Nonuniform Sampling in Multiply Generated Shift-invariant Subspaces of Mixed Lebesgue Spaces

Nan Xing

***Abstract*—This work is devoted to the nonuniform sampling problem in shift-invariant subspaces of mixed Lebesgue spaces. We first define what multiply generated shift-invariant subspaces in mixed Lebesgue spaces $L^{p,q}\left(R^{d+1}\right)$ are. Then we proposed a fast reconstruction algorithm which allows to exactly reconstruct the signals f in the multiply generated shift-invariant subspaces when the sampling set $X=\{(x_j,y_k): k,j\in J\}$ is sufficiently dense.**

***Index Terms*—Mixed Lebesgue spaces; Nonuniform sampling; Shift-invariant subspace.**

I. Introduction

Mixed Lebesgue spaces generalize Lebesgue spaces. It was arised due to considering functions that depend on independent quantities with different properties[1], [2], [4], [3]. For instance, a function which relies on time and spacial variables may attribute mixed Lebesgue spaces. For a function coming from mixed Lebesgue spaces, one can discuss the integrability of each variable separately. This is distinct from Lebesgue spaces which mainly ask the same level of control over all the variables of a function. The flexibility makes these mixed Lebesgue spaces to have a crucial role to play in the study of time based partial differential equations. In this context, we study nonuniform sampling problem in shift-invariant subspaces of mixed Lebesgue spaces.

The sampling theorems are one of the most powerful tools in signal processing and image processing. In 1948, Shannon formally proposed sampling theorem [5], [6]. Shannon sampling theorem shows that for any $f\in L^2(R)$ with $\mathrm{supp}\hat{f}\subseteq[-T,T]$,

$$\begin{aligned} f(x)&=\sum_{n\in Z} f\left(\frac{n}{2T}\right)\frac{\sin\pi(2Tx-n)}{\pi(2Tx-n)}\\ &=\sum_{n\in Z} f\left(\frac{n}{2T}\right)\mathrm{sinc}(2Tx-n), \end{aligned}$$

where the series converges uniformly on compact sets and in $L^2(R)$, and

$$\hat{f}(\xi)=\int_R f(x)e^{-2\pi ix\xi}dx,\qquad \xi\in R.$$

However, in many realistic situations, the sampling set is only a nonuniform sampling set. For instance, the transmission through the internet from satellites only can be viewed as a nonuniform sampling problem, because there exists the loss of data packets in the transmission. In recent years, there are results concerning nonuniform sampling problem [7], [8], [9], [10], [11], [12], [13]. Uniform and nonuniform sampling problems also have been generalized to more general shift-invariant spaces [14], [15], [16], [17], [18], [19] of the form

$$V(\phi)=\left\{\sum_{k\in Z} c(k)\phi(x-k):\{c(k):k\in Z\}\in\ell^2(Z)\right\}.$$

For the sampling problem in shift-invariant subspaces of mixed Lebesgue spaces, Torres and Ward studied uniform sampling problem for band-limited functions in mixed Lebesgue spaces [20], [21]. Li, Liu and Zhang discussed the nonuniform sampling problem in principal shift-invariant spaces of mixed Lebesgue spaces [22]. In this paper, we discuss nonuniform sampling problem in multiply generated shift-invariant subspaces of mixed Lebesgue spaces. We first define what multiply generated shift-invariant subspaces in mixed Lebesgue spaces $L^{p,q}\left(R^{d+1}\right)$ are. Then we proposed a fast reconstruction algorithm which allows to exactly reconstruct the signals f in multiply generated shift-invariant subspaces when the sampling set $X=\{(x_j,y_k):k,j\in J\}$ is sufficiently dense.

The paper is organized as follows. In Section 2, we present the main concepts of mixed Lebesgue spaces and give some valuable preliminary results. In section 3, we define what multiply generated shift-invariant subspaces in mixed Lebesgue spaces $L^{p,q}\left(R^{d+1}\right)$ are and prove some properties of the functions in these shift-invariant subspaces. Section 4 proposes a fast reconstruction algorithm. Finally, concluding remarks are presented in section 5.

II. Definitions and preliminary results

We review a few definitions, notations and results of $L^{p,q}(R^{d+1})$ which will be used throughout this paper.

Definition 2.1 For $1\le p,q<+\infty$. $L^{p,q}=L^{p,q}(R^{d+1})$ consists of all measurable functions $f=f(x,y)$ defined on $R\times R^d$ satisfying

$$\|f\|_{L^{p,q}}=\left[\int_R\left(\int_{R^d}|f(x,y)|^q dy\right)^{\frac{p}{q}}dx\right]^{\frac{1}{p}}<+\infty.$$

The corresponding sequence spaces are defined by

$$\begin{aligned} \ell^{p,q}&=\ell^{p,q}(Z^{d+1})=\Big\{c:\|c\|^p_{\ell^{p,q}}\\ &=\textstyle\sum_{k_1\in Z}\left(\sum_{k_2\in Z^d}|c(k_1,k_2)|^q\right)^{\frac{p}{q}}<+\infty\Big\}. \end{aligned}$$

Nan Xing is with the Department of Basic Education and Research, Changchun Guanghua College, Changchun, Jilin 130033, China

Corresponding author E-mail: xingnanccgh@163.com.

Next, we introduce mixed Wiener amalgam spaces $W(L^{p,q})(R^{d+1})$ for controling the local behavior of function.

Definition 2.2 For $1 \le p, q < \infty$, if a measurable function f satisfies

$$\|f\|^p_{W(L^{p,q})} := \sum_{n\in Z} \sup_{x\in[0,1]} \left[\sum_{l\in Z^d} \sup_{y\in[0,1]^d} |f(x+n, y+l)|^q\right]^{p/q} < \infty,$$

then we say that f belongs to the mixed Wiener amalgam space $W(L^{p,q}) = W(L^{p,q})(R^{d+1})$.

Let $W_0(L^{p,q})$ $(1 \le p, q < \infty)$ denote the space of all continuous functions in $W(L^{p,q})$.

For $1 \le p < \infty$, if a function f satisfies

$$\|f\|^p_{W(L^p)} := \sum_{k\in Z^{d+1}} \operatorname{ess\,sup}_{x\in[0,1]^{d+1}} |f(x+k)|^p < \infty,$$

then we say that f belongs to the Wiener amalgam space $W(L^p) = W(L^p)(R^{d+1})$.

For $p = \infty$, if a measurable function f satisfies

$$\|f\|_{W(L^\infty)} := \sup_{k\in Z^{d+1}} \operatorname{ess\,sup}_{x\in[0,1]^{d+1}} |f(x+k)| < \infty,$$

then we say that f belongs to $W(L^\infty) = W(L^\infty)(R^{d+1})$. Obviously, $W(L^p) \subset W(L^{p,p})$.

Let $W_0(L^p)$ $(1 \le p \le \infty)$ denote the space of all continuous functions in $W(L^p)$.

Let $\mathbf{B}$ be a Banach space. $(\mathbf{B})^{(r)}$ denotes r copies $\mathbf{B} \times \cdots \times \mathbf{B}$ of $\mathbf{B}$. If $C = (c_1, c_2, \cdots, c_r)^T \in (\mathbf{B})^{(r)}$, then we define the norm of C by

$$\|C\|_{(\mathbf{B})^{(r)}} = \left(\sum_{j=1}^r \|c_j\|^2_{\mathbf{B}}\right)^{1/2}.$$

For any sequence $c \in \ell^p$ $(1 \le p \le +\infty)$ and $f \in W(L^1)$, define the semi-discrete convolution of c and f by

$$(c *_{sd} f)(x) = \sum_{k\in Z^{d+1}} c(k) f(x-k).$$

The following is mixed Lebesgue spaces version of Hölder's inequality.

proposition 2.3 [21, Theorem 1.1.3] Assume that $1 \le p, p', q, q' \le \infty$ satisfy $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Then

$$\|fg\|_{L^{1,1}} \le \|f\|_{L^{p,q}} \|g\|_{L^{p',q'}}.$$

III. The shift-invariant subspaces in $L^{p,q}$

For $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T \in W(L^{1,1})^{(r)}$, the multiply generated shift-invariant space in the mixed Legesgue spaces $L^{p,q}$ is given by

$$V_{p,q}(\Phi) = \left\{ \sum_{j=1}^r \sum_{k_1\in Z} \sum_{k_2\in Z^d} c_j(k_1,k_2)\phi_j(\cdot - k_1, \cdot - k_2) : c_j = \left\{c_j(k_1,k_2) : k_1 \in Z, k_2 \in Z^d\right\} \in \ell^{p,q} \right\}.$$

It is easy to see that the three sum pointwisely converges almost everywhere. In fact, for any $1 \le j \le r$, $c_j = \left\{c_j(k_1,k_2) : k_1 \in Z, k_2 \in Z^d\right\} \in \ell^{p,q}$ derives $c_j \in \ell^\infty$. This combines $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T \in W(L^{1,1})^{(r)}$ gives

$$\begin{aligned}
&\sum_{j=1}^r \sum_{k_1\in Z} \sum_{k_2\in Z^d} |c_j(k_1,k_2)\phi_j(x-k_1, y-k_2)| \\
&\le \sum_{j=1}^r \|c_j\|_\infty \sum_{k_1\in Z} \sum_{k_2\in Z^d} |\phi_j(x-k_1, y-k_2)| \\
&\le \sum_{j=1}^r \|c_j\|_\infty \|\phi_j\|_{W(L^{1,1})} \\
&\le \left(\sum_{j=1}^r \|c_j\|^2_\infty\right)^{1/2} \left(\sum_{j=1}^r \|\phi_j\|^2_{W(L^{1,1})}\right)^{1/2} \\
&= \left(\sum_{j=1}^r \|c_j\|^2_\infty\right)^{1/2} \|\Phi\|_{W(L^{1,1})^{(r)}} < \infty \, (a.e.).
\end{aligned}$$

The following theorem gives that the multiply generated shift-invariant space is well-defined in $L^{p,q}$.

Theorem 3.1 Assume that $1 \le p, q < \infty$ and $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T \in W(L^{1,1})^{(r)}$. Then for any $C = (c_1, c_2, \cdots, c_r)^T \in (\ell^{p,q})^{(r)}$, the function

$$f = \sum_{j=1}^r \sum_{k_1\in Z} \sum_{k_2\in Z^d} c_j(k_1,k_2)\phi_j(\cdot - k_1, \cdot - k_2)$$

belongs to $L^{p,q}$ and

$$\|f\|_{L^{p,q}} \le \|C\|_{(\ell^{p,q})^{(r)}} \|\Phi\|_{W(L^{1,1})^{(r)}}.$$

In order to prove Theorem 3.1, we need the following proposition.

Proposition 3.2 [22] Assume that $1 \le p, q < \infty$ and $\phi \in W(L^{1,1})$. Then for any $c \in \ell^{p,q}$, the function $f = \sum_{k_1\in Z} \sum_{k_2\in Z^d} c(k_1,k_2)\phi(\cdot - k_1, \cdot - k_2)$ belongs to $L^{p,q}$ and

$$\|f\|_{L^{p,q}} \le \|c\|_{\ell^{p,q}} \|\phi\|_{W(L^{1,1})}.$$

Now, we give proof of Theorem 3.1.

[**Proof of Theorem 3.1**] Since $\|\cdot\|_{L^{p,q}}$ is a norm, we have

$$\begin{aligned}
\|f\|_{L^{p,q}} &= \left\| \sum_{j=1}^r \sum_{k_1\in Z} \sum_{k_2\in Z^d} c_j(k_1,k_2) \phi_j(\cdot - k_1, \cdot - k_2) \right\|_{L^{p,q}} \\
&\le \sum_{j=1}^r \left\| \sum_{k_1\in Z} \sum_{k_2\in Z^d} c_j(k_1,k_2) \phi_j(\cdot - k_1, \cdot - k_2) \right\|_{L^{p,q}}.
\end{aligned} \tag{1}$$

From Proposition 3.2, for any $1 \le j \le r$

$$\begin{aligned}
&\left\| \sum_{k_1\in Z} \sum_{k_2\in Z^d} c_j(k_1,k_2)\phi_j(\cdot - k_1, \cdot - k_2) \right\|_{L^{p,q}} \\
&\le \|c_j\|_{\ell^{p,q}} \|\phi_j\|_{W(L^{1,1})}.
\end{aligned} \tag{2}$$

Therefore, from (1) and (2), we obain

$$\begin{aligned}
\|f\|_{L^{p,q}} &\le \sum_{j=1}^r \|c_j\|_{\ell^{p,q}} \|\phi_j\|_{W(L^{1,1})} \\
&\le \left(\sum_{j=1}^r \|c_j\|^2_{\ell^{p,q}}\right)^{1/2} \left(\sum_{j=1}^r \|\phi_j\|^2_{W(L^{1,1})}\right)^{1/2} \\
&\le \|C\|_{(\ell^{p,q})^{(r)}} \|\Phi\|_{W(L^{1,1})^{(r)}}.
\end{aligned}$$

In order to obtain the main result in this section, we need to introduce the following two propositions.

Proposition 3.3[24, Theorem 3.3] Assume that $\phi \in W(L^{1,1})$. Then ϕ satisfies

$$\sum_{k\in Z^{d+1}} \left|\widehat{\phi}(\xi + 2\pi k)\right|^2 > 0, \quad \xi \in R^{d+1},$$

if and only if there exists a function $g = \sum_{k_1\in Z}\sum_{k_2\in Z^d} d(k_1,k_2)\phi(\cdot-k_1,\cdot-k_2)$ such that

$$\langle \phi\left(\cdot-\alpha\right), g\rangle = \delta_{0,\alpha}.$$

Here $d=\{d(k_1,k_2): k_1\in Z, k_2\in Z^d\}\in \ell^1$.

Proposition 3.4 [22, Lemma 3.3] The function g in Proposition 3.3 belongs to $W(L^{1,1})$.

Now, we give the main result in this section which shows the norm equivalence of $\|C\|_{(\ell^{p,q})^{(r)}}$, $\|f\|_{L^{p,q}}$ and $\|f\|_{W(L^{p,q})}$. As usual, for quantities X and Y, $X\approx Y$ denotes that there exist constants c_1 and c_2 such that $c_1X\le Y\le c_2X$, and $X\preceq Y$ denotes that there exist constant c such that $X\le cY$.

Theorem 3.5 For $1<p,q<\infty$. Assume that $\Phi=(\phi_1,\phi_2,\cdots,\phi_r)^T\in W(L^1)^{(r)}$ satisfies

$$A\le [\widehat{\Phi},\widehat{\Phi}](\xi),\quad \xi\in R^{d+1},$$

where $A>0$ and $[\widehat{\Phi},\widehat{\Phi}](\xi) = \left(\sum_{k\in Z^{d+1}}\widehat{\phi}_j(\xi+2k\pi)\overline{\widehat{\phi}_{j'}(\xi+2k\pi)}\right)_{1\le j\le r,1\le j'\le r}$.

Then for any $C=(c_1,c_2,\cdots,c_r)^T\in(\ell^{p,q})^{(r)}$ and $f=\sum_{j=1}^r\sum_{k_1\in Z}\sum_{k_2\in Z^d}c_j(k_1,k_2)\phi_j(\cdot-k_1,\cdot-k_2)$, one has

$$\|C\|_{(\ell^{p,q})^{(r)}}\approx\|f\|_{L^{p,q}}\approx\|f\|_{W(L^{p,q})}.$$

Proof: Firstly, we prove $\|C\|_{(\ell^{p,q})^{(r)}}\approx\|f\|_{L^{p,q}}$.

From Theorem 3.1, we get

$$\|f\|_{L^{p,q}}\preceq\|C\|_{(\ell^{p,q})^{(r)}}.$$

Thus we only need to obtain $\|C\|_{(\ell^{p,q})^{(r)}}\preceq\|f\|_{L^{p,q}}$. Since for any $\xi\in R^{d+1}$, $A\le[\widehat{\Phi},\widehat{\Phi}](\xi)$, we get

$$\sum_{k\in Z^{d+1}}\left|\widehat{\phi}_j\left(\xi+2\pi k\right)\right|^2>0,\quad \xi\in R^{d+1}, 1\le j\le r.$$

From Proposition 3.3 and Proposition 3.4, for each ϕ_j $(1\le j\le r)$, there is a function $g_j\in W(L^{1,1})$ such that

$$\langle\phi_j\left(\cdot-\alpha\right),g_j\rangle=\delta_{0,\alpha}.$$

Therefore, for any $1\le j\le r$, $k_1\in Z$ and $k_2\in Z^d$

$$c_j(k_1,k_2)=\int_R\int_{R^d}f(x,y)\overline{g_j\left(x-k_1,y-k_2\right)}dxdy.$$

Let $B=(b_1,b_2,\cdots,b_r)^T\in\left(\ell^{p',q'}\right)^{(r)}$ with $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Thus we have

$$\begin{aligned}|\langle C,B\rangle| &= \left|\sum_{j=1}^r\sum_{k_1\in Z,k_2\in Z^d}c_j(k_1,k_2)\overline{b_j(k_1,k_2)}\right|\\ &= \left|\sum_{j=1}^r\sum_{k_1\in Z,k_2\in Z^d}\overline{b_j(k_1,k_2)}\right.\\ &\quad\left.\int_R\int_{R^d}f(x,y)\overline{g_j\left(x-k_1,y-k_2\right)}dxdy\right|\\ &= \left|\int_R\int_{R^d}f(x,y)\sum_{j=1}^r\sum_{k_1\in Z,k_2\in Z^d}\right.\\ &\quad\left.\overline{b_j(k_1,k_2)g_j\left(x-k_1,y-k_2\right)}dxdy\right|.\end{aligned}$$

From Proposition 2.3 and Theorem 3.1, we have

$$\begin{aligned}|\langle C,B\rangle| &\le \|f\|_{L^{p,q}}\left\|\sum_{j=1}^r\sum_{k_1\in Z,k_2\in Z^d}b_j(k_1,k_2)\right.\\ &\quad\left.g_j\left(x-k_1,y-k_2\right)\right\|_{L^{p',q'}}\\ &\le\|f\|_{L^{p,q}}\|B\|_{\left(\ell^{p',q'}\right)^{(r)}}\|G\|_{W(L^{1,1})^{(r)}}.\end{aligned}$$

Here $G=(g_1,g_2,\cdots,g_r)^T\in W(L^{1,1})^{(r)}$. Therewith

$$\|C\|_{(\ell^{p,q})^{(r)}}\le\|f\|_{L^{p,q}}\|G\|_{W(L^{1,1})^{(r)}},\tag{3}$$

namely $\|C\|_{(\ell^{p,q})^{(r)}}\preceq\|f\|_{L^{p,q}}$. Hence, we get $\|C\|_{(\ell^{p,q})^{(r)}}\approx\|f\|_{L^{p,q}}$.

Next, we prove $\|f\|_{L^{p,q}}\approx\|f\|_{W(L^{p,q})}$.

From the proof of [22, Theorem 3.4], $\|f\|_{L^{p,q}}\preceq\|f\|_{W(L^{p,q})}$ is a well known fact.

Conversely, by Proposition 3.2, $\|C\|_{(\ell^{p,q})^{(r)}}\approx\|f\|_{L^{p,q}}$ and the triangle inequality of norm

$$\begin{aligned}&\|f\|_{W(L^{p,q})}\\ &=\left\|\sum_{j=1}^r\sum_{k_1\in Z}\sum_{k_2\in Z^d}c_j(k_1,k_2)\right.\\ &\quad\left.\phi_j(\cdot-k_1,\cdot-k_2)\right\|_{W(L^{p,q})}\\ &\le\sum_{j=1}^r\left\|\sum_{k_1\in Z}\sum_{k_2\in Z^d}c_j(k_1,k_2)\right.\\ &\quad\left.\phi_j(\cdot-k_1,\cdot-k_2)\right\|_{W(L^{p,q})}\\ &\preceq\sum_{j=1}^r\left\|\sum_{k_1\in Z}\sum_{k_2\in Z^d}c_j(k_1,k_2)\right.\\ &\quad\left.\phi_j(\cdot-k_1,\cdot-k_2)\right\|_{L^{p,q}}\\ &\le\sum_{j=1}^r\|c_j\|_{\ell^{p,q}}\|\phi_j\|_{W(L^{1,1})}\\ &\le\left(\sum_{j=1}^r\|c_j\|^2_{\ell^{p,q}}\right)^{1/2}\left(\sum_{j=1}^r\|\phi_j\|^2_{W(L^{1,1})}\right)^{1/2}\\ &\le\|C\|_{(\ell^{p,q})^{(r)}}\|\Phi\|_{W(L^{1,1})^{(r)}}\\ &\preceq\|C\|_{(\ell^{p,q})^{(r)}}\preceq\|f\|_{L^{p,q}}.\end{aligned}$$

Therefore, we have $\|f\|_{L^{p,q}}\approx\|f\|_{W(L^{p,q})}$. This completes the proof. ∎

IV. Nonuniform sampling in shift-invariant subspaces

In this section, we mainly discuss nonuniform sampling in multiply generated shift-invariant spaces. The main result of this section is a fast reconstruction algorithm which allows to exactly reconstruct the signals f in multiply generated shift-invariant subspaces when the sampling set $X=\{(x_j,y_k): k,j\in J\}$ is sufficiently dense.

Before giving the main result of this section, we first give some definitions.

In order to separate sampling points, we give the following definition.

Definition 4.1 If a set $X=\{(x_k,y_j):k,j\in J\}\subset R^{d+1}$ satisfies $\inf_{(k,j)\ne(k',j')}|(x_k,y_j)-(x_{k'},y_{j'})|=\delta_1>0$ and $\inf_{k\ne k'}|x_k-x_{k'}|=\delta_2>0$, then we say that the set X is strongly-separated. Here $|(x_k,y_j)-(x_{k'},y_{j'})|=\sqrt{(x_k-x_{k'})^2+(y_j-y_{j'})^2}$ and J is a countable index set.

A bounded uniform partition of unity $\{\beta_{j,k}\}_{j,k\in J}$ associated to a strongly-separated sampling set $X=\{(x_j,y_k): j,k\in J\}$ is a set of functions satisfying

1) $0\le\beta_{j,k}\le 1, \forall j,k\in J$,
2) $\mathrm{supp}\beta_{j,k}\subset B_\gamma(x_j,y_k)$,
3) $\sum_{j\in J}\sum_{k\in J}\beta_{j,k}=1$.

Here $B_\gamma(x_j,y_k)$ is the open ball with center (x_j,y_k) and radius γ.

If $f\in W_0(L^{p,q})$, we define

$$Q_Xf=\sum_{j\in J}\sum_{k\in J}f(x_j,y_k)\beta_{j,k}$$

for the quasi-interpolant of the sequence $c_{j,k} = f(x_j, y_k)$.

In order to describe the structure of the sampling set X, we give the following definition.

Definition 4.2 If a set $X = \{(x_j, y_k) : k, j \in J, x_k \in R, y_j \in R^d\}$ satisfies

$$R^{d+1} = \cup_{j,k} B_\gamma(x_j, y_k) \quad \text{for every } \gamma > \gamma_0,$$

then we have the set X is γ_0-dense in R^{d+1}. Here $B_\gamma(x_j, y_k)$ is the open ball with center (x_j, y_k) and radius γ, and J is a countable index set.

The following is the main result of this section. It gives a fast iterative algorithm to reconstruct $f \in V_{p,q}(\Phi)$ from its samples $\{f(x_j, y_k) : j, k \in J\}$.

Theorem 4.3 Assume that $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T \in W_0(L^1)^{(r)}$ and P is a bounded projection from $L^{p,q}$ onto $V_{p,q}(\Phi)$. Then there is a density $\gamma > 0\,(\gamma = \gamma(p, q, P))$ such that any $f \in V_{p,q}(\Phi)$ can be reconstructed from its samples $\{f(x_j, y_k) : (x_j, y_k) \in X\}$ on any γ-dense set $X = \{(x_j, y_k) : j, k \in J\}$ by the following iterative algorithm:

$$\begin{cases} f_1 = & PQ_X f \\ f_{n+1} = & PQ_X(f - f_n) + f_n. \end{cases} \tag{4}$$

The iterates f_n converges to f in $L^{p,q}$ norms and uniformly. Furthermore, the convergence is geometric, namely,

$$\|f - f_n\|_{L^{p,q}} \le M\alpha^n$$

for some $\alpha = \alpha(\gamma) < 1$ and $M < \infty$.

Before proving Theorem 4.3, we introduce some useful results.

Let f be a continuous function. The oscillation (or modulus of continuity) of f is given by $\text{osc}_\delta(f)(x) = \sup_{|y|\le\delta} |f(x + y) - f(x)|$. Let $F = (f_1, f_2, \cdots, f_r)^T$ be a continuous vector function. The oscillation (or modulus of continuity) of F is given by $\text{osc}_\delta(F)(x) = (\text{osc}_\delta(f_1), \text{osc}_\delta(f_2), \cdots, \text{osc}_\delta(f_r))^T$.

Proposition 4.4 [22, Lemma 4.3] If $\phi \in W_0(L^1)$, then $\text{osc}_\delta(\phi) \in W_0(L^1)$.

The following is the vector version of Proposition 4.4.

Lemma 4.5 If $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T \in W_0(L^1)^{(r)}$, then $\text{osc}_\delta(\Phi) \in W_0(L^1)^{(r)}$.

Proof: In view of proposition 4.4, for any $1 \le j \le r$, we have $\text{osc}_\delta(\phi_j) \in W_0(L^1)$. Therefore

$$\begin{aligned} \|\text{osc}_\delta(\Phi)\|_{W_0(L^1)^{(r)}} &= \left(\sum_{j=1}^r \|\text{osc}_\delta(\phi_j)\|^2_{W_0(L^1)}\right)^{1/2} \\ &\le \sqrt{r} \max_{1\le j\le r} \|\text{osc}_\delta(\phi_j)\|_{W_0(L^1)} < \infty. \end{aligned}$$

Namely, we have $\text{osc}_\delta(\Phi) \in W_0(L^1)^{(r)}$. ∎

Proposition 4.6 [22, Lemma 4.4] If $\phi \in W_0(L^1)$, then $\lim_{\delta\to0} \|\text{osc}_\delta(\phi)\|_{W(L^1)} = 0$.

The following lemma is a generalization of Proposition 4.6.

Lemma 4.7 If $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T \in W_0(L^1)^{(r)}$, then $\lim_{\delta\to0} \|\text{osc}_\delta(\Phi)\|_{W(L^1)^{(r)}} = 0$.

Proof: According to proposition 4.6, for any $1 \le j \le r$, we obtain $\lim_{\delta\to0} \|\text{osc}_\delta(\phi_j)\|_{W(L^1)} = 0$. So,

$$\begin{aligned} \lim_{\delta\to0} \quad & \|\text{osc}_\delta(\Phi)\|_{W_0(L^1)^{(r)}} \\ = \quad & \lim_{\delta\to0} \left(\sum_{j=1}^r \|\text{osc}_\delta(\phi_j)\|^2_{W_0(L^1)}\right)^{1/2} \\ = \quad & \left(\lim_{\delta\to0} \sum_{j=1}^r \|\text{osc}_\delta(\phi_j)\|^2_{W_0(L^1)}\right)^{1/2} \\ = \quad & \left(\sum_{j=1}^r \lim_{\delta\to0} \|\text{osc}_\delta(\phi_j)\|^2_{W_0(L^1)}\right)^{1/2} \\ = \quad & \left(\sum_{j=1}^r 0\right)^{1/2} = 0. \end{aligned}$$

∎

In order to prove Theorem 4.3, we need the following two lemmas.

Lemma 4.8 Assume that $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T \in W_0(L^1)^{(r)}$ and $f \in V_{p,q}(\Phi)$. Then the oscillation (or modulus of continuity) $\text{osc}_\delta(f)$ belongs to $L^{p,q}$. Moreover for all $\epsilon > 0$, there exists $\delta_0 > 0$ such that

$$\|\text{osc}_\delta(f)\|_{L^{p,q}} \le \epsilon\|f\|_{L^{p,q}}$$

uniformly for all $0 < \delta < \delta_0$ and $f \in V_{p,q}(\Phi)$.

Proof: Letting $f = \sum_{j=1}^r \sum_{k\in Z^{d+1}} c_j(k)\phi_j(\cdot - k) \in V^{p,q}(\Phi)$, then

$$\begin{aligned} \text{osc}_\delta(f)(x) &= \sup_{|y|\le\delta} |f(x + y) - f(x)| \\ &\le \sum_{j=1}^r \sup_{|y|\le\delta} \sum_{k\in Z^{d+1}} |c_j(k)| \\ &| \quad \phi_j(x + y - k) - \phi_j(x - k)| \\ &\le \sum_{j=1}^r \sum_{k\in Z^{d+1}} |c_j(k)| \\ \sup & \quad {}_{|y|\le\delta} |\phi_j(x - k + y) - \phi_j(x - k)| \\ &= \sum_{j=1}^r \sum_{k\in Z^{d+1}} |c_j(k)| \text{osc}_\delta(\phi_j)(x - k) \\ &= \sum_{j=1}^r [|c_j| *_{sd} \text{osc}_\delta(\phi_j)](x). \end{aligned}$$

Here $|c_j| = \{|c_j(k)| : k \in Z^{d+1}\}$. In view of Theorem 3.1, Theorem 3.5 and Lemma 4.5, there exists $M > 0$ such that

$$\begin{aligned} \|\text{osc}_\delta(f)\|_{L^{p,q}} &\le \left\|\sum_{j=1}^r |c_j| *_{sd} \text{osc}_\delta(\phi_j)\right\|_{L^{p,q}} \\ &\le \||C|\|_{(\ell^{p,q})^{(r)}} \|\text{osc}_\delta(\Phi)\|_{W(L^{1,1})^{(r)}} \\ &= \|C\|_{(\ell^{p,q})^{(r)}} \|\text{osc}_\delta(\Phi)\|_{W(L^{1,1})^{(r)}} \\ &\le M\|f\|_{L^{p,q}} \|\text{osc}_\delta(\Phi)\|_{W(L^{1,1})^{(r)}}. \end{aligned} \tag{5}$$

Here $C = (c_1, c_2, \cdots, c_r)^T$ and $|C| = (|c_1|, |c_2|, \cdots, |c_r|)^T$. By Lemma 4.7, for any $\epsilon > 0$, there exists $\delta_0 > 0$ such that

$$\|\text{osc}_\delta(\Phi)\|_{W(L^1)^{(r)}} < \frac{\epsilon}{M}, \quad \forall\, 0 < \delta \le \delta_0.$$

This combines (5) yields $\|\text{osc}_\delta(f)\|_{L^{p,q}} \le \epsilon\|f\|_{L^{p,q}}$. ∎

Lemma 4.9 Let $\Phi = (\phi_1, \phi_2, \cdots, \phi_r)^T \in W_0(L^1)^{(r)}$ and P be any bounded projection from $L^{p,q}$ onto $V_{p,q}(\Phi)$. Then there is $\gamma_0 = \gamma_0(p, q, P)$ such that the operator $I - PQ_X$ is a contraction on $V_{p,q}(\Phi)$ for every strongly-separated γ-dense set X with $\gamma \leq \gamma_0$.

Proof: Let $f \in V_{p,q}(\Phi)$, we have

$$\begin{aligned} \|f - PQ_X f\|_{L^{p,q}} &= \|Pf - PQ_X f\|_{L^{p,q}} \\ &\leq \|P\|_{op}\|f - Q_X f\|_{L^{p,q}} \\ &\leq \|P\|_{op}\|\mathrm{osc}_\gamma f\|_{L^{p,q}} \\ &\leq \epsilon\|P\|_{op}\|f\|_{L^{p,q}}, \end{aligned}$$

where the last inequality holds according to Lemma 4.8. We can choose a γ_0 such that for any $\gamma < \gamma_0$, $\epsilon\|P\|_{op} < 1$. Therefore, we get a contraction. ∎

We are now ready to prove the main result in this section.

Proof of Theorem 4.3*:* For convenience, let $e_n = f - f_n$ be the error after n iterations. From (4),

$$\begin{aligned} e_{n+1} &= f - f_{n+1} \\ &= f - f_n - PQ_X(f - f_n) \\ &= (I - PQ_X)e_n. \end{aligned}$$

From Lemma 4.9, we can choose a small γ satisfies that $\|I - PQ_X\|_{op} = \alpha < 1$. Then we obtain

$$\|e_{n+1}\|_{L^{p,q}} \leq \alpha\|e_n\|_{L^{p,q}} \leq \alpha^n\|e_0\|_{L^{p,q}}.$$

Where $\|e_n\|_{L^{p,q}} \to 0$, when $n \to \infty$. This completes the proof. ∎

V. Conclusion

In this paper, we discuss nonuniform sampling problem in shift-invariant subspaces of mixed Lebesgue spaces. We first define that multiply generated shift-invariant subspaces in mixed Lebesgue spaces $L^{p,q}\left(R^{d+1}\right)$. Then we proposed a fast reconstruction algorithm. When the sampling set $X = \{(x_j, y_k) : k, j \in J\}$ is sufficiently dense, this fast reconstruction algorithm allows to exactly reconstruct the signals f in the multiply generated shift-invariant subspaces. Studying nonuniform sampling problem in multiply generated vector shift-invariant subspaces of mixed Lebesgue spaces is the goal of future work.

References

[1] A. Benedek, R. Panzone, "The space L^p with mixed norm," *Duke Math. J.*, vol. 28(3), pp. 301-324, 1961.

[2] A. Benedek, A. P. Calderón, R. Panzone, "Convolution operators on Banach space valued functions," *Proc. Nat. Acad. Sci. USA*, vol. 48(3), pp. 356-365, 1962.

[3] D. L. Fernandez, "Vector-valued singular integral operators on L^p-spaces with mixed norms and applications," *Pac. J. Math.*, vol. 129(2), pp. 257-275, 1987.

[4] J. L. Francia, F. J. Ruiz, J. L. Torrea, "Calderón–Zygmund theory for operator-valued kernels," *Adv. Math.*, vol. 62(1), pp. 7-48, 1986.

[5] C. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37(1), pp. 10-21, 1949.

[6] C. Shannon, "Communication in the presence of noise," *Proc. IEEE*, vol. 72(9), pp. 447-457, 1998.

[7] Q. Sun, "Nonuniform average sampling and reconstruction of signals with finite rate of innovation," *SIAM J. Math. Anal.*, vol. 38(5), pp. 1389-1422, 2006.

[8] J.J. Benedetto, P.J.S.G. Ferreira, *Modern sampling theory, Mathematics and applications*, Birkhäuser, Boston, 2001.

[9] F.Marvasti, *Nonuniform sampling, theory and practice*, Kluwer Academic/Plenum Publishers, New York, 2001.

[10] R. Venkataramani, Y. Bresler, "Sampling theorems for uniform and periodic nonuniform MIMO sampling of multiband signals," *IEEE Trans. Signal Process.*, vol. 51(12), pp. 3152-3163, 2003.

[11] C.Heil, *Harmonic Analysis and Applications*, Birkhäuser, Boston, 2006.

[12] W. Sun, X. Zhou, "Sampling theorem for multiwavelet subspaces," *Chinese Sci. Bull.*, vol. 44(14), pp. 1283-1286, 1999.

[13] X. Zhou, W. Sun, "On the sampling theorem for wavelet subspaces," *J. Fourier Anal. Appl.*, vol. 5(4), pp. 347-354, 1999.

[14] A. Aldroubi, H. Feichtinger, "Exact iterative reconstruction algorithm for multivariate irregularly sampled functions in spline-like spaces: The L^p theory," *Proc. Amer. Math. Soc.*, vol. 126(9), pp. 2677-2686, 1998.

[15] A. Aldroubi, K. Gröchenig, "Beurling-Landau-type theorems for nonuniform sampling in shift-invariant spaces," *J. Fourier Anal. Appl.*, vol. 6(1), pp. 91-101, 2000.

[16] A. Aldroubi, K. Gröchenig, "Nonuniform sampling and reconstruction in shift-invariant spaces," *SIAM Rev.*, vol. 43(4), pp. 585-562, 2001.

[17] A. Aldroubi, M. Unser, "Sampling procedure in function spaces and asymptotic equivalence with Shannon's sampling theory," *Numer. Funct. Anal. Optim.*, vol. 15(1), pp. 1-21, 1994.

[18] P. P. Vaidyanathan, B. Vrcelj, "Biorthogonal partners and applications," *IEEE Trans. Signal Process.*, vol. 49, pp. 1013-1027, 2001.

[19] Q. Zhang, W. Sun, "Invariance of shift invariant spaces," *Sci. China Seri.*, vol. A 55(7), pp. 1395-1401, 2012.

[20] R. Torres, E. Ward, Leibniz's Rule, , "Sampling and wavelets on mixed Lebesgue spaces," *J. Fourier Anal. Appl.*, vol. 21(5), pp. 1053-1076, 2015

[21] E. Ward, *New estimates in harmonic analysis for mixed Lebesgue spaces*, Ph.D. Thesis, University of Kansas, 2010.

[22] R. Li, B. Liu, R. Liu, Q. Zhang, "Nonuniform sampling in principal shift-invariant subspaces of mixed Lebesgue spaces $L^{p,q}(R^{d+1})$," *J. Math. Anal. Appl.*, vol. 453(2), pp. 928-941, 2017.

[23] Z. P. BAI, Y. F. LI, " Study on the Length of the Near -Fire Source Area in Long and Narrow Space," *Engineering Letters*, vol. 28, no.1, pp100-104, 2020.

[24] R.Q. Jia, C.A. Micchelli, " Using the refinement equation for the construction of pre-wavelets II: Powers of two. P.J. Laurent, A. Le Mhaut, L.L. Schumaker (Eds.)," *Curves and Surfaces*, Academic Press, New York, pp. 209-246, 1991.

[25] Jinjie Wen, Zhengxu Zhao, Yang Guo, and Qian Zhong, "Research into Net Assessment of System Lifecycle for Teleoperation of Deep Space Mission," *IAENG International Journal of Computer Science*, vol. 47, no.1, pp1-10, 2020.