The Distributional Solution of the Fractional-order Descriptor Linear Time-Invariant System and Its Application in Fractional Circuits

Zaiyong Feng, MingZhong Chen, Linghua Ye, Lingling Wu

***Abstract*—The distributional solution of the Fractional-Order Descriptor Linear Time-Invariant System (FODLTIS) and its application in the fractional circuits are studied. Based on proposing the definition of ${}_0^C D_t^{\alpha}\delta(t)$, the Laplace transform of ${}_0^C D_t^{\alpha}\delta(t)$ is investigated. Next, with regard to FODLTIS, the system is decomposed into fast subsystem and slow subsystem by restricted equivalent transformation. Solution for the slow subsystem is known. Using Laplace transform of ${}_0^C D_t^{\alpha}\delta(t)$, distributional solution for the fast subsystem is derived. Combining solutions of the slow subsystem and the fast subsystem, the distributional solution of FODLTIS is successfully obtained. The structure of the distributional solution of the system shows that the superposition principle for the system still holds for FODLTIS. Finally, the distributional solution of FODLTIS is applied in the fractional RC circuit, fractional RL circuit, and the fractional LC circuit with ideal operational amplifier. Numerical solution and figures are made to verify the correctness and stability of the solution.**

***Index Terms* —FODLTIS, Dirac delta function, Caputo fractional derivatives, Distributional solution, The Fractional Circuits.**

I. Introduction

THE fractional-order system is getting more and more attention from academia and engineering fields. The theoretical research on fractional-order system mainly focuses on the solutions and the qualitative properties such as stability of the system [1-6]. The fractional-order system has been successfully applied in many fields [7-11]. Meanwhile, the descriptor system includes both the differential equations and the algebraic equations [12-13]. Many researchers have carried out in-depth studies on the solution, controllability, observability and admissibility of the descriptor linear system in [12-15].

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The Fractional-Order Descriptor System (FODS) inherits the characteristics of both the fractional-order system and the descriptor system. Studies on FODS started from 2010, focusing on: (1) Properties of the system. In 2010, N'Doye I, Zasadzinski M proposed FODS model without input, studied the asymptotic stability of the system by using linear matrix inequalities [16]. Subsequently, studies on the stability, stabilization, admissibility of the system appeared one after another [17-20]. (2) Studies on the system solutions. Kaczorek T, Feng Z Y, Chen N, etc. in [21-23] studied the equivalent standard form of the fractional-order descriptor linear time-invariant system, and discussed different solutions of the system; (3) Applications of the system, such as fractional-order circuits in [24-25].

Study on the distributional solution of FODS is not yet started. The ordinary descriptor linear system could be decomposed into the slow subsystem and the fast subsystem. The slow subsystem has a classical solution while the fast subsystem has the distributional solution consisting of the Dirac function $\delta(t)$ and its integer-order derivatives [26]. Similarly, the fast subsystem of the FODS should also have the distributional solution consisting of $\delta(t)$ and its fractional derivatives. However, major existing studies [21-23] didn't involve the distributional solution of the system, although it's significant in system analysis and control.

In this paper, we studied the distributional solution of the basic FODS——the Fractional-Order Descriptor Linear Time-Invariant System (FODLTIS). We discussed the Caputo fractional derivative of $\delta(t)$ and its Laplace transform, based on which, the distributional solution of FODLTIS was obtained. In the part of application, the distributional solution of FODLTIS was applied in the fractional RC circuit, fractional RL circuit, and the fractional LC circuit with ideal operational amplifier. Numerical solution and figures confirmed the correctness and stability of the proposed solution.

II. Description of the Problem

Consider the following FODLTIS（1） [23-24,27]:

$$\begin{cases} E\,{}_0^C D_t^{\alpha}\boldsymbol{x}(t) = A\boldsymbol{x}(t)+B\boldsymbol{u}(t),\ \ \boldsymbol{x}(0)=\boldsymbol{x}_0; & (1.1) \\ \boldsymbol{y}(t) = C\boldsymbol{x}(t)+D\boldsymbol{u}(t); & (1.2) \end{cases} \tag{1}$$

Where $\boldsymbol{x}(t), \boldsymbol{y}(t), \boldsymbol{u}(t)$ are the state variable, output variable and control input variable respectively, $\boldsymbol{x}(t)\in R^n$, $\boldsymbol{y}(t)\in R^m$, $\boldsymbol{u}(t)\in R^r$. The matrices $E, A\in R^{n\times n}$, $B\in R^{n\times r}$, $C\in R^{m\times n}$, $D\in R^{m\times r}$. The fractional derivative ${}_0^C D_t^{\alpha}\boldsymbol{x}(t)$ adopts the

Caputo fractional derivative, and $0<\alpha<1$. Considering the existence and uniqueness of the solution of the system (1) [22], it is assumed that the matrix pair (E,A) is regular. We mainly investigate the distributional solution of (1).

III. Definitions and Preliminaries

The Dirac function $\delta(t)$ and its derivative $\delta^{(i)}(t),(i\in N)$ are also known as distribution. Instead of classical function in traditional sense, $\delta(t)$ is defined as follows [28] :

Definition 1 $\delta(t)=\begin{cases}0, & t\neq 0\\ \infty & t=0\end{cases}, \int_{-\infty}^{+\infty}\delta(t)dt=1.$

Important properties of $\delta(t)$ are listed below [28-29]:

Property 1 Suppose $f(t)$ is continuous within the interval $[a, b]$ containing 0, then $\int_a^b f(t)\delta(t)dt=f(0)$.

Property 2 $\delta^{(i)}(t)$ is the ith-order derivative $(i\in N)$ of $\delta(t)$, similarly, $\delta^{(i)}(t)=0, t\neq 0$. If $f(t)$ is continuous within the interval $[a, b]$ containing 0, and ith-order differentiable at $t=0$, then $\int_a^b f(t)\delta^{(i)}(t)dt=(-1)^i f^{(i)}(0)$.

Property 3 $\mathscr{L}\left(\delta^{(i)}(t)\right)=s^i, i\in N$, particularly, $\mathscr{L}(\delta(t))=1$.

The full definition of the function $f(t)$'s Caputo fractional derivative ${}_a^C D_t^\alpha f(t)$ is as follows:

Definition 2 [1] If $f(t)$ has continuous derivatives up to the m-order, then

$$
{}_a^C D_t^\alpha f(t)=\begin{cases}
{}_a I_t^{-\alpha} f(t)=\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_a^t (t-\tau)^{-\alpha-1} f(\tau)d\tau, & (\alpha<0)\\
f^{(m)}(t), & (\alpha=m, m\in N)\\
{}_a^C D_t^{\alpha-m} f^{(m)}(t)=\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_a^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, & \\
 & (0\le m-1<\alpha<m,\ m\in N^+).
\end{cases}
$$

When $0<\alpha<1$, ${}_a^C D_t^\alpha f(t)=\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_a^t \frac{f'(\tau)}{(t-\tau)^\alpha}d\tau$.

The Laplace transform of ${}_a^C D_t^\alpha f(t)$ only involves the initial value of the integer-order derivative of $f(t)$, this brings great convenience to its application [1].

Property 4 $\mathscr{L}\left\{{}_0^C D_t^p f(t)\right\}=s^p F(s)-\sum_{k=0}^{m-1} s^{p-k-1} f^{(k)}(0)$,

$0\le m-1<p<m$, where $F(s)=\mathscr{L}(f(t))$.

Property 5 gives the Laplace transform of $t^\alpha\ (\alpha>-1)$ [29].

Property 5 When $\alpha>-1$, $\mathscr{L}(t^\alpha)=\dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$.

Denoting $-\alpha-1$ as β, equivalently we have **Property 5'**:

Property 5' When $\beta<0$, $\mathscr{L}^{-1}(s^\beta)=\mathscr{L}^{-1}\left(\dfrac{1}{s^{-\beta}}\right)=\dfrac{t^{-\beta-1}}{\Gamma(-\beta)}$.

IV. ${}_0^C D_t^\alpha \delta(t)$ and Its Laplace Transformation

There has not been research on the Caputo fractional derivative of $\delta(t)$ till now. As in definition 2, the definition of ${}_0^C D_t^\alpha \delta(t)$ is as below:

Definition 3

$$
{}_0^C D_t^\alpha \delta(t)=\begin{cases}
{}_0 I_t^{-\alpha}\delta(t)=\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_0^t (t-\tau)^{-\alpha-1}\delta(\tau)d\tau, & (\alpha<0)\\
\delta^{(m)}(t), & (\alpha=m, m\in N)\\
{}_0^C D_t^{\alpha-m}\delta^{(m)}(t)=\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}\delta^{(m)}(\tau)d\tau, & \\
 & (0\le m-1<\alpha<m, m\in N^+).
\end{cases}
$$

Theorem 1 shows us that ${}_0^C D_t^\alpha \delta(t)$ could be expressed in analytical form.

Theorem 1 ${}_0^C D_t^{(\alpha)}\delta(t)=\dfrac{1}{\Gamma(-\alpha)}\dfrac{1}{t^{\alpha+1}}, (\alpha\notin N)$.

Proof: We can prove it by classifying the different values of α in definition 3.

(1) When $\alpha<0$, from Definition 3 and Property 1,we have:

$$
{}_0^C D_t^{(\alpha)}\delta(t)=\frac{1}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\delta(\tau)d\tau=\left.\frac{1}{\Gamma(-\alpha)}(t-\tau)^{-\alpha-1}\right|_{\tau=0}
$$

$$
=\frac{1}{\Gamma(-\alpha)}\frac{1}{t^{\alpha+1}}
$$

(2) When $0\le m-1<\alpha<m, m\in N^+$, from Definition 3 and Property 1, we have:

$$
{}_0^C D_t^{(\alpha)}\delta(t)={}_0^C D_t^{(\alpha-m)} D^{(m)}\delta(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\delta^{(m)}(\tau)d\tau
$$

$$
=\left.\frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m(t-\tau)^{m-\alpha-1}}{d\tau^m}\right|_{\tau=0}=\left.\frac{(-1)^{m-1}(m-\alpha-1)}{\Gamma(m-\alpha)}\frac{d^{m-1}(t-\tau)^{m-\alpha-2}}{d\tau^{m-1}}\right|_{\tau=0}
$$

$$
=\left.\frac{(-1)^{m-1}}{\Gamma(m-\alpha-1)}\frac{d^{m-1}(t-\tau)^{-(\alpha+1)+m-1}}{d\tau^{m-1}}\right|_{\tau=0}=\left.\frac{(-1)^{m-m}}{\Gamma(m-\alpha-m)}(t-\tau)^{-(\alpha+1)+m-m}\right|_{\tau=0}
$$

$$
=\frac{1}{\Gamma(-\alpha)}\frac{1}{t^{\alpha+1}}
$$

Using above definitions and properties, we can obtain the Laplace transform of ${}_0^C D_t^\alpha \delta(t)$.

Theorem 2 $\mathscr{L}\left[{}_0^C D_t^\alpha\delta(t)\right]=s^\alpha\ \left(\mathscr{L}^{-1}(s^\alpha)={}_0^C D_t^\alpha \delta(t)\right), \alpha\in R.$

Proof: (1) When $\alpha<0$,

$$
\mathscr{L}\left[{}_0^C D_t^\alpha\delta(t)\right]=\mathscr{L}\left[\frac{1}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}\delta(\tau)d\tau\right]
$$

$$
=\frac{1}{\Gamma(-\alpha)}\mathscr{L}\left[t^{-\alpha-1}*\delta(t)\right]=\frac{1}{\Gamma(-\alpha)}\mathscr{L}\left(t^{-\alpha-1}\right)\cdot\mathscr{L}\left[\delta(t)\right]
$$

Because $-\alpha-1>-1$, According to Property 5 and Property 3, we obtain: $\mathscr{L}\left[{}_0^C D_t^\alpha\delta(t)\right]=\dfrac{1}{\Gamma(-\alpha)}\cdot\dfrac{\Gamma(-\alpha)}{s^{-\alpha}}\times 1=s^\alpha$.

(2) When $\alpha=m, m\in N$, $\mathscr{L}\left(\delta^{(m)}(t)\right)=s^m$ from Property 3.

(3) When $0\le m-1<\alpha<m, m\in N^+$,

$$
\mathscr{L}\left[{}_0^C D_t^\alpha\delta(t)\right]=\mathscr{L}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\delta^{(m)}(\tau)d\tau\right]
$$

$$
=\frac{1}{\Gamma(m-\alpha)}\mathscr{L}\left(t^{m-\alpha-1}\right)\cdot\mathscr{L}\left[\delta^{(m)}(t)\right]
$$

Because $m-1<\alpha<m$, $-1<m-\alpha-1<0$, according to Properties 3 and 5, we can obtain

$$\mathscr{L}\left[{}_0^C D_t^{\alpha}\delta(t)\right]=\frac{1}{\Gamma(m-\alpha)}\mathscr{L}\left(t^{m-\alpha-1}\right)\cdot\mathscr{L}\left[\delta^{(m)}(t)\right]$$
$$=\frac{1}{\Gamma(m-\alpha)}\cdot\frac{\Gamma(m-\alpha)}{s^{m-\alpha}}\cdot s^m=s^{\alpha}.$$

Proof completed.

Remark 1 Property 3 gives $\mathscr{L}\left(\delta^{(i)}(t)\right)=s^i\ (i\in N)$, Theorem 2 proves $\mathscr{L}\left[{}_0^C D_t^{\alpha}\delta(t)\right]=s^{\alpha}\ (\alpha\in R)$. Obviously, Theorem 2 is the generalization of Property 3

Remark 2 In fact, when $\alpha<0$, $\mathscr{L}^{-1}\left(s^{\alpha}\right)={}_0^C D_t^{\alpha}\delta(t)={}_0I_t^{-\alpha}\delta(t)$ $=\frac{1}{\Gamma(-\alpha)}\int_0^t(t-\tau)^{-\alpha-1}\delta(\tau)d\tau=\frac{t^{-\alpha-1}}{\Gamma(-\alpha)}$. Theorem 2 degenerates to Property 5'. When $\alpha=i\in N$, ${}_0^C D_t^{\alpha}\delta(t)=\delta^{(i)}(t)$, Theorem 2 comes to Property 3. Therefore, Theorem 2 gives novel conclusion that $\mathscr{L}^{-1}\left(s^{\alpha}\right)={}_0^C D_t^{\alpha}\delta(t)\ (\alpha>0,\alpha\notin N)$.

V. Distributional Solution of the Fractional Descriptor Linear Time-Invariant System

The key to solving FODLTIS (1) is to obtain the solution of the subsystem (1.1). Since the coefficient matrix pair (E,A) is regular, There exist invertible matrices P_1, P_2, such that the matrix pair (E,A) has the following canonical form after restricted equivalent transformation [22]:

$$(E,A)\sim P_1(E,A)P_2=(P_1EP_2,P_1AP_2)=\left(\begin{bmatrix}I&0\\0&N\end{bmatrix},\begin{bmatrix}A_1&0\\0&I\end{bmatrix}\right).$$

Here, N is a nilpotent matrix, if its index is h, then $N^i\neq0$, $N^h=0,\ i=1,2,\cdots h-1$.

Left multiplying P_1 on the system (1.1), letting $\boldsymbol{x}=P_2[\boldsymbol{x}_1,\boldsymbol{x}_2]^T$, then system (1.1) may be equivalently transformed into the following canonical form (2) $(0<\alpha<1)$

$$\begin{cases}{}_0^C D_t^{(\alpha)}\boldsymbol{x}_1(t)=A_1\boldsymbol{x}_1(t)+B_1\boldsymbol{u}(t),\ \boldsymbol{x}_1(0)=\boldsymbol{x}_{10};\quad(2.1)\\ N\,{}_0^C D_t^{(\alpha)}\boldsymbol{x}_2(t)=\boldsymbol{x}_2(t)+B_2\boldsymbol{u}(t);\ \boldsymbol{x}_2(0)=\boldsymbol{x}_{20};\quad(2.2)\end{cases}\qquad(2)$$

According to the ordinary descriptor linear system, we refer to the subsystem (2.1) of (2) as the slow subsystem, the subsystem (2.2) as the fast subsystem. Next, we will discuss the solutions of the slow subsystem and fast subsystem of the system (2) respectively.

The slow subsystem (2.1) is a fractional linear system. Kaczorek T proposed Lemma 1 as below in 2008 [21]:

Lemma 1 The solution of slow subsystem (2.1) has the following form:

$$\boldsymbol{x}_1(t)=x_{1i}(t,x_{10})+x_{1u}(t,u)=\Phi_0(t)\boldsymbol{x}_{10}+\int_0^t\Phi(t-\tau)B_1\boldsymbol{u}(\tau)d\tau\quad(3)$$

where $\Phi_0(t)=\sum_{k=0}^{\infty}\frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)}=E_{\alpha}\left(A_1t^{\alpha}\right),\ \Phi(t)=\sum_{k=0}^{\infty}\frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}$.

$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}$ is the Mittage-Leffler function of single parameter, and the proof of Lemma 1 is given in [21].

When $\alpha=1$, $\Phi_0(t)=\Phi(t)=e^{A_1t}$, solution (3) degenerates to the classical solution of ordinary linear systems.

As for the solution of the fast subsystem (2.2), we give Theorem 3 as below:

Theorem 3 Distributional solution of the fast subsystem (2.2) of FODLTIS (2) is as below:

$$\boldsymbol{x}_2(t,u,x_{20})=\boldsymbol{x}_{2i}(t,x_{20})+\boldsymbol{x}_{2u}(t,u)\qquad(4)$$

$\boldsymbol{x}_{2i}(t,x_{20})$ is the zero-input response, and

$$\boldsymbol{x}_{2i}(t,x_{20})=-\sum_{k=1}^{h-1}N^k\,{}_0^C D_t^{k\alpha-1}\delta(t)\boldsymbol{x}_{20}\qquad(4.1)$$

$\boldsymbol{x}_{2u}(t,u)$ is the zero-state response, and

$$\boldsymbol{x}_{2u}(t,u)=-\sum_{k=0}^{h-1}N^kB_2\left[{}_0^C D_t^{k\alpha}\boldsymbol{u}(t)+\sum_{i=0}^{l_k-1}{}_0^C D_t^{k\alpha-1-i}\delta(t)u^{(i)}(0)\right]\qquad(4.2)$$

h is the index of the nilpotent matrix N, $l_k=\lceil k\alpha\rceil$, $k=0,1,2,\cdots,h-1$.

Proof: Applying the Laplace transform to the fast subsystem (2.2), we obtain:

$$N\left(\boldsymbol{s}^{\alpha}\boldsymbol{X}_2(s)-\boldsymbol{s}^{\alpha-1}\boldsymbol{x}_2(0)\right)=I\boldsymbol{X}_2(s)+B_2\boldsymbol{U}(s)$$

After transposition and sorting out:

$$\left(N\boldsymbol{s}^{\alpha}-I\right)\boldsymbol{X}_2(s)=N\boldsymbol{s}^{\alpha-1}\boldsymbol{x}_2(0)+B_2\boldsymbol{U}(s)$$

Thus: $\boldsymbol{X}_2(s)=\left(N\boldsymbol{s}^{\alpha}-I\right)^{-1}\left[N\boldsymbol{s}^{\alpha-1}\boldsymbol{x}_2(0)+B_2\boldsymbol{U}(s)\right]$ (5)

Expanding $\left(N\boldsymbol{s}^{\alpha}-I\right)^{-1}$ by the Neumann series, namely, $\left(N\boldsymbol{s}^{\alpha}-I\right)^{-1}=-\sum_{k=0}^{\infty}N^k\boldsymbol{s}^{\alpha k}$. Since h is the index of the nilpotent matrix N, the above expansion may be simplified to the sum of the finite terms $\left(N\boldsymbol{s}^{\alpha}-I\right)^{-1}=-\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{\alpha k}$. After substituting it into (5), we have:

$$\boldsymbol{X}_2(s)=-\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{\alpha k}\left[N\boldsymbol{s}^{\alpha-1}\boldsymbol{x}_{20}+B_2\boldsymbol{U}(s)\right]$$
$$=-\sum_{k=0}^{h-1}N^{k+1}\boldsymbol{s}^{(k+1)\alpha-1}\boldsymbol{x}_{20}-\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{k\alpha}B_2\boldsymbol{U}(s)$$
$$=-\sum_{k=1}^{h-1}N^k\boldsymbol{s}^{k\alpha-1}\boldsymbol{x}_{20}-\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{k\alpha}B_2\boldsymbol{U}(s)$$

Applying the inverse Laplace transformation to the above equation with Theorem 2 gives:

$$\boldsymbol{x}_2(t)=\mathscr{L}^{-1}\left\{-\sum_{k=1}^{h-1}N^k\boldsymbol{s}^{k\alpha-1}\boldsymbol{x}_{20}-\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{k\alpha}B_2\boldsymbol{U}(s)\right\}$$
$$=-\sum_{k=1}^{h-1}N^k\mathscr{L}^{-1}\left[\boldsymbol{s}^{k\alpha-1}\right]\boldsymbol{x}_{20}-\mathscr{L}^{-1}\left[\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{k\alpha}B_2\boldsymbol{U}(s)\right]$$
$$=-\sum_{k=1}^{h-1}N^k\,{}_0^C D_t^{k\alpha-1}\delta(t)\boldsymbol{x}_{20}-\mathscr{L}^{-1}\left[\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{k\alpha}B_2\boldsymbol{U}(s)\right]\qquad(6)$$

We calculate $\mathscr{L}^{-1}\left[\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{k\alpha}B_2\boldsymbol{U}(s)\right]$, since

$$\mathscr{L}\left[{}_0^C D_t^{k\alpha}\boldsymbol{u}(t)\right]=\boldsymbol{s}^{k\alpha}\boldsymbol{U}(s)-\sum_{i=0}^{l_k-1}\boldsymbol{s}^{k\alpha-1-i}u^{(i)}(0),\left(l_k-1<k\alpha\le l_k,l_k\in N^+\right)$$

thus $\mathscr{L}^{-1}\left[\boldsymbol{s}^{k\alpha}\boldsymbol{U}(s)\right]=\mathscr{L}^{-1}\left\{\mathscr{L}\left[{}_0^C D_t^{k\alpha}\boldsymbol{u}(t)\right]+\sum_{i=0}^{l_k-1}\boldsymbol{s}^{k\alpha-1-i}u^{(i)}(0)\right\}$

$$={}_0^C D_t^{k\alpha}\boldsymbol{u}(t)+\sum_{i=0}^{l_k-1}{}_0^C D_t^{k\alpha-1-i}\delta(t)u^{(i)}(0)\qquad(7)$$

Hence, $\mathscr{L}^{-1}\left[\sum_{k=0}^{h-1}N^k\boldsymbol{s}^{k\alpha}B_2\boldsymbol{U}(s)\right]$

$$=\sum_{k=0}^{h-1} N^k B_2 \left[{}_0^C D_t^{k\alpha} \boldsymbol{u}(t) + \sum_{i=0}^{l_k-1} {}_0^C D_t^{k\alpha\text{-}1\text{-}i} \delta(t) u^{(i)}(0) \right].$$

Substituting the above equation into (6), we can obtain the solution of the fast subsystem (2.2):

$$\boldsymbol{x}_2(t) = -\sum_{k=1}^{h-1} N^k\, {}_0^C D_t^{k\alpha-1} \delta(t) \boldsymbol{x}_{20}$$
$$-\sum_{k=0}^{h-1} N^k B_2 \left[{}_0^C D_t^{k\alpha} \boldsymbol{u}(t) + \sum_{i=0}^{l_k-1} {}_0^C D_t^{k\alpha\text{-}1\text{-}i} \delta(t) u^{(i)}(0) \right]$$

Obviously, $\boldsymbol{x}_2(t,u,x_{20}) = \boldsymbol{x}_{2i}(t,x_{20}) + \boldsymbol{x}_{2u}(t,u)$, where

$$\boldsymbol{x}_{2i}(t,x_{20}) = -\sum_{k=1}^{h-1} N^k\, {}_0^C D_t^{k\alpha-1} \delta(t) \boldsymbol{x}_{20}$$

$$\boldsymbol{x}_{2u}(t,u) = -\sum_{k=0}^{h-1} N^k B_2 \left[{}_0^C D_t^{k\alpha} \boldsymbol{u}(t) + \sum_{i=0}^{l_k-1} {}_0^C D_t^{k\alpha\text{-}1\text{-}i} \delta(t) u^{(i)}(0) \right]$$

Proof completed.

The solution (4) contains linear combinations of the Caputo fractional derivative of distributional function $\delta(t)$ and the control input $\boldsymbol{u}(t)$. Therefore, (4) is distributional solution.

Considering Lemma 1, Theorem 3 and the restricted equivalent relation between system (1.1) and system (2), We have Theorem 4 for system (1.1):

Theorem 4 After the restricted equivalent transformation, the FODLTIS (1.1) is equivalent to the system (2). System (1.1) has the following distributional solution (8):

$$\boldsymbol{x}(t,x_0,u) = P_2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P_2 \begin{bmatrix} x_{1i}(t,x_{10}) + x_{1u}(t,u) \\ x_{2i}(t,x_{20}) + x_{2u}(t,u) \end{bmatrix}. \qquad (8)$$

Where $\boldsymbol{x}_{1i}(t,x_{10}) = \Phi_0(t) x_{10} = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)} x_{10} = E_\alpha(A_1 t^\alpha) x_{10}$ is the response of slow subsystem (2.1) to the initial value $\boldsymbol{x}_{10}$.

$\boldsymbol{x}_{1u}(t,u) = \int_0^t \Phi(t-\tau) B_1 \boldsymbol{u}(\tau) d\tau$, $\Phi(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}$ is the response of the slow subsystem (2.1) to the input $\boldsymbol{u}(t)$.

$\boldsymbol{x}_{2i}(t,x_{20}) = -\sum_{k=1}^{h-1} N^k\, {}_0^C D_t^{k\alpha-1} \delta(t) \boldsymbol{x}_{20}$ is the response of the fast subsystem (2.2) to the initial value $\boldsymbol{x}_{20}$.

$\boldsymbol{x}_{2u}(t,u) = -\sum_{k=0}^{h-1} N^k B_2 \left[{}_0^C D_t^{k\alpha} \boldsymbol{u}(t) + \sum_{i=0}^{l_k-1} {}_0^C D_t^{k\alpha\text{-}1\text{-}i} \delta(t) u^{(i)}(0) \right]$ is the response of the fast subsystem (2.2) to the input $\boldsymbol{u}(t)$.

Remark 3 Theorem 4 shows us that the superposition principle [12] still holds for FODLTIS. The solution (8), i.e., the full response of (1.1), may be decomposed into:

$$\boldsymbol{x}(t,x_0,u) = P_2 \begin{bmatrix} x_{1i}(t,x_{10}) + x_{1u}(t,u) \\ x_{2i}(t,x_{20}) + x_{2u}(t,u) \end{bmatrix} = P_2 \begin{bmatrix} x_{1i}(t,x_{10}) \\ x_{2i}(t,x_{20}) \end{bmatrix} + P_2 \begin{bmatrix} x_{1u}(t,u) \\ x_{2u}(t,u) \end{bmatrix}$$

The first part of the solution only reflects the effect of the system's initial state x_0. It is zero-input response, physically characterizing the system's free motion under the initial state. The second part of the solution only reflects the effect of the system's input $\boldsymbol{u}(t)$. It is zero-state response, physically characterizing the system's forced motion under the external input. Therefore, the full response of the FODLTIS is also the superposition of zero-input response and zero-state response. It's easy to see that the superposition principle of solution for linear system still holds for FODLTIS.

Remark 4 The studies of [21, 23-24, 30] investigated the solution of FODLTIS. However, the distributional solution of the system was not discussed. Besides, some key results of these studies were doubtful. For example, in the literature [24], when the inverse Laplace transformation was performed to the formula (17) to obtain the system solution (18), $\mathscr{L}^{-1}\left[\boldsymbol{s}^{i\alpha} \boldsymbol{U}(s)\right] = \frac{d^{i\alpha}}{dt^{i\alpha}} u(t)$ was taken as a result, which actually missed the second term on the right side of our formula (7), so the inverse transformation was incomplete. In formula (18) of the same literature, the correctness of the conclusion that $\mathscr{L}^{-1}\left[\boldsymbol{s}^{(i+1)\alpha-1} x_{20}\right] = \frac{d^{(i+1)\alpha-1}}{dt^{(i+1)\alpha-1}} x_{20}$ is also questionable.

VI. Application in The Fractional Circuits

In 1990s, Westerlaud pointed out the fractional nature of capacitors and inductors [31-32].

Let the order of the fractional capacitor is α, then the Volt-Ampere relation of the capacitor under the correlation reference direction of current and voltage satisfies [21, 24]:

$$i_C(t) = C_\alpha \frac{d^\alpha u_C(t)}{dt^\alpha} = C_\alpha\, {}_0^C D_t^\alpha u_C(t) \qquad (9)$$

In which, i_C, u_C are the Current and Voltage of the fractional capacitor respectively.

Similarly, let the order of the fractional inductor is β, then the Volt-Ampere relation of the inductor satisfies [21, 24]:

$$u_I(t) = L_\beta \frac{d^\beta i_I(t)}{dt^\beta} = L_\beta\, {}_0^C D_t^\alpha i_I(t) \qquad (10)$$

In which, i_I, u_I are the Current and Voltage of the fractional inductor respectively.

Example 1. Consider the fractional RC circuit shown in Fig 1. Letting $R=2\Omega$, $C_{1,\alpha} = C_{2,\alpha} = C_{3,\alpha} = 5F/s^{1-\alpha}$, $\alpha=0.5$, it could be modeled as follows according to Kirchhoff's law:

$$\begin{bmatrix} 10 & 0 & 0 \\ 5 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{d^{0.5}u_1}{dt^{0.5}} \\ \frac{d^{0.5}u_2}{dt^{0.5}} \\ \frac{d^{0.5}u_3}{dt^{0.5}} \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad (11)$$

Obviously, the model is a FODLTIS. The state vector is $X = [u_1, u_2, u_3]^T$, and the input vector is $U = [e_1, e_2]^T$.

Taking $P_1 = \frac{1}{40}\begin{bmatrix} 4 & 0 & -2 \\ 2 & -4 & -1 \\ 0 & 0 & 40 \end{bmatrix}$, $P_2 = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -2 & -1 \end{bmatrix}$, system (11) could be transformed into canonical form (12)：

$$\begin{cases} I_{2\times2}\begin{bmatrix}\dfrac{d^{0.5}\widehat{u_1}}{dt^{0.5}}\\ \dfrac{d^{0.5}\widehat{u_2}}{dt^{0.5}}\end{bmatrix} = -\begin{bmatrix}\dfrac{1}{10} & \dfrac{1}{10}\\ \dfrac{1}{20} & \dfrac{1}{20}\end{bmatrix}\begin{bmatrix}\dfrac{d^{0.5}\widehat{u_1}}{dt^{0.5}}\\ \dfrac{d^{0.5}\widehat{u_2}}{dt^{0.5}}\end{bmatrix} + \begin{bmatrix}\dfrac{1}{10} & \dfrac{-1}{20}\\ \dfrac{1}{20} & \dfrac{-1}{40}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} & (12.1)\\ N_{1\times1}\dfrac{d^{0.5}\widehat{u_3}}{dt^{0.5}} = \widehat{u_3} + [0\,1]\begin{bmatrix}e_1\\ e_2\end{bmatrix} & (12.2)\end{cases}$$

in which $N_{1\times1}=0$.

Suppose the initial state value and control input of system (12) are $\widehat{X_0}=[\widehat{u_{10}},\widehat{u_{20}},\widehat{u_{30}}]^T=[2,1,1]^T$, $U_0=[e_1,e_2]^T=[1,2]^T$. According to Lemma 1 and theorem 3, we can obtain that the solution of (12) is

$$\begin{bmatrix}\widehat{u_1}\\ \widehat{u_2}\\ \widehat{u_3}\end{bmatrix} = \begin{bmatrix} 2\sum_{k=0}^{\infty}\left(-\dfrac{3}{20}t^{1/2}\right)^k \dfrac{1}{\Gamma\left(\frac{1}{2}k+1\right)}\\ \sum_{k=0}^{\infty}\left(-\dfrac{3}{20}t^{1/2}\right)^k \dfrac{1}{\Gamma\left(\frac{1}{2}k+1\right)}\\ -2\end{bmatrix} = \begin{bmatrix}2E_{\frac{1}{2}}\left(-\dfrac{3}{20}t^{1/2}\right)\\ E_{\frac{1}{2}}\left(-\dfrac{3}{20}t^{1/2}\right)\\ -2\end{bmatrix} \quad (13)$$

Obviously, $\widehat{u_3}=-2$ satisfies the fast subsystem (12.2). Since the coefficients of the first equation in slow subsystem (12.1) is twice as much as the corresponding coefficients of the second one, and the initial value $\widehat{u_{10}}=2$ is twice as much as the initial value $\widehat{u_{20}}=1$, thus $\widehat{u_1}=2\widehat{u_2}$.

Next, we verify the correctness of the solution $\widehat{u_1}$ and $\widehat{u_2}$ by comparing the analytical solution with the numerical solution. Considering that $\widehat{u_1}=2\widehat{u_2}$, we just verify the correctness of $\widehat{u_2}=E_{\frac{1}{2}}\left(-\dfrac{3}{20}t^{1/2}\right)$.

Substituting $\widehat{u_1}=2\widehat{u_2}$, $U_0=[e_1,e_2]^T=[1,2]^T$ into the second equation in (12.1), we derive:

$$\frac{d^{0.5}\widehat{u_2}}{dt^{0.5}} = -\frac{3}{20}\widehat{u_2},\ \widehat{u_{20}}=1. \quad (14)$$

In order to make (14) suitable to be solved with Grunwald-letnikov fractional derivative [1] which is often used in numerical calculation, we need to zero the initial value of the variable. We make an auxiliary function $y=\widehat{u_2}-1$ such that the initial value $y_0=0$. And (14) is equivalent to the following :

$$\frac{d^{0.5}y}{dt^{0.5}} = -\frac{3}{20}y-\frac{3}{20},\ y_0=0. \quad (15)$$

Taking Grunwald-letnikov fractional derivative

$${}_0^GD_t^{\alpha}y(t)\approx\frac{\sum_{j=0}^{\left[\frac{t}{h}\right]}(-1)^j\binom{\alpha}{j}y(t-jh)}{h^{\alpha}}$$

into (15), we obtain (16):

$$y(t)=\frac{-1}{\dfrac{1}{h^{0.5}}+\dfrac{3}{20}}\left[\frac{3}{20}+\frac{1}{h^{0.5}}\sum_{j=1}^{\left[\frac{t}{h}\right]}w_j y(t-jh)\right] \quad (16)$$

Letting step h=0.001, $w_0=1, w_j=\left(1-\dfrac{\alpha+1}{j}\right), j=2,3,\cdots$, $\alpha=0.5, t\in[0,30](s)$, the numerical results of $y(t)$ can be obtained by Matlab programming. Numerical solution $\widehat{u_2}=y+1$ is listed in Table 1 comparing with the analytical one. Table 1 shows that the numerical solution of $\widehat{u_2}$ is in good agreement with the analytical solution. The difference between them is as small as 10^{-4}, and as time goes on, the difference gets smaller and smaller. Therefore, $\widehat{u_2}=E_{\frac{1}{2}}\left(-\dfrac{3}{20}t^{1/2}\right)$ is the right solution of system (12). Consequently, the correctness of $\widehat{u_1}=2\widehat{u_2}$ is also confirmed. From Theorem 4**,** solution of the fractional RC circuit (11) is

$$X=\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}=P_2\begin{bmatrix}\widehat{u_1}\\ \widehat{u_2}\\ \widehat{u_3}\end{bmatrix}=\begin{bmatrix}2E_{\frac{1}{2}}\left(-\dfrac{3}{20}t^{1/2}\right)\\ E_{\frac{1}{2}}\left(-\dfrac{3}{20}t^{1/2}\right)+1\\ -E_{\frac{1}{2}}\left(-\dfrac{3}{20}t^{1/2}\right)+1\end{bmatrix} \quad (17)$$

Example 2. Consider the fractional RL circuit shown in Fig 2 with R_1=R_2=$5\Omega, L_{1,\alpha}=L_{2,\alpha}=1H/s^{1-\alpha}, L_{3,\alpha}=2H/s^{1-\alpha}$. The circuit could be modeled by FODLTIS as follows:

$$\begin{bmatrix}1&0&2\\0&1&2\\0&0&0\end{bmatrix}\begin{bmatrix}\dfrac{d^{\alpha}i_1}{dt^{\alpha}}\\ \dfrac{d^{\alpha}i_2}{dt^{\alpha}}\\ \dfrac{d^{\alpha}i_3}{dt^{\alpha}}\end{bmatrix}=\begin{bmatrix}-5&0&0\\0&-5&0\\1&1&-1\end{bmatrix}\begin{bmatrix}i_1\\ i_2\\ i_3\end{bmatrix}+\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}e_1\\ e_2\end{bmatrix} \quad (18)$$

Letting input and initial value of the circuit after restricted transformation are $U=\begin{bmatrix}e_1\\ e_2\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$, $\boldsymbol{x_0}=\begin{bmatrix}x_{10}\\ x_{20}\\ x_{30}\end{bmatrix}=\begin{bmatrix}1\\3\\0\end{bmatrix}$.

We can give the solution of (18) from Theorem 4:

$$\boldsymbol{x}(t,x_0,u)=\begin{bmatrix}i_1\\ i_2\\ i_3\end{bmatrix}=\begin{bmatrix}\dfrac{2}{5}-\dfrac{1}{5}E_{\alpha}\left(-t^{\alpha}\right)-\dfrac{4}{5}E_{\alpha}\left(-5t^{\alpha}\right)\\ \dfrac{4}{5}-\dfrac{1}{5}E_{\alpha}\left(-t^{\alpha}\right)+\dfrac{4}{5}E_{\alpha}\left(-5t^{\alpha}\right)\\ \dfrac{6}{5}-\dfrac{2}{5}E_{\alpha}\left(-t^{\alpha}\right)\end{bmatrix} \quad (19)$$

Figures 4-6 show us the full response of i_1, i_2 and i_3 under α=0.2, α=0.5 and α=0.8 respectively. Obviously, the full response of i_1, i_2 and i_3 under each different fractional order α is stable.

Example 3. Consider the fractional order LC circuit with ideal operational amplifier shown in Fig 3. Letting $C_\alpha = 4F/s^{1-\alpha}, L_\alpha = 3H/s^{1-\alpha}$, α=0.4. The circuit is driven by sine function signal $u = e(t) = \begin{cases} \sin 2t & t \ge 0 \\ 0 & t < 0 \end{cases}$. The circuit system could be modeled as follows:

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{d^{0.4}u_1}{dt^{0.4}} \\ \frac{d^{0.4}i_1}{dt^{0.4}} \\ \frac{d^{0.4}u_2}{dt^{0.4}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ i_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} e(t) \quad (20)$$

$N = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$ is a nilpotent matrix which index is 3, hence system (20) is a pure fast subsystem of FODLTIS. The state vector is $\boldsymbol{x} = [x_1, x_2, x_3]^T = [u_1, i_1, u_2]^T$, the input is $u = e(t)$. According to Theorem 3**,** Solution of (20) is:

$$\boldsymbol{x}(t, x_0, u) = x_i(t, x_0) + x_u(t, u) \quad (21)$$

in which:

$$\boldsymbol{x}_i(t, x_0) = -N\,{}_0^C D_t^{(\alpha-1)}\delta(t)\boldsymbol{x}_0 - N^2\,{}_0^C D_t^{(2\alpha-1)}\delta(t)\boldsymbol{x}_0$$

$$= \begin{pmatrix} -3\,{}_0^C D_t^{(-0.6)}\delta(t) i_1(0) - 12\,{}_0^C D_t^{(-0.2)}\delta(t) u_2(0) \\ -4\,{}_0^C D_t^{(-0.6)}\delta(t) u_2(0) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-3i_1(0)}{\Gamma(0.6)t^{0.4}} + \frac{-12u_2(0)}{\Gamma(0.2)t^{0.8}} \\ \frac{-4u_2(0)}{\Gamma(0.6)t^{0.4}} \\ 0 \end{pmatrix} \quad (22)$$

$$\boldsymbol{x}_u(t, u) = -\sum_{k=0}^{h-1} N^k B\left[{}_0^C D_t^{(k\alpha)}\boldsymbol{u}(t) + \sum_{i=0}^{l_k-1} {}_0^C D_t^{(k\alpha\text{-}1\text{-}i)}\delta(t) u^{(i)}(0)\right]$$

$$= \begin{pmatrix} 12\left[{}_0^C D_t^{(0.8)}\boldsymbol{e}(t) + {}_0^C D_t^{(-0.2)}\delta(t) e(0)\right] \\ 4\left[{}_0^C D_t^{(0.4)}\boldsymbol{e}(t) + {}_0^C D_t^{(-0.6)}\delta(t) e(0)\right] \\ e(t) \end{pmatrix}$$

Taking $u = e(t) = \begin{cases} \sin 2t & t \ge 0 \\ 0 & t < 0 \end{cases}$ into above equation, we get: $\boldsymbol{x}_u(t, u) = \begin{pmatrix} 12\,{}_0^C D_t^{(0.8)}(\sin 2t) \\ 4\,{}_0^C D_t^{(0.4)}(\sin 2t) \\ \sin 2t \end{pmatrix}$.

Hence, solution of the fractional circuit system (20) is:

$$\boldsymbol{x}(t, x_0, u) = x_i(t, x_0) + x_u(t, u)$$

$$= \begin{pmatrix} \frac{-3i_1(0)}{\Gamma(0.6)t^{0.4}} + \frac{-12u_2(0)}{\Gamma(0.2)t^{0.8}} + 12\,{}_0^C D_t^{(0.8)}(\sin 2t) \\ \frac{-4u_2(0)}{\Gamma(0.6)t^{0.4}} + 4\,{}_0^C D_t^{(0.4)}(\sin 2t) \\ \sin 2t \end{pmatrix} \quad (23)$$

The first part of $\boldsymbol{x}(t, x_0, u)$, i.e., $x_i(t, x_0)$ reflects the effect of the circuit's initial state x_0, it will soon converge to 0 as t increases. For the second part of $\boldsymbol{x}(t, x_0, u)$, i.e., $x_u(t, u)$, $D_t^{(0.8)}(\sin 2t)$ and ${}_0^C D_t^{(0.4)}(\sin 2t)$ can be represented by Mittag-Leffler function of two parameters $E_{\alpha,\beta}(z)$ [33]:

$${}_0^C D_t^{(0.8)}(\sin 2t) = \frac{t^{0.2}}{j}\left[E_{1,1.2}(2jt) - E_{1,1.2}(-2jt)\right]$$

$${}_0^C D_t^{(0.4)}(\sin 2t) = \frac{t^{0.6}}{j}\left[E_{1,1.6}(2jt) - E_{1,1.6}(-2jt)\right]$$

In which, $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, j^2 = -1$. As a result,

$$\boldsymbol{x}_u(t, u) = \begin{bmatrix} x_{u1} \\ x_{u2} \\ x_{u3} \end{bmatrix} = \begin{pmatrix} \frac{12t^{0.2}}{j}\left[E_{1,1.2}(2jt) - E_{1,1.2}(-2jt)\right] \\ \frac{4t^{0.6}}{j}\left[E_{1,1.6}(2jt) - E_{1,1.6}(-2jt)\right] \\ \sin 2t \end{pmatrix} \quad (24)$$

Fig 7 shows us the trajectory of x_{u1}, x_{u2} and x_{u3} under $\alpha = 0.4$, and the trajectory of x_{u1}, x_{u2} and x_{u3} under different α are shown in Figures 8-10 respectively. We can see that the input signal $u = e(t)$ is of sine function, and each component of the response $\boldsymbol{x}_u(t, u)$ is also a sine function. The variable x_{u3} reflects the control input of the system itself, i.e. $x_{u3}=u(t)=\sin 2t$. The angular frequencies of x_{u1} and x_{u2} are almost invariant under different fractional order α.The phase and the amplitude of x_{u2} vary slightly with fractional order α whereas the phase and the amplitude of x_{u1} vary relatively large with different α. Generally, the bigger the fractional order α is, the larger the amplitude of x_{u1} will become.

VII. CONCLUSION

The paper mainly discussed the distributional solution of the fractional-order descriptor linear time-invariant system and its application in the fractional circuits. After the definition of ${}_0^C D_t^{\alpha}\delta(t)$ is proposed, the Laplace transformation of ${}_0^C D_t^{\alpha}\delta(t)$ is studied, based on which the distributional solution of FODLTIS is successfully derived. Comparing with the solution of the ordinary descriptor linear system, the distributional solution of FODLTIS is more complex. However, the system's full response is still the superposition of zero-input response and zero-state response, thus the superposition principle of linear systems still holds. Finally, Simulation results verified that the application of the main conclusions in fractional circuits is effective and feasible.

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Appendixes of Tables and Figures

TABLE 1

Numerical Solution and Analytical Solution of $\widehat{u_2}$

t (s)	Numerical Solution	Analytical Solution	Difference（10^{-3}）
2	0.995278977286736	0.994670047618361	0.608929668375557
4	0.991173130169418	0.990796504242218	0.376625927200669
6	0.988438292975577	0.988143340255794	0.294952719782637
8	0.986245324384164	0.985994979242011	0.250345142153274
10	0.984364510147293	0.984143235572738	0.221274574554342
12	0.982693209434557	0.982492790473761	0.200418960796256
14	0.981175047740785	0.980990526234903	0.184521505881552
16	0.979775129033565	0.979603243190511	0.171885843053832
18	0.978470078757652	0.978308548319914	0.161530437738366
20	0.977243333786610	0.977090490900568	0.152842886042226
22	0.976082651274841	0.975937232540657	0.145418734184388
24	0.974978678941983	0.974839700228323	0.138978713660309
26	0.973924081501859	0.973790758533952	0.133322967906579
28	0.972912979725363	0.972784675526529	0.128304198833917
30	0.971940575400325	0.971816764263626	0.123811136699237

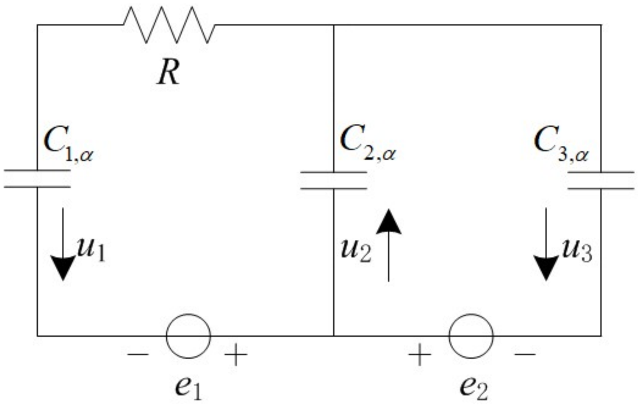

Fig 1. The Fractional RC Circuit

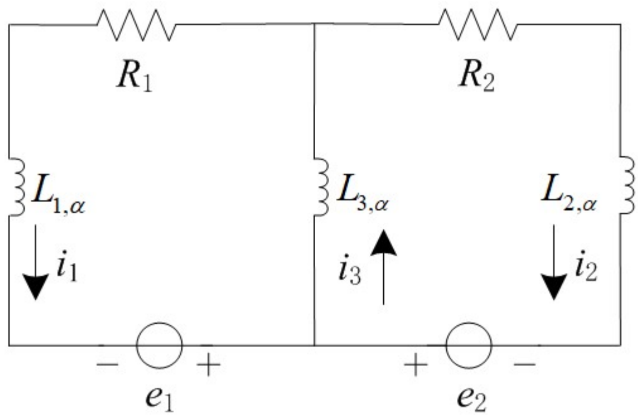

Fig 2. The Fractional RL Circuit

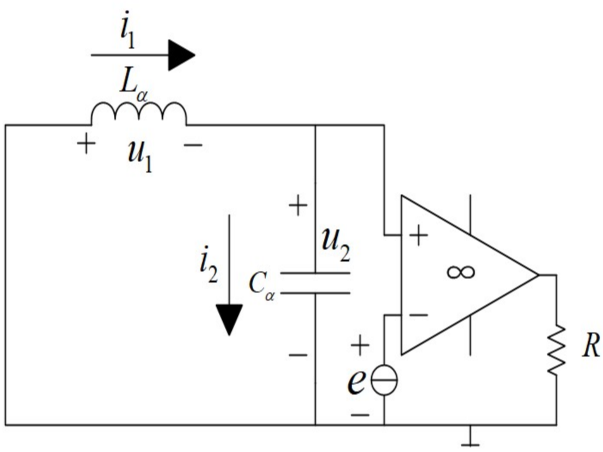

Fig 3. The Fractional LC Circuit with Ideal Operational Amplifier

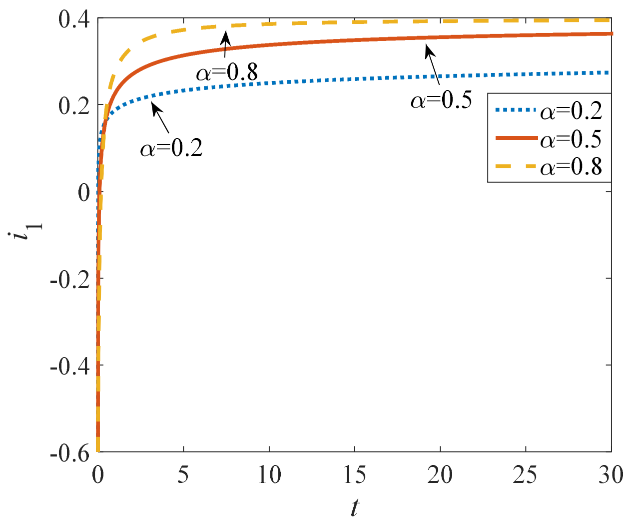

Fig 4. Trajectory of i_1 under different α

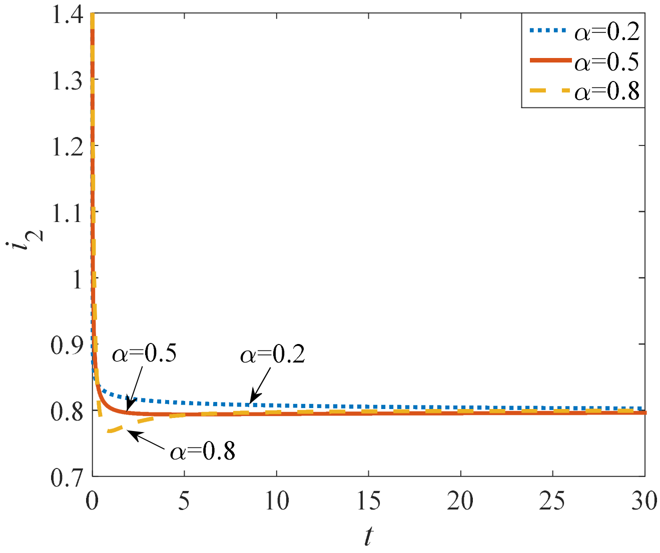

Fig 5. Trajectory of i_2 under different α

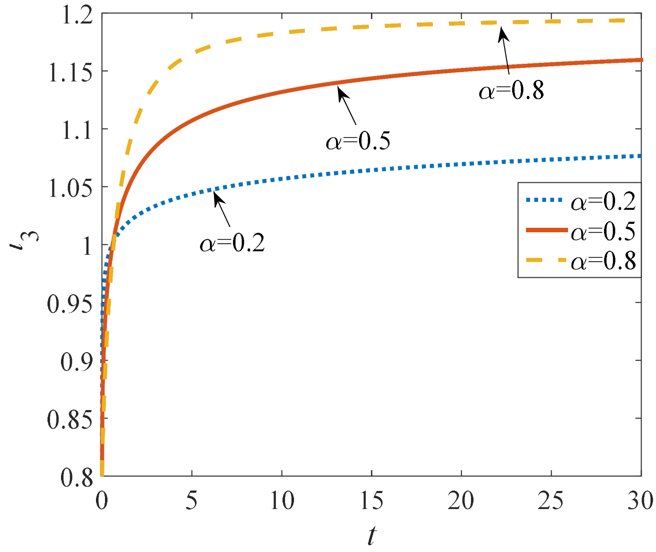

Fig 6. Trajectory of i_3 under different α

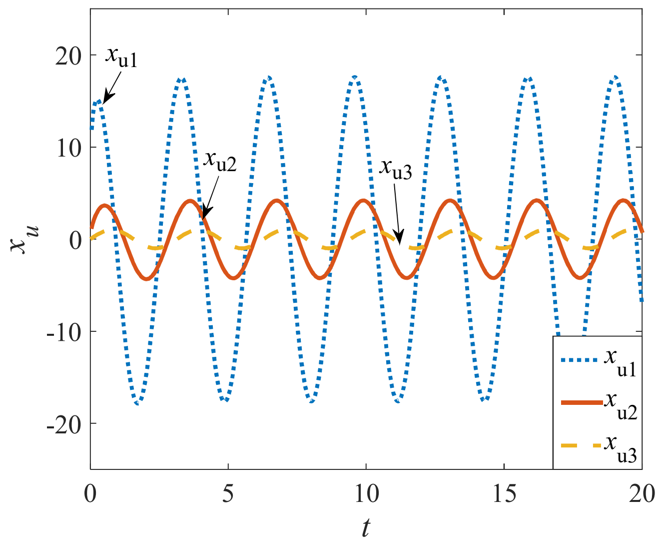

Fig 7. Trajectory of x_{u1}, x_{u2} and x_{u3} under α=0.4

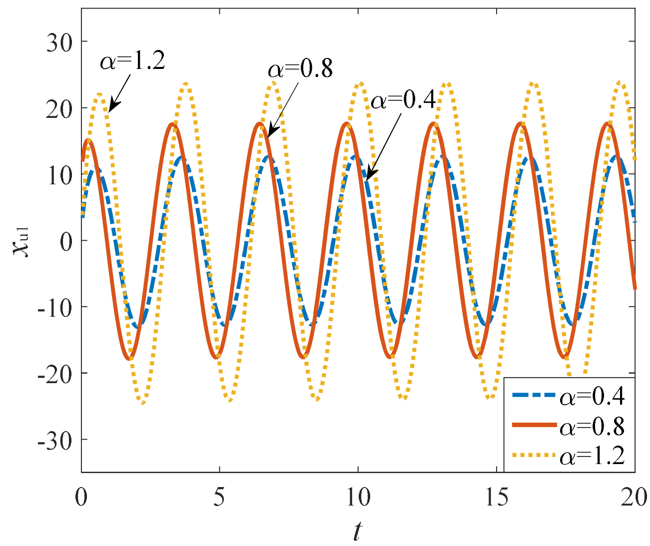

Fig 8. Trajectory of x_{u1} under different α

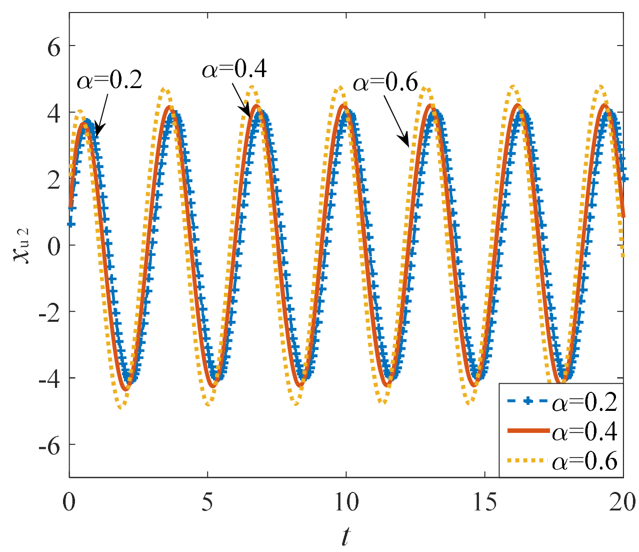

Fig 9. Trajectory of x_{u2} under different α

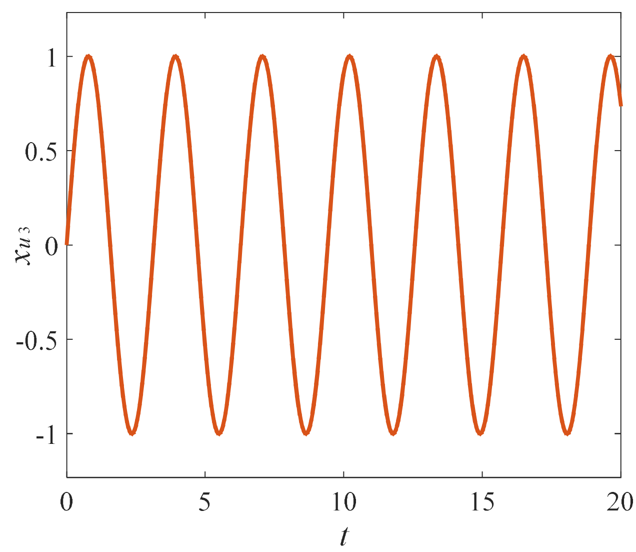

Fig 10. Trajectory of x_{u3}