) = \int_0^\eta a(t)D_{0+}^\beta u(t)dt,
\]

where \(D_{0+}^\alpha\) is the standard Riemann-Liouville fractional derivative, \(0 < \beta < 1, 0 \leq \gamma < \alpha - 1, \eta \in (0, 1), f \in C([0, 1] \times R^+ \times R^+, a(t) \in L^1[0, 1]\cap C(0, 1)\) is nonnegative.

The study of differentiation and integration to a fractional order has caught importance and popularity among researchers compared to classical differentiation and integration. Fractional operators used to illustrate better the reality of real-world phenomena with the hereditary property [1-3]. Existence of solutions is the basis of the theory of fractional differential equation. Most of the previous literature deals with the existence of solutions for fractional differential equations boundary value problems by the use of techniques of nonlinear analysis, see [6-36] and the references therein.

For example, in [17], Henderson and Luca considered the existence of positive solutions for the following fractional differential equation boundary value problems

\[
D_{0+}^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad n - 1 < \alpha \leq n,
\]

\[
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,
\]

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\[
D_{0+}^\alpha u(1) = \sum_{i=1}^m \eta_i D_{0+}^\beta u(\xi_i),
\]

where \(p \in [1, n - 2], q \in [0, p]\).

See also [33] where, the authors studied the following fractional differential equation with infinite-point boundary value conditions

\[
D_{0+}^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad n - 1 < \alpha \leq n,
\]

\[
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \int_0^1 u(t)dt.
\]

When \(1 \leq \beta < \alpha - 1\), Zhang and Zhong [35] investigated the existence of triple positive solutions for the fractional boundary value problem

\[
D_{0+}^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad n - 1 < \alpha \leq n,
\]

\[
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \int_0^1 u(t)dt,
\]

by using the Leggett-Williams and Krasnosel’skii fixed point theorems.

Recently, [32] presented the existence and multiplicity of positive solutions for a class of singular fractional nonlocal boundary value problems

\[
D_{0+}^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad n - 1 < \alpha \leq n,
\]

\[
u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \int_0^1 u(t)dt.
\]

All the above work was done under the assumption that \(f\) is allowed to depend just on \(u\), while the first order derivative \(u'\) is not involved explicitly in the nonlinear term \(f\). As we know, when the nonlinear term \(f\) is involved in the first-order derivative, difficulties arise immediately. In this work, we use the monotone iterative technique to overcome these difficulties. To the best knowledge of the authors, no work...
has been done for boundary value problem (1), (2) by use of the monotone iterative technique. The aim of this work is to fill the gap in the literature.

The paper is organized as follows. In section 2, we give some necessary concepts and results. Section 3 is devoted to study two existence results when \( f = f(t, u) \). The first one uses the Schauder fixed point theorem, while in the second one, existence result is obtained via the classical Krasnosel’skii fixed point theorem. In section 4, the existence result when the nonlinearity \( f \) depends on the solution and its first derivative is established by using the monotone iterative technique.

In this paper, \( E := C[0, 1] \) denotes the Banach space of all continuous functions on \( [0, 1] \) with the norm \( ||u||_0 = \max\{|u(x)|, 0 \leq x \leq 1\} \) and \( E^1 := C^1[0, 1] \) will refer to the Banach space of continuously differentiable functions on \( [0, 1] \) equipped with the norm \( ||u||_1 = \max\{|u(x)|, 0 \leq x \leq 1\} \).

II. THE PRELIMINARY LEMMAS

To reformulate the problem (1), (2) into a fixed point theorem, we present some necessary definitions and lemmas from conformable fractional calculus theory in this section.

**Definition 2.1** [32] The fractional integral of order \( \alpha > 0 \) for function \( u : (0, +\infty) \to R \) is given by
\[
I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds
\]
provided that the right hand side is point-wise defined on \( (0, +\infty) \).

**Definition 2.2** [32] The Riemann-Liouville fractional derivative of order \( \alpha > 0 \) for a function \( u : (0, +\infty) \to R \) is given by
\[
D_{0+}^\alpha u(t) = \frac{d^n}{dt^n} \left[ \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u(s) ds \right]
\]
where \( n = [\alpha]+1 \), \([\alpha]\) denotes the integer part of real number \( \alpha \), provided that the right hand side is point-wise defined on \( (0, +\infty) \).

**Lemma 2.1** [32] Let \( \alpha > 0 \), then the following equality holds for \( u \in L(0, 1), D_{0+}^\alpha u \in L(0, 1) \)
\[
I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}
\]
where \( c_i \in R, \ i = 1, 2, \ldots, n, n-1 < \alpha \leq n \).

**Lemma 2.2** [32] Assume that \( g \in L(0, 1) \) and \( \alpha > \beta > 0 \). Then
\[
D_{0+}^\beta I_{0+}^\alpha g(s) ds = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \int_0^s (t-s)^{\alpha-\beta-1} g(s) ds,
\]

**Lemma 2.3** [32] Assume that \( a \in L^1[0, 1] \cap C(0, 1) \),
\[
\Delta := \Gamma(\alpha-\gamma) - \Gamma(\alpha-\beta) \int_0^1 a(t) t^{\alpha-\gamma-1} dt \neq 0.
\]
Then for any \( y \in L[0, 1] \cap C(0, 1) \), the unique solution of the boundary value problem
\[
D_{0+}^\alpha u(t) + y(t) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3,
\]
\[
u(0) = u'(0) = 0, \quad D_{0+}^\beta u(1) = \int_0^1 a(t) D_{0+}^\gamma u(t) dt,
\]
is
\[
u(t) = \int_0^1 G(t, s)y(s) ds,
\]
Ω₁ ⊂ Ω₂ ⊂ Ω₂. Let \( T : K \cap (Ω₂ \setminus Ω₁) \to K \) be a completely continuous operator, such that:
(a) either \( \| Tv \| \leq \| v \| \) for \( v \in K \cap \partial Ω₁ \) and \( \| Tv \| \geq \| v \| \) for \( v \in K \cap \partial Ω₂ \),
(b) or \( \| Tv \| \geq \| v \| \) for \( v \in K \cap \partial Ω₁ \) and \( \| Tv \| \leq \| v \| \) for \( v \in K \cap \partial Ω₂ \).

Then \( T \) has at least a fixed point in \( K \cap (Ω₂ \setminus Ω₁) \).

3.1 An existence result by the Schauder fixed point theorem

Theorem 3.4 Suppose that
(1) \( f(t, \cdot) \) is nondecreasing on \( R^+ \), for all \( t \in [0, 1] \);
(2) Assume that there exists \( R > 0 \) such that
\[
\int_0^1 f(t, R) dt \leq \frac{R}{\Gamma(\alpha)} + \frac{1}{\Delta} \int_0^\rho a(t) dt. \tag{12}
\]

Then fractional boundary value problem (1), (2) has at least one nonnegative solution \( u \) such that \( \| u \| \leq R \).

Proof: For \( u \in \Omega = \{ u \in E : \| u \| \leq R \} \), from Lemma 2.5, we have
\[
\left\| (Lu)(t) \right\| \leq \int_0^1 G(t,s)f(s,u(s))ds \leq \int_0^1 G(t,s)f(s,u(s))ds \leq \int_0^1 G(t,s)f(s,R)ds \leq \int_0^1 t^{\alpha-1} \left( \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + h(s) \right) f(s,R)ds \leq \frac{1}{\Gamma(\alpha)} \int_0^\rho a(t) dt \int_0^1 f(s,R)ds \leq R. \]

So, \( \| Lu \| \leq R \). Therefore, the operator \( L \) maps \( B \) into itself. Hence, by applying Theorem 3.2 and Lemma 3.1, \( L \) has a fixed point \( u \) in \( B \).

3.2 An existence result by the classical Krasnosel’skii fixed point theorem

Construct the following cone
\[
P = \{ u \in E : u(t) \geq \beta \| u \| t^{\alpha-1}, \ t \in [0, 1] \}. \tag{13}
\]

Theorem 3.5 Assume that there exist \( \rho \in (0, 1), q₁, q₂ \in C([0, 1], R^+) \), nondecreasing functions \( \varphi₁, \varphi₂ \in C(R, R^+) \) and \( r₀, R₀ > 0 \) with \( r₀ \neq R₀ \) such that
(A) \( 0 \leq f(t, u) \leq q₁(t)\varphi₁(u) \) for all \( t \in [0, 1], \ 0 \leq u \leq r₀ \) with
\[
M\varphi₁(r₀) \leq \frac{r₀}{\Gamma(\alpha)} + \frac{1}{\Delta} \int_0^\rho a(t) dt. \tag{14}
\]
(B) \( f(t, u) \geq q₂(t)\varphi₂(u) \) for all \( t \in [0, 1], \ \beta \rho^{\alpha-1} R₀ \leq u \leq R₀ \) with
\[
\beta m\varphi₂(\beta R₀\rho^{\alpha-1}) \left( \frac{1-\rho}{\alpha-\beta} \right) \Gamma(\alpha) \geq R₀. \tag{15}
\]

Then fractional boundary value problem (1), (2) has a positive solution satisfying
\[
\min(r₀, R₀) \leq \| u \| \leq \max(r₀, R₀). \tag{16}
\]

Here \( m = \min \{ q₂(t), \ t \in [\rho, 1] \} \) and \( M = \max \{ q₁(t), \ t \in [0, 1] \} \).

Proof: (a) Let the open set \( B₁ = \{ u \in E : \| u \| < r₀ \} \) and \( u \in P \cap \partial B₁ \). Then for any \( t \in [0, 1] \), and since \( \varphi₁ \) is nondecreasing, we have
\[
\left\| (Lu)(t) \right\| \leq \int_0^1 G(t,s)f(s,u(s))ds \leq \int_0^1 G(t,s)q₁(s)\varphi₁(u)ds \leq \int_0^1 G(t,s)q₁(s)\varphi₁(\| u \|)ds \leq \int_0^1 G(t,s)q₁(\| u \|)ds \leq \frac{1}{\Gamma(\alpha)} \int_0^\rho a(t) dt \int_0^1 f(s,R)ds \leq r₀ = \| u \|.
\]

So, \( \| Lu \| \leq \| u \| \), for all \( u \in P \cap \partial B₁ \).

(b) Let the open set \( B₂ = \{ u \in E : \| u \| < R₀ \} \) and \( u \in P \cap \partial B₂ \). So, Lemma 2.5 yields
\[
u(t) \geq \beta R₀\rho^{\alpha-1}, \ \forall t \in [0, 1].
\]

Then for any \( t \in [0, 1] \), and since \( \varphi₂ \) is nondecreasing, we get
\[
\| Lu \| \geq \max_{t \in [0, 1]} \beta \rho^{\alpha-1} \int_0^1 \Phi₂(s)f(s,u(s))ds \geq \beta \rho^{\alpha-1} \int_0^1 \Phi₂(s)q₂(s)\varphi₂(u)ds \geq \beta \rho^{\alpha-1} \int_0^1 \Phi₂(s)\varphi₂(\beta R₀\rho^{\alpha-1})ds \geq \beta \rho^{\alpha-1} \int_0^1 \Phi₂(s)ds \geq \beta \rho^{\alpha-1} \int_0^1 \left( \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \right) ds \geq \beta \rho^{\alpha-1} \int_0^1 \left( \frac{1-\rho}{\alpha-\beta} \right) \Gamma(\alpha) \geq R₀ = \| u \|.
\]

Then, \( \| Lu \| \geq \| u \| \), for all \( u \in P \cap \partial B₂ \). Moreover, from Lemma 2.5, we get \( L(P) \subset P \). Then fractional boundary value problem (1), (2) has a positive solution satisfying
\[
\min(r₀, R₀) \leq \| u \| \leq \max(r₀, R₀).
\]

IV. THE CASE \( f = f(t, u, v) \)

In this sequel, we denote by \( K \) the positive cone of \( E^1 \) given by
\[
K = \{ u \in E^1 : u(t) \geq 0, \ t \in [0, 1] \}. \tag{17}
\]
Lemma 4.1 The Green function \(G(t, s)\) defined by (6) satisfies

\[
\frac{\partial G(t, s)}{\partial t} \leq (\alpha - 1)t^{\alpha - 2} \left[ \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} + h(s) \right].
\]

Proof: It follows from (6) that

\[
\frac{\partial G(t, s)}{\partial t} = \frac{\partial G_1(t, s)}{\partial t} + (\alpha - 1)h(s)t^{\alpha - 2}.
\]

(7) implies that

\[
\frac{\partial G_1(t, s)}{\partial t} = \frac{1}{\Gamma(\alpha)} \begin{cases}
(\alpha - 1)t^{\alpha - 2}(1 - s)^{\alpha - 1} - \alpha(1 - s)^{\alpha - 2} & 0 \leq t \leq s \leq 1, \\
(\alpha - 1)t^{\alpha - 2}(1 - s)^{-1} & 0 \leq s \leq t \\
-\alpha(1 - t)^{\alpha - 2} & 0 \leq t \leq 1.
\end{cases}
\]

So, we have

\[
\frac{\partial G(t, s)}{\partial t} \leq (\alpha - 1)t^{\alpha - 2} \left[ \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} + h(s) \right].
\]

Lemma 4.2 The operator \(T : K \rightarrow K\) is completely continuous.

Proof: First, using Lemma 2.5, we get \(T(K) \subseteq K\), and each fixed point of \(T\) is a solution of problem (1),(2). We claim that \(T : K \rightarrow K\) is completely continuous. The continuity of \(T\) is obvious since \(f\) is continuous. Now, we prove \(T\) is compact.

Let \(\Omega \subseteq K\) be an bounded set. Then, there exists \(R > 0\), such that \(\Omega \subseteq \{u \in K\|u\| \leq R\}, f \in C([0, 1] \times R^+ \times R, R^+)\) implies there exists \(\Psi_R(t) \in C(0, 1)\) such that

\[
f(t, u, v) \leq \Psi_R(t), \quad \forall t \in [0, 1], u \in [0, R], v \in [-R, R].
\]

For any \(u \in \Omega\), we obtain

\[
0 \leq (Tu)(t) = \int_0^1 G(t, s)f(s, u(s), u'(s))ds
\]

\[
\leq \int_0^1 t^{\alpha - 2} \left[ \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} + h(s) \right] \Psi_R(s) ds
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \frac{(\alpha - \gamma)}{\Delta} \int_0^\eta a(t) dt
\]

\[
\frac{1}{\Gamma(\alpha)} \Psi_R(s) ds
\]

\[
= : M.
\]

From the definition of \(T\), we get

\[
\|(Tu)\|_0 \leq M.
\]

On the other hand, for all \(u \in \Omega\), using Lemma 4.1, we find

\[
(Tu)'(t) = \int_0^1 \frac{\partial G(t, s)}{\partial t} f(s, u(s), u'(s))ds
\]

\[
\leq \int_0^1 \frac{\partial G_1(t, s)}{\partial t} \Psi_R(s) ds
\]

\[
\leq \int_0^1 (\alpha - 1)t^{\alpha - 2} \left[ \frac{(1 - s)^{\alpha - 1}}{\Gamma(\alpha)} + h(s) \right] \Psi_R(s) ds
\]

\[
\leq (\alpha - 1) \frac{1}{\Gamma(\alpha)} \frac{(\alpha - \gamma)}{\Delta} \int_0^\eta a(t) dt
\]

\[
\frac{1}{\Gamma(\alpha)} \Psi_R(s) ds
\]

\[
=: (\alpha - 1)M.
\]

Thus, we get

\[
\|(Tu)'\|_0 \leq (\alpha - 1)\mathcal{M}.
\]

(20)

In view of the above two equations (19),(20), we get \(T\Omega\) is uniformly bounded.

It is clear that \(G(t, s)\) is uniformly continuous on \([0, 1] \times [0, 1]\). This means for any \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any \(t_1, t_2 \in [0, 1], |t_1 - t_2| < \delta, s \in [0, 1]\), one has

\[
|G(t_2, s) - G(t_1, s)| < \frac{\varepsilon}{\Psi_R(s) + 1}.
\]

consequently,

\[
|(Tu)(t_2) - (Tu)(t_1)| \leq \int_0^1 |G(t_2, s) - G(t_1, s)|f(s, u(s), u'(s))ds
\]

\[
< \int_0^1 \frac{\varepsilon}{\Psi_R(s) + 1} \Psi_R(s) ds < \varepsilon.
\]

Similarly, since \(\frac{\partial G(t, s)}{\partial t}\) is uniformly continuous on \([0, 1] \times [0, 1]\), we can prove

\[
|(Tu)'(t_2) - (Tu)'(t_1)| < \varepsilon.
\]

This means that \(T\Omega\) is equicontinuous. By the Arzela-Ascoli theorem, we know that \(T : K \rightarrow K\) is completely continuous.

Theorem 4.3 Assume that there exists \(a > 0\), such that

\[
(C_1) f(t, x_1, y_1) \leq f(t, x_2, y_2),
\]

for any \(0 \leq t \leq 1, 0 \leq x_1 \leq x_2 \leq (\alpha - 1)a, 0 \leq |y_1| \leq |y_2| \leq (\alpha - 1)a;

\[
(C_2) \max_{0 \leq t \leq 1} f(t, (\alpha - 1)a, (\alpha - 1)a) \leq \frac{1}{\Gamma(\alpha)} \frac{1}{\Delta} \int_0^\eta a(t) dt,
\]

\[
(C_3) f(t, 0, 0) \neq 0 \text{ for } 0 \leq t \leq 1.
\]

Then the fractional boundary value problem (1),(2) has one positive solution \(\omega^* \in K\) such that \(0 < \omega^* \leq (\alpha - 1)a\), \(0 < (\omega^*)' \leq (\alpha - 1)a\) and \(\lim_{n \to \infty} T^n \omega_0 = \omega^*, \lim_{n \to \infty} T^n (\omega^*)' = (\omega^*)'\) where

\[
\omega_0(t) = at^{\alpha - 1}, \quad 0 \leq t \leq 1.
\]

Proof: We write

\[
K_{(\alpha - 1)a} = \{u \in K\|u\| \leq (\alpha - 1)a\}
\]

and

\[
K_{(\alpha - 1)a} = \{u \in K\|u\| \leq (\alpha - 1)a\}.
\]

We first claim \(T : K_{(\alpha - 1)a} \rightarrow K_{(\alpha - 1)a}\). Let \(u \in K_{(\alpha - 1)a}\), then

\[
0 \leq u(t) \leq \max_{0 \leq t \leq 1} |u(t)| \leq \|u\| \leq (\alpha - 1)a,
\]

\[
|u'(t)| \leq \max_{0 \leq t \leq 1} |u'(t)| \leq \|u\| \leq a < (\alpha - 1)a.
\]

So, from assumptions \((C_1)\) and \((C_2)\), we get

\[
0 \leq f(t, u(t), u'(t)) \leq f(t, (\alpha - 1)a, (\alpha - 1)a)
\]

\[
\leq \max_{0 \leq t \leq 1} f(t, (\alpha - 1)a, (\alpha - 1)a)
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \frac{1}{\Delta} \int_0^\eta a(t) dt,
\]

\[
0 \leq t \leq 1.
\]

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Therefore, for \( u \in K_{(\alpha-1)a} \), according to Lemma 4.1, we get the following estimates

\[
|T(u')(t)| = \left| \int_0^1 G(t,s)f(s,u(s),u'(s))ds \right|
\leq \int_0^1 G(t,s)\left| f(s,u(s),u'(s)) \right|ds
\leq \int_0^1 G(t,s)\left| f(s,(\alpha-1)a,(\alpha-1)a) \right|ds
\leq \int_0^1 \alpha^{-1} \left( 1-s \right)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \Gamma(\alpha) \int_0^a a(t)dt \right)
\int_0^1 f(s,(\alpha-1)a,(\alpha-1)a)ds
\leq (\alpha-1)a.
\]

Thus, we have

\[
\|Tu\| \leq (\alpha-1)a.
\]

This means \( T : K_{(\alpha-1)a} \to K_{(\alpha-1)a} \). Denote \( \omega_0(t) = at^{\alpha-1}, \quad 0 \leq t \leq 1, \)

Let \( \omega_1 = T\omega_0 \), then \( \omega_1 \in K_{(\alpha-1)a} \), we write

\[
\omega_{n+1} = T\omega_n = T^{n+1}\omega_0, \quad (n = 0, 1, 2, \cdots). \tag{24}
\]

Since \( T : K_{(\alpha-1)a} \to K_{(\alpha-1)a} \), we have \( \omega_n \in T K_{(\alpha-1)a} \subseteq K_{(\alpha-1)a}, \quad n = 0, 1, 2, \cdots \). \( T \) is completely continuous implies \( \{\omega_n\}_{n=0}^{\infty} \) is a sequentially compact set,

\[
\omega_1(t) = T\omega_0(t) = \int_0^1 G(t,s)f(s,\omega_0(s),\omega_0'(s))ds
\leq \int_0^1 G(t,s)\left| f(s,(\alpha-1)a,(\alpha-1)a) \right|ds
\leq \int_0^1 t^{\alpha-1} \left( 1-s \right)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \Gamma(\alpha) \int_0^a a(t)dt \right)
\int_0^1 f(s,(\alpha-1)a,(\alpha-1)a)ds
= \alpha^{-1} \omega_0(t).
\]

\[
|\omega_1'(t)| = |(T\omega_0)'(t)|
\leq \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,\omega_0(s),\omega_0'(s))ds
\leq \int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,(\alpha-1)a,(\alpha-1)a)ds
\leq \int_0^1 \alpha^{-1} \left( 1-s \right)^{\alpha-1} \left( \frac{1}{\Gamma(\alpha)} + \frac{1}{\Delta} \Gamma(\alpha) \int_0^a a(t)dt \right)
\int_0^1 f(s,(\alpha-1)a,(\alpha-1)a)ds
= a(\alpha-1)t^{\alpha-2} = \omega_0'(t).
\]

then we have

\[
\omega_1(t) \leq \omega_0(t), \quad |\omega_1'(t)| \leq |\omega_0'(t)|, \quad 0 \leq t \leq 1.
\]

Thus,

\[
\omega_2(t) = T\omega_1(t) \leq T\omega_0(t) = \omega_1(t), \quad 0 \leq t \leq 1,
\]

\[
|\omega_2'(t)| = |(T\omega_1)'(t)| \leq |(T\omega_0)'(t)| = |\omega_0'(t)|, \quad 0 \leq t \leq 1.
\]

Hence by induction, we have

\[
\omega_{n+1} \leq \omega_n, \quad |\omega_{n+1}'(t)| \leq |\omega_n'(t)|, \quad 0 \leq t \leq 1, \quad n = 1, 2, \cdots.
\]

Thus, there exists \( \omega^* \in K_{(\alpha-1)a} \) such that \( \omega_n \to \omega^* \). Letting \( n \to \infty \) in (24), we obtain \( T\omega^* = \omega^* \) since \( T \) is continuous. If \( f(t,0,0) \neq 0, \quad 0 \leq t \leq 1 \), then the zero function is not the solution of (1),(2). Therefore, \( \omega^* \) is a positive solution of (1),(2).

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