A Study on Z-Soft Fuzzy Rough Sets in BCI-Algebras

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Abstract—**In this paper, the notion of Z-soft fuzzy rough BCI-algebras (ideals) of BCI-algebras is introduced, which is an extended notion of soft rough fuzzy BCI-algebras (ideals) of BCI-algebras. We first apply Z-soft fuzzy rough sets to BCI-algebras. In addition, we study roughness in BCI-algebras with respects to a Z-soft fuzzy approximation space. Moreover, some new Z-soft fuzzy rough operations over BCI-algebras are explored. In particular, Z-lower and Z-upper soft fuzzy rough BCI-algebras (ideals) over BCI-algebras are investigated.**

Index Terms—**BCI-algebra; Pseudo fuzzy soft set; Z-soft fuzzy rough set; Z-soft fuzzy rough BCI-algebra (ideal).**

I. Introduction

It is well known that many our traditional tools for modeling, reasoning and computing are crisp, deterministic and precise in character. However, in order to solve the complicated problems in biology, engineering, medical science and many other fields that contain uncertainties, we cannot successfully use traditional mathematical tools, because of various types of uncertainties existing in these problems. There have been a great amount of research and applications in the literature concerning some special tools such as fuzzy set theory [29], intuitionistic fuzzy set theory [7], rough set theory [22] and so on. However, as pointed out in [21] that all of these theories have their advantage as well as inherent limitations in dealing with uncertainties. A major problem shared with these theories is their incompatibility with the parameterization tools.

In 1999, Molodtsov [21] put forward the concept of soft set as a new mathematical tool for dealing with uncertainties. As reviewed in [21], a wide range of applications of soft sets have been developed in many different areas, including the smoothness of functions, game theory, operations researches, Riemann integration, probability theory and so on. In recent years, this theory has potential applications, such as, see [2], [6], [8], [9], [13], [20], [30], [31], [32].

Rough set theory was introduced by Pawlak [22] for dealing with some inexact and uncertain information systems. It is well known that rough set theory has been successfully applied to expert systems, signal analysis, machine learning, intelligent systems, decision analysis and many other fields. In general, the Pawlak's rough set theory is established by an equivalence relation. However, in the real world, the equivalence relation is very restrictive in some concrete and useful applications. For this reason, some researchers put forward some more general models, see [5], [23], [25], [26], [27], [34]. At the same time, it invoked an issue concerning possible relations between rough set theory and related algebraic structure. It is worth noting that Yao [28] gave some constructive and algebraic methods on rough sets and Davvaz [10] investigated the relationships between rough sets and rings. Recently, Ali et al. [4] studied roughness in hemirings.

As far as known, in the field of logic, the research of t-norm based on logical systems has become increasingly more important. It is well known that BCK and BCI-algebras are two classes of algebras of logic which were introduced by Imai and Iseki [11], [12]. We know that these two classes of logical algebras have been studied by many researchers, see [15], [17], [18]. Most of the algebras associated with the t-norm based on logic, such as MTL-algebras, BL-algebras and MV-algebras which are extensions of BCK-algebras. This shows that BCK/BCI-algebras are considerably general structures, which means that it is an important topic on these two kinds of logical algebras. In particular, Ma and Zhan [20] put forward rough soft BCI-algebras by means of an ideal of the BCI-algebras. In 2017, Zhan and Zhu [33] applied Z-soft rough fuzzy sets to hemirings and studied Z-soft rough fuzzy ideals of hemirings and considered their application in decision making. Ma et al. [19] applied Z-soft fuzzy rough ideals to hemirings. Zhu and Hu [35] investigated soft rough fuzzy lattices (ideals, filters) over lattices. In the same year, Zhu and Hu [36] applied Z-soft rough fuzzy sets to BCI-algebras. Moreover, they studied roughness in BCI-algebras with respects to a Z-soft approximation space and explored some new Z-soft rough fuzzy operations over BCI-algebras. In particular, they also investigated Z-lower and Z-upper soft rough fuzzy BCI-algebras (ideals). In 2019, Zhu [37] considered soft fuzzy rough rings (ideals) of rings and their application in decision making.

The aim of this paper is to provide a framework by combining soft sets, rough sets, fuzzy sets and BCI-algebras all together. We propose the concept of Z-soft fuzzy rough BCI-algebras (ideals) by applying Z-soft fuzzy rough sets. It is worth noting that the notion of Z-soft fuzzy rough BCI-algebras (ideals) is an extended notion of soft rough fuzzy BCI-algebras (ideals) of BCI-algebras, which have been studied in [36]. Thus, the conclusions obtained in this paper are the generalizations of [36].

The rest of this paper is organized as follows: In Section II, we recall some fundamental concepts and results on BCI-algebras, soft sets, fuzzy sets and rough sets. In Section III, we study some new operations on Z-soft fuzzy rough sets over BCI-algebras. In Section IV, we investigate some characterizations of Z-soft fuzzy rough BCI-algebras (ideals). Finally, our researches are concluded in Section V.

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II. Preliminaries

In this section, we will review some basis notions about BCI-algebras, soft sets, fuzzy sets and rough sets.

Recalled that a BCI-algebras [12] is an algebra $(X,*,0)$ of type (2,0) satisfying the following axioms:
(1) $((x*y)*(x*z))*(z*y)=0$,
(2) $(x*(x*y))*y=0$,
(3) $x*x=0$,
(4) $x*y=0$ and $y*x=0$ imply $x=y$,
for all $x,y,z\in X$.

In a BCI-algebra X, we can define a partial order $\leq$ by putting $x\leq y$ if and only if $x*y=0$. In this paper, unless otherwise stated, X is always a BCI-algebra.

A non-empty subset S of X is called a subalgebra of X if $x*y\in S$ whenever $x,y\in S$. A non-empty subset I of X is called an ideal of X, denoted by $I\lhd X$, if it satisfies: (1) $0\in I$; (2) $x*y\in I$ and $y\in I$ imply $x\in I$ for all $x,y\in X$.

Definition 2.1: [29] Let U be a set. A fuzzy set over U is a function $\mu:U\to[0,1]$.

Definition 2.2: [14] (i) A fuzzy set μ over X is called a fuzzy BCI-algebra of X if, for all $x,y\in X$, it satisfies:
$\mu(x*y)\geq\mu(x)\wedge\mu(y)$.
(ii) A fuzzy set μ over X is called a fuzzy ideal of X if, for all $x,y\in X$, it satisfies:
(F1) $\mu(0)\geq\mu(x)$;
(F2) $\mu(x)\geq\mu(x*y)\wedge\mu(y)$.

Remark 2.3: [36] Let μ be a fuzzy set over X. Then $\mu_t=\{x\in X|\mu(x)\geq t\}$, $t\in[0,1]$. A fuzzy set μ of X is a fuzzy BCI-algebra (ideal) of X if and only if every non-empty set μ_t is a BCI-algebra (ideal) of X for all $t\in[0,1]$.

Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set over U and $F(U)$ denote the all fuzzy sets over U. Then we recall the concepts of soft sets, fuzzy soft sets and pseudo fuzzy soft sets as follows.

Definition 2.4: [21] A pair $\mathfrak{S}=(F,A)$ is called a soft set over U, where $A\subseteq E$ and $F:A\to P(U)$ is a set-valued mapping.

In other words, a soft set over U is a parameterized family of subsets of U. For any parameter $x\in A, F(x)$ can be considered as the set of x-approximate elements of the soft set (F,A).

Definition 2.5: [1] A pair $(\widetilde{F},A)$ is called a fuzzy soft set over U, where $A\subseteq E$ and $\widetilde{F}:A\to F(U)$ is a mapping.

Definition 2.6: [14] Let $(\widetilde{F},A)$ be a fuzzy soft set over X. Then $(\widetilde{F},A)$ is called a fuzzy soft BCI-algebra (ideal) over X if $\widetilde{F}(x)$ is a fuzzy BCI-algebra (ideal) of X for all $x\in A$.

Definition 2.7: [22] Let R be an equivalence relation on the universe U and (U,R) be a Pawlak approximation space. A subset $X\subseteq U$ is called definable if $\underline{R}(X)=\overline{R}(X)$; in the opposite case, i.e., if $\underline{R}(X)-\overline{R}(X)\neq\emptyset$, X is said to be a rough set, where the two operators are defined as:

$$\underline{R}(X)=\{x\in U|[x]_R\subseteq X\},$$

$$\overline{R}(X)=\{x\in U|[x]_R\cap X\neq\emptyset\}.$$

Definition 2.8: [24] A pair (F^{-1},A) is called a pseudo soft set over U, where $A\subseteq E$ and $F^{-1}:U\to P(A)$.

Definition 2.9: [24] A pair (F^{-1},A) is called a pseudo fuzzy soft set over U, where $A\subseteq E$ and $F^{-1}:U\to F(A)$.

In [19], based on the concept of pseudo fuzzy soft set, Ma et al. defined a new kind of soft fuzzy rough approximation operators as follows.

Definition 2.10: [19] Let (F^{-1},A) be a pseudo fuzzy soft set over U. We call the triple (U,A,F^{-1}) the soft fuzzy approximation space. For any $\mu\in F(U)$, the Z-lower and Z-upper soft fuzzy rough approximations of μ are denoted by $\underline{\mu_{F^{-1}}}$ and $\overline{\mu_{F^{-1}}}$, respectively, which are fuzzy sets over U given by

$$\underline{\mu_{F^{-1}}}(x)=\bigwedge\{\mu(z)|z\in U,F^{-1}(z)(e)=F^{-1}(x)(e),\ \forall e\in A\},$$

$$\overline{\mu_{F^{-1}}}(x)=\bigvee\{\mu(z)|z\in U,F^{-1}(z)(e)=F^{-1}(x)(e),\ \forall e\in A\}$$

for all $x\in U$.

The operators $\underline{\mu_{F^{-1}}}$ and $\overline{\mu_{F^{-1}}}$ are called the Z-lower and Z-upper soft fuzzy rough approximation operators of fuzzy set μ, respectively. In particular, if $\underline{\mu_{F^{-1}}}=\overline{\mu_{F^{-1}}}$, μ is said to be Z-soft fuzzy definable; otherwise, μ is called a Z-soft fuzzy rough set.

III. Z-soft fuzzy rough sets over BCI-algebras

In this section, we study some new operations of Z-soft fuzzy rough sets over BCI-algebras. In addition, we give some examples to illustrate it. Firstly, we propose the concept of Z-soft fuzzy rough sets over BCI-algebras.

Definition 3.1: Let (F^{-1},A) be a pseudo fuzzy soft set over a BCI-algebra X. We call the triple (X,A,F^{-1}) the soft fuzzy approximation space. For any $\mu\in F(X)$, the Z-lower and Z-upper soft fuzzy rough approximations of μ are denoted by $\underline{\mu_{F^{-1}}}$ and $\overline{\mu_{F^{-1}}}$, respectively, which are fuzzy sets over X given by

$$\underline{\mu_{F^{-1}}}(x)=\bigwedge\{\mu(z)|z\in X,F^{-1}(z)(e)=F^{-1}(x)(e),\ \forall e\in A\},$$

$$\overline{\mu_{F^{-1}}}(x)=\bigvee\{\mu(z)|z\in X,F^{-1}(z)(e)=F^{-1}(x)(e),\ \forall e\in A\}$$

for all $x\in X$.

The operators $\underline{\mu_{F^{-1}}}$ and $\overline{\mu_{F^{-1}}}$ are called the Z-lower and Z-upper soft fuzzy rough approximation operators of fuzzy set μ, respectively. In particular, if $\underline{\mu_{F^{-1}}}=\overline{\mu_{F^{-1}}}$, μ is said to be Z-soft fuzzy definable; otherwise, μ is called a Z-soft fuzzy rough set over X.

Now, in order to illustrate the roughness in BCI-algebra X with respect to a soft fuzzy approximation space over BCI-algebras, firstly, we introduce two special kinds of pseudo fuzzy soft sets over BCI-algebras.

Definition 3.2: Let $\mathfrak{S}=(F^{-1},A)$ be a pseudo fuzzy soft set over a BCI-algebra X. Then $\mathfrak{S}$ is said to be a C-pseudo fuzzy soft set over X if $\forall e\in A$, $F^{-1}(u)(e)=F^{-1}(v)(e)$ and $F^{-1}(m)(e)=F^{-1}(n)(e)$ imply $F^{-1}(u*m)(e)=F^{-1}(v*n)(e)$, for all $u,v,m,n\in X$.

Example 3.3: Let $X=\{0,a,b,c\}$ be a BCI-algebra in Table 2.

Table 2 BCI-algebra X

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a pseudo fuzzy soft set $\mathfrak{S}=(F^{-1},A)$ over X which is given by Table 3.

Table 3 Pseudo fuzzy soft set $\mathfrak{S}$

	0	a	b	c
e_1	0.1	0.1	0.1	0.2
e_2	0.2	0.2	0.2	0.1
e_3	0.4	0.4	0.4	0.4

Then the mapping $F^{-1} : X \to F(A)$ over a soft fuzzy approximation space (X, A, F^{-1}) is given by $F^{-1}(0) = F^{-1}(a) = F^{-1}(b) = \left\{\frac{0.1}{e_1}, \frac{0.2}{e_2}, \frac{0.4}{e_3}\right\}$, $F^{-1}(c) = \left\{\frac{0.2}{e_1}, \frac{0.1}{e_2}, \frac{0.4}{e_3}\right\}$. Then we can check that $\mathfrak{S}$ is not a C-soft set over X. In fact, $\forall e \in A$, $F^{-1}(0)(e) = F^{-1}(a)(e)$ and $F^{-1}(c)(e) = F^{-1}(c)(e)$ but $F^{-1}(0 * c)(e) = F^{-1}(c)(e) \neq F^{-1}(b)(e) = F^{-1}(a * c)(e)$.

Example 3.4: We consider the BCI-algebra X in Example 3.3. $\mathfrak{S} = (F^{-1}, A)$ is a pseudo fuzzy soft set over X which is given by Table 4.

Table 4 Pseudo fuzzy soft set $\mathfrak{S}$

	0	a	b	c
e_1	0.1	0.1	0.4	0.4
e_2	0.2	0.2	0.1	0.1
e_3	0.3	0.3	0.2	0.2

Then the mapping $F^{-1} : X \to F(A)$ over a soft fuzzy approximation space (X, A, F^{-1}) is given by $F^{-1}(0) = F^{-1}(a) = \left\{\frac{0.1}{e_1}, \frac{0.2}{e_2}, \frac{0.3}{e_3}\right\}$, $F^{-1}(b) = F^{-1}(c) = \left\{\frac{0.4}{e_1}, \frac{0.1}{e_2}, \frac{0.2}{e_3}\right\}$. Then it is easy to check that $\mathfrak{S}$ is a C-pseudo fuzzy soft set over X.

The following definition is from Zadeh's expansion principle.

Definition 3.5: Let μ and ν be two fuzzy sets of a BCI-algebra X. Define $\mu * \nu$ of μ and ν as follows:

$$(\mu*\nu)(x) = \begin{cases} \bigvee\limits_{x=a*b} \mu(a) \wedge \nu(b) & a, b \in X, \text{ s.t. } x = a * b, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in X$.

Proposition 3.6: Let $\mathfrak{S} = (F^{-1}, A)$ be a C-pseudo fuzzy soft set over a BCI-algebra X and (X, A, F^{-1}) be a soft fuzzy approximation space. If μ and ν are any two fuzzy sets of X, then

$$\overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}} \subseteq \overline{(\mu * \nu)_{F^{-1}}}.$$

Proof. Putting $x = a * b, a, b \in X$. Then

$$\begin{aligned}
&(\overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}})(x) \\
&= \bigvee_{x=a*b} \overline{\mu_{F^{-1}}}(a) \wedge \overline{\nu_{F^{-1}}}(b) \\
&= \bigvee_{x=a*b} \bigvee\{\mu(c) | c \in X, F^{-1}(c)(e) = F^{-1}(a)(e), \forall e \in A\} \\
&\quad \wedge \bigvee\{\nu(d) | d \in X, F^{-1}(d)(e) = F^{-1}(b)(e), \forall e \in A\}.
\end{aligned}$$

Since $\mathfrak{S} = (F^{-1}, A)$ is a C-pseudo fuzzy soft set over X, we have

$$\begin{aligned}
&\bigvee_{x=a*b} \bigvee\{\mu(c) | c \in X, F^{-1}(c)(e) = F^{-1}(a)(e), \forall e \in A\} \\
&\quad \wedge \bigvee\{\nu(d) | d \in X, F^{-1}(d)(e) = F^{-1}(b)(e), \forall e \in A\} \\
&= \bigvee_{x=a*b} \bigvee\{\mu(c) \wedge \nu(d) | F^{-1}(c)(e) = F^{-1}(a)(e), \\
&\qquad F^{-1}(d)(e) = F^{-1}(b)(e), \forall e \in A\} \\
&\leq \bigvee_{x=a*b} \bigvee\{\mu(c) \wedge \nu(d) | F^{-1}(c * d)(e) = F^{-1}(a * b)(e) \\
&\qquad = F^{-1}(x)(e), \forall e \in A\} \\
&= \bigvee\{\mu(c) \wedge \nu(d) | F^{-1}(c * d)(e) = F^{-1}(x)(e), \forall e \in A\} \\
&\leq \bigvee\{(\mu * \nu)(y) | F^{-1}(y)(e) = F^{-1}(x)(e), \forall e \in A\} \\
&= \overline{(\mu * \nu)_{F^{-1}}}(x).
\end{aligned}$$

It follows that $(\overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}})(x) \leq \overline{(\mu * \nu)_{F^{-1}}}(x)$, i.e., $\overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}} \subseteq \overline{(\mu * \nu)_{F^{-1}}}$.
□

The following example shows that the containment in Proposition 3.6 is proper.

Example 3.7: Let $X = \{0, a, b, c\}$ be a BCI-algebra in Table 5.

Table 5 BCI-algebra X

$*$	0	a	b	c
0	0	0	0	c
a	a	0	a	c
b	b	b	0	0
c	c	c	c	0

Define a pseudo fuzzy soft set $\mathfrak{S} = (F^{-1}, A)$ over X which is given by Table 6.

Table 6 Pseudo fuzzy soft set $\mathfrak{S}$

	0	a	b	c
e_1	0.1	0.1	0.1	0.4
e_2	0.2	0.2	0.2	0.2
e_3	0.4	0.4	0.4	0.1
e_4	0.3	0.3	0.3	0.3

Then the mapping $F^{-1} : X \to F(A)$ over a soft fuzzy approximation space (X, A, F^{-1}) is given by $F^{-1}(0) = F^{-1}(a) = F^{-1}(b) = \left\{\frac{0.1}{e_1}, \frac{0.2}{e_2}, \frac{0.4}{e_3}, \frac{0.3}{e_3}\right\}$,
$F^{-1}(c) = \left\{\frac{0.4}{e_1}, \frac{0.2}{e_2}, \frac{0.1}{e_3}, \frac{0.3}{e_3}\right\}$. Then we can check that $\mathfrak{S}$ is a C-pseudo fuzzy soft set over X.

If we take $\mu = \left\{\frac{0.4}{0}, \frac{0.3}{c}\right\}$ and $\nu = \left\{\frac{0.5}{c}\right\}$, then $\overline{\mu_{F^{-1}}} = \left\{\frac{0.4}{0}, \frac{0.4}{a}, \frac{0.4}{b}, \frac{0.3}{c}\right\}$ and $\overline{\nu_{F^{-1}}} = \left\{\frac{0.5}{c}\right\}$. So $\overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}} = \left\{\frac{0.3}{0}, \frac{0.4}{c}\right\}$ and $\overline{(\mu * \nu)_{F^{-1}}} = \left\{\frac{0.3}{0}, \frac{0.3}{a}, \frac{0.3}{b}, \frac{0.4}{c}\right\}$. Thus, $\overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}} \subsetneqq \overline{(\mu * \nu)_{F^{-1}}}$.

Definition 3.8: Let $\mathfrak{S} = (F^{-1}, A)$ be a C-pseudo fuzzy soft set over a BCI-algebra X. Then $\mathfrak{S}$ is called a CC-pseudo fuzzy soft set over X if $\forall e \in A$, $a \in X$ and $F^{-1}(a)(e) = F^{-1}(m * n)(e)$, there exist $u, v \in X$ such that $F^{-1}(m)(e) = F^{-1}(u)(e)$ and $F^{-1}(n)(e) = F^{-1}(v)(e)$ satisfying $a = u * v$.

Example 3.9: (i) We consider the BCI-algebra in Example 3.7. The pair $\mathfrak{S} = (F^{-1}, A)$ is a pseudo fuzzy soft set over X which is given by Table 6. Then we can check that $\mathfrak{S}$ is a C-pseudo fuzzy soft set over X. Nevertheless, $\mathfrak{S}$ is not a CC-pseudo fuzzy soft set over X. In fact, for $\forall e \in A$, $F^{-1}(a)(e) = F^{-1}(c * c)(e)$, we can only take $F^{-1}(c)(e) = F^{-1}(c)(e)$ and $F^{-1}(c)(e) = F^{-1}(c)(e)$, however, $a \neq c * c$.

(ii) The pair $\mathfrak{S} = (F^{-1}, A)$ in Example 3.3 is a CC-pseudo fuzzy soft set over X.

If we strength the condition, we can obtain the following result.

Proposition 3.10: Let $\mathfrak{S} = (F^{-1}, A)$ be a CC-pseudo fuzzy soft set over a BCI-algebra X and (X, A, F^{-1}) be a soft fuzzy approximation space. If μ and ν are any two fuzzy sets of X, then

$$\overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}} = \overline{(\mu * \nu)_{F^{-1}}}.$$

Proof. It follows from Proposition 3.6 that we only need to show $\overline{(\mu*\nu)_{F^{-1}}} \subseteq \overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}}$.

$$\begin{aligned}
&\overline{(\mu*\nu)_{F^{-1}}}(x)\\
&= \bigvee\{(\mu*\nu)(y) | F^{-1}(y)(e) = F^{-1}(x)(e), \forall e \in A\}\\
&= \bigvee\{\bigvee_{y=a*b} \mu(a)\wedge\nu(b) | F^{-1}(a*b)(e) = F^{-1}(x)(e),\\
&\forall e \in A\}.
\end{aligned}$$

Since $\mathfrak{S} = (F^{-1}, A)$ is a CC-pseudo fuzzy soft set over X, there exist $c, d \in X$ such that $F^{-1}(a)(e) = F^{-1}(c)(e)$ and $F^{-1}(b)(e) = F^{-1}(d)(e), \forall e \in A$ satisfying $x = c*d$. So we have

$$\begin{aligned}
&\bigvee\{\bigvee_{y=a*b} \mu(a)\wedge\nu(b) | F^{-1}(a*b)(e) = F^{-1}(x)(e),\\
&\qquad \forall e \in A\}\\
&= \bigvee_{y=a*b}\{\bigvee \mu(a)\wedge\nu(b) | F^{-1}(a)(e) = F^{-1}(c)(e),\\
&\qquad F^{-1}(b)(e) = F^{-1}(d)(e), \forall e \in A\}\\
&= \bigvee_{y=a*b} \bigvee\{\mu(a) | F^{-1}(a)(e) = F^{-1}(c)(e), \forall e \in A\}\\
&\qquad \wedge \bigvee\{\nu(b) | F^{-1}(b)(e) = F^{-1}(d)(e), \forall e \in A\}\\
&= \bigvee\{\mu(a) | F^{-1}(a)(e) = F^{-1}(c)(e), \forall e \in A\}\\
&\qquad \wedge \bigvee\{\nu(b) | F^{-1}(b)(e) = F^{-1}(d)(e), \forall e \in A\}\\
&= \overline{\mu_{F^{-1}}}(c) \wedge \overline{\nu_{F^{-1}}}(d)\\
&\le \bigvee_{x=a'*b'} \overline{\mu_{F^{-1}}}(a') \wedge \overline{\nu_{F^{-1}}}(b')\\
&= (\overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}})(x).
\end{aligned}$$

Thus, $\overline{(\mu*\nu)_{F^{-1}}} \subseteq \overline{\mu_{F^{-1}}} * \overline{\nu_{F^{-1}}}$. $\square$

Next, we consider Z-lower soft fuzzy rough approximations over BCI-algebras.

Proposition 3.11: Let $\mathfrak{S} = (F^{-1}, A)$ be a CC-pseudo fuzzy soft set over a BCI-algebra X and (X, A, F^{-1}) be a soft fuzzy approximation space. If μ and ν are any two fuzzy sets of X, then

$$\underline{\mu_{F^{-1}}} * \underline{\nu_{F^{-1}}} \subseteq \underline{(\mu*\nu)_{F^{-1}}}.$$

Proof. Let $x = a*b, a, b \in X$. Then

$$\begin{aligned}
&(\underline{\mu_{F^{-1}}} * \underline{\nu_{F^{-1}}})(x)\\
&= \bigvee_{x=a*b} \underline{\mu_{F^{-1}}}(a) \wedge \underline{\nu_{F^{-1}}}(b)\\
&= \bigvee_{x=a*b} \bigwedge\{\mu(c) | c \in X, F^{-1}(c)(e) = F^{-1}(a)(e), \forall e \in A\}\\
&\quad \wedge \bigwedge\{\nu(d) | d \in X, F^{-1}(d)(e) = F^{-1}(b)(e), \forall e \in A\}.
\end{aligned}$$

Since $\mathfrak{S} = (F^{-1}, A)$ is a CC-soft set over X, we have

$$\begin{aligned}
&\bigvee_{x=a*b} \bigwedge\{\mu(c) | c \in X, F^{-1}(c)(e) = F^{-1}(a)(e), \forall e \in A\}\\
&\quad \wedge \bigwedge\{\nu(d) | d \in X, F^{-1}(d)(e) = F^{-1}(b)(e), \forall e \in A\}\\
&= \bigvee_{x=a*b} \bigwedge\{\mu(c)\wedge\nu(d) | F^{-1}(c)(e) = F^{-1}(a)(e),\\
&\quad F^{-1}(d)(e) = F^{-1}(b)(e), \forall e \in A\}\\
&\le \bigvee_{x=a*b} \bigwedge\{\mu(c)\wedge\nu(d) | F^{-1}(c*d)(e) = F^{-1}(a*b)(e)\\
&\qquad = F^{-1}(x)(e), \forall e \in A\}\\
&= \bigwedge\{(\mu*\nu)(y) | F^{-1}(y)(e) = F^{-1}(x)(e), \forall e \in A\}\\
&= \underline{(\mu*\nu)_{F^{-1}}}(x).
\end{aligned}$$

Thus, $(\underline{\mu_{F^{-1}}} * \underline{\nu_{F^{-1}}})(x) \le \underline{(\mu*\nu)_{F^{-1}}}(x)$, i.e., $\underline{\mu_{F^{-1}}} * \underline{\nu_{F^{-1}}} \subseteq \underline{(\mu*\nu)_{F^{-1}}}$.
$\square$

The following example shows that Proposition 3.11 is not true if $\mathfrak{S} = (F^{-1}, A)$ is not a CC-pseudo fuzzy soft set over X.

Example 3.12: Let $X = \{0, a, b, c, d\}$ be a BCI-algebra in Table 7.

Table 7 BCI-algebra X

$*$	0	a	b	c	d
0	0	0	0	c	c
a	a	0	a	c	c
b	b	b	0	c	c
c	c	c	c	0	0
d	d	c	d	a	0

Define a pseudo fuzzy soft set $\mathfrak{S} = (F^{-1}, A)$ over X which is given by Table 8.

Table 8 Pseudo fuzzy soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	0.1	0.1	0.1	0.1	0.6
e_2	0.5	0.1	0.6	0.6	0.5
e_3	0.4	0.4	0.4	0.4	0.1
e_4	0.1	0.3	0.1	0.1	0.6

Then the mapping $F^{-1} : X \to F(A)$ over soft fuzzy approximation space (X, A, F^{-1}) is given by $F^{-1}(0) = \left\{\frac{0.1}{e_1}, \frac{0.5}{e_2}, \frac{0.4}{e_3}, \frac{0.1}{e_4}\right\}$, $F^{-1}(a) = \left\{\frac{0.1}{e_1}, \frac{0.1}{e_2}, \frac{0.4}{e_3}, \frac{0.3}{e_4}\right\}$, $F^{-1}(b) = F^{-1}(c) = \left\{\frac{0.1}{e_1}, \frac{0.6}{e_2}, \frac{0.4}{e_3}, \frac{0.1}{e_4}\right\}$, $F^{-1}(d) = \left\{\frac{0.6}{e_1}, \frac{0.5}{e_2}, \frac{0.1}{e_3}, \frac{0.6}{e_4}\right\}$. Then we can check that $\mathfrak{S}$ is not a CC-pseudo fuzzy soft set over X.

If we take $\mu = \left\{\frac{0.2}{0}, \frac{0.3}{b}\right\}$ and $\nu = \left\{\frac{0.4}{b}, \frac{0.5}{d}\right\}$, then $\underline{\mu_{F^{-1}}} = \left\{\frac{0.2}{0}\right\}$ and $\underline{\nu_{F^{-1}}} = \left\{\frac{0.5}{d}\right\}$. So $\underline{\mu_{F^{-1}}} * \underline{\nu_{F^{-1}}} = \left\{\frac{0.2}{c}\right\}$ and $\underline{(\mu*\nu)_{F^{-1}}} = \left\{\frac{0.3}{0}\right\}$. This means that $\underline{\mu_{F^{-1}}} * \underline{\nu_{F^{-1}}} \not\subseteq \underline{(\mu*\nu)_{F^{-1}}}$.

The following example shows that the containment in Proposition 3.11 is proper.

Example 3.13: Consider the BCI-algebra X and the pseudo fuzzy soft set $\mathfrak{S} = (F^{-1}, A)$ in Example 3.3. Then we know that $\mathfrak{S}$ is a CC-pseudo fuzzy soft set over X. If we take $\mu = \left\{\frac{0.2}{0}, \frac{0.3}{a}, \frac{0.4}{b}\right\}$ and $\nu = \left\{\frac{0.3}{0}, \frac{0.2}{b}, \frac{0.3}{c}\right\}$, then $\underline{\mu_{F^{-1}}} = \left\{\frac{0.2}{0}, \frac{0.2}{a}\right\}$ and $\underline{\nu_{F^{-1}}} = \left\{\frac{0.2}{b}, \frac{0.2}{c}\right\}$. So $\underline{\mu_{F^{-1}}} * \underline{\nu_{F^{-1}}} = \left\{\frac{0.2}{b}, \frac{0.2}{c}\right\}$. On the other hand, $\underline{(\mu*\nu)_{F^{-1}}} = \left\{\frac{0.2}{0}, \frac{0.3}{a}, \frac{0.3}{b}, \frac{0.2}{c}\right\}$. This means that $\underline{\mu_{F^{-1}}} * \underline{\nu_{F^{-1}}} \subsetneqq \underline{(\mu*\nu)_{F^{-1}}}$.

IV. Characterizations of Z-soft fuzzy rough BCI-algebras (ideals) of BCI-algebras

In this section, by means of the concepts of C-pseudo fuzzy soft sets, CC-pseudo fuzzy soft sets and ZC-pseudo fuzzy soft sets, we give some Characterizations of Z-soft fuzzy rough BCI-algebras (ideals) of BCI-algebras. Firstly, we introduce the notion of Z-soft fuzzy rough BCI-algebras (ideals) of BCI-algebras.

Definition 4.1: Let X be a BCI-algebra. For any fuzzy set $\mu \in F(X)$,
(1) μ is called a Z-lower (upper) soft fuzzy rough BCI-algebra (ideal) with respect to X if $\underline{\mu_{F^{-1}}}$ ($\overline{\mu_{F^{-1}}}$) is a fuzzy BCI-algebra (ideal) of X;
(2) μ is called a Z-soft fuzzy rough BCI-algebra (ideal) with respect to X if $\underline{\mu_{F^{-1}}}$ and $\overline{\mu_{F^{-1}}}$ are fuzzy BCI-algebras (ideals) of X.

Example 4.2: Let $X = \{0, a, b, c, d\}$ be a BCI-algebra in Table 9.

Table 9 BCI-algebra X

$*$	0	a	b	c	d
0	0	0	0	c	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	d	a	0

Define a pseudo fuzzy soft set $\mathfrak{S} = (F^{-1}, A)$ over X which is given by Table 10.

Table 10 Pseudo fuzzy soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	0.1	0.5	0.4	0.5	0.4
e_2	0.5	0.4	0.5	0.4	0.5
e_3	0.4	0.3	0.7	0.3	0.7

Then the mapping $F^{-1} : X \to F(A)$ over a soft fuzzy approximation space (X, A, F^{-1}) is given by $F^{-1}(0) = \left\{\frac{0.1}{e_1}, \frac{0.5}{e_2}, \frac{0.4}{e_3}\right\}$, $F^{-1}(a) = F^{-1}(c) = \left\{\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.3}{e_3}\right\}$, $F^{-1}(b) = F^{-1}(d) = \left\{\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}\right\}$. It follows from Definition 4.1 that for a fuzzy set $\mu = \left\{\frac{0.2}{0}, \frac{0.3}{b}, \frac{0.4}{c}, \frac{0.5}{d}\right\}$,

$$\underline{\mu_{F^{-1}}} = \left\{\frac{0.2}{0}, \frac{0.2}{b}, \frac{0.3}{d}\right\} \text{ and } \overline{\mu_{F^{-1}}} = \left\{\frac{0.2}{0}, \frac{0.5}{b}, \frac{0.5}{d}\right\}.$$

It is easy to check that $\underline{\mu_{F^{-1}}}$ and $\overline{\mu_{F^{-1}}}$ are fuzzy BCI-algebras of X. In other words, μ is a Z-soft fuzzy rough BCI-algebra of X.

Example 4.3: Let $X = \{0, a, b, c\}$ be a BCI-algebra in Table 11.

Table 11 BCI-algebra X

$*$	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

Define a pseudo fuzzy soft set $\mathfrak{S} = (F^{-1}, A)$ over X which is given by Table 12.

Table 12 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	0.1	0.1	0.4	0.4
e_2	0.4	0.4	0.4	0.4
e_3	0.1	0.1	0.3	0.3

Then the mapping $F^{-1} : X \to F(A)$ over a soft fuzzy approximation space (X, A, F^{-1}) is given by $F^{-1}(0) = F^{-1}(a) = \left\{\frac{0.1}{e_1}, \frac{0.4}{e_2}, \frac{0.1}{e_3}\right\}$, $F^{-1}(b) = F^{-1}(c) = \left\{\frac{0.4}{e_1}, \frac{0.4}{e_2}, \frac{0.3}{e_3}\right\}$. It follows from Definition 4.1 that for a fuzzy set $\mu = \left\{\frac{0.5}{0}, \frac{0.2}{a}, \frac{0.3}{b}\right\}$,

$$\underline{\mu_{F^{-1}}} = \left\{\frac{0.2}{0}, \frac{0.2}{a}\right\} \text{ and } \overline{\mu_{F^{-1}}} = \left\{\frac{0.5}{0}, \frac{0.5}{a}, \frac{0.3}{b}, \frac{0.3}{c}\right\}.$$

It is easy to check that $\underline{\mu_{F^{-1}}}$ and $\overline{\mu_{F^{-1}}}$ are fuzzy ideals of X. In other words, μ is a Z-soft fuzzy rough ideal of X.

Proposition 4.4: Suppose that $\mathfrak{S} = (F^{-1}, A)$ is a pseudo fuzzy soft set over a BCI-algebra X, (X, A, F^{-1}) is a soft fuzzy approximation space and μ and ν are Z-lower soft fuzzy rough BCI-algebras (ideals) of X. Then $\mu \cap \nu$ is a Z-lower soft fuzzy rough BCI-algebra (ideal) of X.

Proof. It follows from Definition 4.1 that $\underline{\mu_{F^{-1}}}$ and $\underline{\nu_{F^{-1}}}$ are fuzzy BCI-algebras (ideals) of X. So $\underline{\mu_{F^{-1}}} \cap \underline{\nu_{F^{-1}}}$ is a fuzzy BCI-algebra (ideal) of X. It is clear that $\overline{(\mu \cap \nu)_{F^{-1}}} = \underline{\mu_{F^{-1}}} \cap \underline{\nu_{F^{-1}}}$, i.e., $\underline{(\mu \cap \nu)_{F^{-1}}}$ is also a fuzzy BCI-algebra (ideal) of X. Hence, $\mu \cap \nu$ is a Z-lower soft fuzzy rough BCI-algebra (ideal) of X. $\square$

In general, $\mu \cap \nu$ is not a Z-upper soft fuzzy rough BCI-algebra (ideal) of X if μ and ν are Z-upper soft fuzzy rough BCI-algebras (ideals) of X. Actually, we have the following example.

Example 4.5: Consider the BCI-algebra X and the pseudo fuzzy soft set $\mathfrak{S} = (F^{-1}, A)$ in Example 4.3. Let $\mu = \left\{\frac{0.2}{0}, \frac{0.3}{c}\right\}$ and $\nu = \left\{\frac{0.4}{a}, \frac{0.5}{c}\right\}$. Then $\underline{\mu_{F^{-1}}} = \left\{\frac{0.2}{0}, \frac{0.2}{a}, \frac{0.3}{b}, \frac{0.3}{c}\right\}$ and $\overline{\nu_{F^{-1}}} = \left\{\frac{0.4}{0}, \frac{0.4}{a}, \frac{0.5}{b}, \frac{0.5}{c}\right\}$. It is easy to check that $\overline{\mu_{F^{-1}}}$ and $\overline{\mu_{F^{-1}}}$ are fuzzy BCI-algebras. That is, μ and ν are Z-upper soft fuzzy rough BCI-algebras of X. However, $\overline{(\mu \cap \nu)_{\varphi}} = \left\{\frac{0.3}{b}, \frac{0.3}{c}\right\}$ is not a fuzzy BCI-algebra of X.

Proposition 4.6: Let $\mathfrak{S} = (F^{-1}, A)$ be a C-pseudo fuzzy soft set over a BCI-algebra X and (X, A, F^{-1}) be a soft fuzzy approximation space. If μ is a fuzzy BCI-algebra of X, then μ is a Z-upper soft fuzzy rough BCI-algebra of X.

Proof. Since $\mathfrak{S} = (F^{-1}, A)$ is a C-pseudo fuzzy soft set over X and μ is a fuzzy BCI-algebra of X,

$$\begin{aligned}
&\overline{\mu_{F^{-1}}}(x) \wedge \overline{\mu_{F^{-1}}}(y) \\
&= \bigvee\{\mu(a) | F^{-1}(a)(e) = F^{-1}(x)(e), \forall e \in A\} \\
&\quad \wedge \bigvee\{\mu(b) | F^{-1}(b)(e) = F^{-1}(y)(e), \forall e \in A\} \\
&\leq \bigvee\{\mu(a) \wedge \mu(b) | F^{-1}(a * b)(e) = F^{-1}(x * y)(e), \\
&\qquad \forall e \in A\} \\
&\leq \bigvee\{\mu(a * b) | F^{-1}(a * b) = F^{-1}(x * y)\} \\
&= \overline{\mu_{F^{-1}}}(x * y),
\end{aligned}$$

for any $x, y \in X$, which implies that $\overline{\mu_{F^{-1}}}$ is a fuzzy BCI-algebra of X. Thus, μ is a Z-upper soft fuzzy rough BCI-algebra of X. $\square$

Proposition 4.7: Let $\mathfrak{S} = (F^{-1}, A)$ be a CC-pseudo fuzzy soft set over a BCI-algebra X and (X, A, F^{-1}) be a soft fuzzy approximation space. If μ is a fuzzy BCI-algebra of X, then μ is a Z-lower soft fuzzy rough BCI-algebra of X.

Proof. Since $\mathfrak{S} = (F^{-1}, A)$ is a C-pseudo fuzzy soft set over X and μ is a fuzzy BCI-algebra of X,

$$\begin{aligned}
&\underline{\mu_{F^{-1}}}(x) \wedge \underline{\mu_{F^{-1}}}(y) \\
&= \bigwedge\{\mu(a) | F^{-1}(a)(e) = F^{-1}(x)(e), \forall e \in A\} \\
&\quad \wedge \bigwedge\{\mu(b) | F^{-1}(b)(e) = F^{-1}(y)(e), \forall e \in A\} \\
&= \bigwedge\{\mu(a) \wedge \mu(b) | F^{-1}(a * b)(e) = F^{-1}(x * y)(e), \forall e \in A\} \\
&\leq \bigwedge\{\mu(a * b) | F^{-1}(a * b)(e) = F^{-1}(x * y)(e, \forall e \in A)\} \\
&= \underline{\mu_{F^{-1}}}(x * y),
\end{aligned}$$

for any $x, y \in X$, which implies that $\underline{\mu_{F^{-1}}}$ is a fuzzy BCI-algebra of X. Thus, μ is a Z-lower soft fuzzy rough BCI-algebra of X. $\square$

The above two propositions show that any Z-soft fuzzy rough BCI-algebra is a generalization of a fuzzy BCI-algebra of X.

Definition 4.8: Let $\mathfrak{S} = (F^{-1}, A)$ be a C-pseudo fuzzy soft set over a BCI-algebra X, where $F^{-1} : X \to F(A)$. Then $\mathfrak{S}$ is called a ZC-pseudo fuzzy soft set over X if I is a subalgebra of X and $\forall e \in A, a, b \in X$, $F^{-1}(a)(e) = F^{-1}(b)(e)$ if and only if there exist $i_1, i_2 \in I$ such that $a * i_1 = b * i_2$.

Theorem 4.9: Let $\mathfrak{S} = (F^{-1}, A)$ be a ZC-pseudo fuzzy soft set over an associative BCI-algebra X, (X, A, F^{-1}) be a soft fuzzy approximation space. If μ is a fuzzy ideal of X and $\mu(x * y) \geq \mu(x) \wedge \mu(y)$ for any $x, y \in X$, then μ is a Z-upper soft fuzzy rough ideal of X.

Proof. It follows from Proposition 4.6 that $\overline{\mu_{F^{-1}}}(x * y) \geq \overline{\mu_{F^{-1}}}(x) \wedge \overline{\mu_{F^{-1}}}(y)$ for any $x, y \in X$. Further, $\overline{\mu_{F^{-1}}}(0) = \overline{\mu_{\varphi}}(x * x) \geq \overline{\mu_{F^{-1}}}(x) \wedge \overline{\mu_{F^{-1}}}(x) = \overline{\mu_{F^{-1}}}(x)$ for all $x \in X$. This shows that (F1) holds. Now we prove (F2) holds. Assume that there exist $x, y \in X$ such that

$\overline{\mu_{F^{-1}}}(x) < \overline{\mu_{F^{-1}}}(x*y) \wedge \overline{\mu_{F^{-1}}}(y)$, choose $t \in [0,1]$ such that $\overline{\mu_{F^{-1}}}(x) < t < \overline{\mu_{F^{-1}}}(x*y) \wedge \overline{\mu_{F^{-1}}}(y)$. Since $\mu \subseteq \overline{\mu_{F^{-1}}}$, we have $\mu(x) \subseteq \overline{\mu_{F^{-1}}}(x)$. Then

$$\begin{aligned}&\overline{\mu_{F^{-1}}}(x*y) \wedge \overline{\mu_{F^{-1}}}(y)\\&= \bigvee\{\mu(a)|F^{-1}(a)(e) = F^{-1}(x*y)(e), \forall e \in A\}\\&\quad \wedge \bigvee\{\mu(b)|F^{-1}(b)(e) = F^{-1}(y)(e), \forall e \in A\}\\&= \bigvee\{\mu(a) \wedge \mu(b)|F^{-1}(a)(e) = F^{-1}(x*y)(e),\\&\quad F^{-1}(b)(e) = F^{-1}(y)(e), \forall e \in A\}.\end{aligned}$$

Since μ is a fuzzy BCI-algebra of X, it follows from Remark 2.3 that μ_t is a subalgebra of X. Because $F^{-1}(a)(e) = F^{-1}(x*y)(e), F^{-1}(b)(e) = F^{-1}(y)(e), \forall e \in A$ and $\mathfrak{S} = (F^{-1}, A)$ is a ZC-soft set over X, there exist $i_1, i_2, i_3, i_4 \in \mu_t$ such that $a*i_1 = (x*y)*i_2, b*i_3 = y*i_4$. Since $[(x*y)*i_2]*i_4 = [(x*y)*i_2]*i_4$, and X is an associative BCI-algebra, we have $[(x*y)*i_4]*i_2 = [(x*y)*i_2]*i_4$, that is

$$[x*(y*i_4)]*i_2 = [(x*y)*i_2]*i_4,$$

$$[x*(b*i_3)]*i_2 = (a*i_1)*i_4,$$

$$[(x*b)*i_3]*i_2 = (a*i_1)*i_4,$$

$$(x*b)*(i_3*i_2) = a*(i_1*i_4).$$

It follows from Definition 4.8 that $F^{-1}(x*b)(e) = F^{-1}(a)(e), \forall e \in A$, where $i_3*i_2 \in \mu_t, i_1*i_4 \in \mu_t$. Thus

$$\begin{aligned}&\overline{\mu_{F^{-1}}}(x*y) \wedge \overline{\mu_{F^{-1}}}(y)\\&= \bigvee\{\mu(a) \wedge \mu(b)|F^{-1}(a)(e) = F^{-1}(x*y)(e),\\&\quad F^{-1}(b)(e) = F^{-1}(y)(e), \forall e \in A\}\\&= \bigvee\{\mu(a) \wedge \mu(b)|(x*b)*(i_3*i_2) = a*(i_1*i_4)\}\\&= \bigvee\{\mu(a) \wedge \mu(b)|F^{-1}(x*b)(e) = F^{-1}(a)(e), \forall e \in A\}\\&> t.\end{aligned}$$

This means that there exist $a_0, b_0 \in X$ such that $\mu(a_0) \wedge \mu(b_0) > t$ satisfying $F^{-1}(x*b_0)(e) = F^{-1}(a_0)(e), \forall e \in A$. So there exist $i_5, i_6 \in \mu_t$ such that $(x*b_0)*i_5 = a_0*i_6$ and $\mu(a_0) \wedge \mu(b_0) > t$. That is $x*(b_0*i_5) = a_0*i_6$ and $\mu(a_0) \wedge \mu(b_0) > t$. By the hypothesis, μ is a fuzzy ideal of X and then

$$\begin{aligned}\mu(x) &\geq \mu(a_0*i_6) \wedge \mu(b_0*i_5)\\&\geq \mu(a_0) \wedge \mu(i_6) \wedge \mu(b_0) \wedge \mu(i_5)\\&\geq t.\end{aligned}$$

This is a contradiction.
Hence, $\overline{\mu_{F^{-1}}}(x) \geq \overline{\mu_{F^{-1}}}(x*y) \wedge \overline{\mu_{F^{-1}}}(y)$ for all $x, y \in X$. This implies that $\overline{\mu_{F^{-1}}}$ is a fuzzy ideal of X, that is, μ is a Z-upper soft fuzzy rough ideal of X. □

Remark 4.10: Let $\mathfrak{S} = (F, A)$ be a CC-pseudo fuzzy soft set over a BCI-algebra X. If for all $a \in X$ and $m, n \in X$, $F^{-1}(a)(e) = F^{-1}(m*n)(e), \forall e \in A$ if and only if for each $F^{-1}(m)(e) = F^{-1}(u)(e)$ and $F^{-1}(n)(e) = F^{-1}(v)(e)$, we have $u*v = a$, $u, v \in X$.

Theorem 4.11: Let $\mathfrak{S} = (F^{-1}, A)$ be a CC-pseudo fuzzy soft set over X and (X, A, F^{-1}) be a soft fuzzy approximation space. If μ is a fuzzy ideal of X and $\mu(x*y) \geq \mu(x) \wedge \mu(y)$ for any $x, y \in X$, then μ is a Z-lower soft fuzzy rough ideal of X.

Proof. It follows from Proposition 4.7 that $\underline{\mu_{F^{-1}}}(x*y) \geq \underline{\mu_{F^{-1}}}(x) \wedge \underline{\mu_{F^{-1}}}(y)$ for any $x, y \in X$. Further, $\underline{\mu_{F^{-1}}}(0) = \underline{\mu_{F^{-1}}}(x*x) \geq \underline{\mu_{F^{-1}}}(x) \wedge \underline{\mu_{F^{-1}}}(x) = \underline{\mu_{F^{-1}}}(x)$ for all $x \in X$. This shows that (F1) holds. Now we prove (F2) holds. For all $x, y \in X$,

$$\begin{aligned}&\underline{\mu_{F^{-1}}}(x*y) \wedge \underline{\mu_{F^{-1}}}(y)\\&= \bigwedge\{\mu(a)|F^{-1}(a)(e) = F^{-1}(x*y)(e), \forall e \in A\}\\&\quad \wedge \bigwedge\{\mu(b)|F^{-1}(b)(e) = F^{-1}(y)(e), \forall e \in A\}\\&= \bigwedge\{\mu(c) \wedge \mu(d)|F^{-1}(a)(e) = F^{-1}(x*y)(e),\\&\quad F^{-1}(b)(e) = F^{-1}(y)(e), \forall e \in A\}.\end{aligned}$$

Since $\mathfrak{S} = (F^{-1}, A)$ is a CC-pseudo fuzzy soft set over X, it follows from Remark 4.10 that $a = c*d$ for all $F^{-1}(x)(e) = F^{-1}(c)(e), F^{-1}(y)(e) = F^{-1}(d)(e), \forall e \in A$, where $c, d \in X$. Thus

$$\begin{aligned}&\underline{\mu_{F^{-1}}}(x*y) \wedge \underline{\mu_{F^{-1}}}(y)\\&= \bigwedge\{\mu(a) \wedge \mu(b)|F^{-1}(x)(e) = F^{-1}(c)(e),\\&F^{-1}(y)(e) = F^{-1}(d)(e) = F^{-1}(b)(e), \forall e \in A, a = c*d\}\\&= \bigwedge\{\mu(c*d) \wedge \mu(b)|F^{-1}(x)(e) = F^{-1}(c)(e),\\&F^{-1}(b)(e) = F^{-1}(d)(e), \forall e \in A, a = c*d\}\\&= \bigwedge\{\mu(c*d)|F^{-1}(x)(e) = F^{-1}(c)(e), \forall e \in A\}\\&\wedge \bigwedge\{\mu(b)|F^{-1}(b)(e) = F^{-1}(d)(e), \forall e \in A\}\\&= \bigwedge\{\mu(c*d)|F^{-1}(x)(e) = F^{-1}(c)(e), \forall e \in A\}\\&\qquad \wedge \underline{\mu_{F^{-1}}}(d).\end{aligned}$$

Since $\underline{\mu_{F^{-1}}} \subseteq \mu$ and μ is a fuzzy ideal of X, we have

$$\begin{aligned}&\bigwedge\{\mu(c*d)|F^{-1}(x)(e) = F^{-1}(c)(e), \forall e \in A\} \wedge \underline{\mu_{F^{-1}}}(d)\\&\leq \bigwedge\{\mu(c*d) \wedge \mu(d)|F^{-1}(x)(e) = F^{-1}(c)(e), \forall e \in A\}\\&\leq \bigwedge\{\mu(c)|F^{-1}(c)(e) = F^{-1}(x)(e), \forall e \in A\}\\&= \underline{\mu_{F^{-1}}}(x).\end{aligned}$$

Thus, $\underline{\mu_{F^{-1}}}(x) \geq \underline{\mu_{F^{-1}}}(x*y) \wedge \underline{\mu_{F^{-1}}}(y)$ for all $x, y \in X$. This implies that $\underline{\mu_{F^{-1}}}$ is a fuzzy ideal of X, that is, μ is a Z-lower soft fuzzy rough ideal of X. □

V. Conclusions

In this paper, we propose the concept of Z-soft fuzzy rough sets over BCI-algebras. In addition, we study the roughness in BCI-algebras with respect to soft fuzzy approximation spaces. Meanwhile, we explore some new Z-soft fuzzy rough operations over BCI-algebras. In particular, we investigate Z-lower and Z-upper soft fuzzy rough BCI-algebras (ideals) over BCI-algebras.

As an extension of this work, the following topics may be considered:
(1) Constructing Z-soft fuzzy rough sets to other algebras, such as hyperrings, BL-algebras, EQ-algebras and so on;
(2) Investigating decision making methods based on Z-soft fuzzy rough sets;
(3) Establishing Z-soft fuzzy rough sets to some areas of applications, such as information sciences, intelligent systems and so on.

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