Note on the Conformable Fractional Derivatives and Integrals of Complex-valued Functions of a Real Variable

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***Abstract*— Recently, the most important properties of the conformable fractional derivative and integral have been introduced. In this paper, some interesting results of real fractional Calculus are extended to the context of the complex-valued functions of a real variable. Finally, using all obtained results the complex conformable integral is defined, and some of its most important properties are established.**

***Index Terms*— Conformable fractional derivative, Conformable fractional integral, Complex-valued functions of a real variable, Complex fractional Calculus**

I. Introduction

FOR many years, several definitions of fractional derivative have been introduced by various researchers. The most common fractional derivatives are Riemann-Liouville and Caputo fractional derivative. For more information about the characteristics of Riemann-Caputo, Caputo and other related fractional definitions, we refer to [1].

Recently, Khalil et al. introduced a new definition of fractional derivative called the conformable fractional derivative [2]. Unlike other definitions, this new definition satisfies the formulas of derivative of product and quotient of two functions and has a simpler chain rule than other definitions. In addition to the conformable fractional derivative definition, the conformable integral definition, Rolle theorem, and Mean value theorem for conformable fractional differentiable functions were given in literature. In [3], Abdeljawad improves this new theory. For instance, definitions of left and right conformable fractional derivatives and fractional integrals of higher order (i.e. of order $\alpha > 1$), fractional power series expansion, fractional Laplace transform definition, fractional integration by parts formulas, chain rule and Grönwall inequality are provided by Abdeljawad.

In [4] the conformable partial derivative of the order $\alpha \in (0,1]$ of the real value of several variables and conformable gradient vector are defined as well as a conformable version of Clairaut´s Theorem for partial derivatives of conformable fractional orders is proven in [4].

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In [5], conformable Jacobian matrix is defined; chain rule for multivariable conformable derivative is given; relation between conformable Jacobian matrix and conformable partial derivatives is revealed.

In [6], two new results on homogeneous functions involving their conformable partial derivatives are introduced, specifically, the homogeneity of the conformable partial derivatives of a homogeneous function and the conformable version of Euler´s Theorem.

In [7], [8], a theory of fractional analytic functions in the conformable sense is developed. In short time, many studies about the theory and applications of the fractional differential equations which based on this new fractional derivative definition were conducted in [9], [10], [11], [13], [13], [14], [15], [16], [17], [18], [19], [20] and [21].

The paper is organized as follows. In Section 2, the main concepts of the conformable fractional calculus are presented. In Section 3, the conformable fractional derivative of the order $\alpha \in (0,1]$ of the complex-valued functions of a real variable is defined and its most important properties are introduced. In Section 4, the real conformable fractional integral is extended to the context of the complex-valued functions of a real variable. Also, some results of the classical integral calculus such as the Second Fundamental Theorem or a property of moduli of integrals are established. In Section 5, conformable integrals of complex-valued functions of a complex variable are define on curves in the complex plane. In addition, some of the most important properties of these integrals are established.

II. Basic Definitions and Tools

Definition 2.1. Given a function $f:[0,\infty) \to R$. Then the conformable fractional derivative of f of order α, [2], is defined by

$$(T_\alpha f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{1}$$

for all $t > 0$, $0 < \alpha \le 1$. If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t \to 0^+}(T_\alpha f)(t)$ exist, then it is defined as

$$(T_\alpha f)(0) = \lim_{t \to 0^+}(T_\alpha f)(t) \tag{2}$$

As a consequence of the above definition, the following useful theorem is obtained.

Theorem 2.1. [2]. If a function $f:[0,\infty) \to R$ is α-differentiable at $t_0 > 0$, $0 < \alpha \le 1$, then f is continuous at t_0.

It is easily shown that T_α satisfies the following properties:

Theorem 2.2. [2]. Let $0 < \alpha \le 1$ and f , g be α -differentiable at a point $t > 0$. Then

(i) $T_\alpha(af + bg) = a\,(T_\alpha f) + b\,(T_\alpha g), \forall a, b \in R.$

(ii) $T_\alpha(t^p) = pt^{p-\alpha}, \forall p \in R.$

(iii) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

(iv) $T_\alpha(fg) = f(T_\alpha g) + g(T_\alpha f)$.

(v) $T_\alpha\left(\frac{f}{g}\right) = \frac{g(T_\alpha f) - f(T_\alpha g)}{g^2}$.

(vi) If, in addition, f is differentiable, then $(T_\alpha f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

The conformable fractional derivative of certain functions for above definition is given as follows, [2]:

(i) $T_\alpha(1) = 0$.

(ii) $T_\alpha(sin(at)) = at^{1-\alpha}cos(at),\ a \in R$.

(iii) $T_\alpha(cos(at)) = -at^{1-\alpha}sin(at),\ a \in R$.

(iv) $T_\alpha(e^{at}) = ae^{at},\ a \in R$.

Further, many functions behave as in the usual derivative. Here are some formulas, [2]:

(i) $T_\alpha\left(\frac{1}{\alpha}t^\alpha\right) = 1$

(ii) $T_\alpha\left(e^{\frac{1}{\alpha}t^\alpha}\right) = e^{\frac{1}{\alpha}t^\alpha}$

(iii) $T_\alpha\left(\sin\left(\frac{1}{\alpha}t^\alpha\right)\right) = cos\left(\frac{1}{\alpha}t^\alpha\right)$

(iv) $T_\alpha\left(\cos\left(\frac{1}{\alpha}t^\alpha\right)\right) = -sin\left(\frac{1}{\alpha}t^\alpha\right)$

Remark 2.1. One should notice that a function could be α-differentiable at a point but not differentiable. For example, take $f(t) = 3\sqrt[3]{t}$. Then $\left(T_{\frac{1}{3}}f\right)(0) = \lim_{t\to0^+}\left(T_{\frac{1}{3}}f\right)(t) = 1$, where $\left(T_{\frac{1}{3}}f\right)(t) = 1$, for $t > 0$. But $\frac{df}{dt}(0)$ does not exist.

Definition 2.2. The (left) conformable derivative starting from a of a function $f: [a, \infty) \to R$ of f of order $0 < \alpha \le 1$, [3], is defined by

$$(T_\alpha^a f)(t) = \lim_{\epsilon\to0}\frac{f(t+\varepsilon(t-a)^{1-\alpha}) - f(t)}{\varepsilon} \tag{3}$$

When $a = 0$, it is written as $(T_\alpha f)(t)$. If f is α -differentiable in some (a, b), then define

$$(T_\alpha^a f)(a) = \lim_{t\to a^+}(T_\alpha^a f)(t) \tag{4}$$

Note that if f is differentiable, then $(T_\alpha^a f)(a) = (t-a)^{1-\alpha}\frac{df}{dt}(t)$. Theorem 2.2 holds for Definition 2.2 when changing by $(t - a)$.

Theorem 2.3. ***(Chain Rule)***.[3]. Assume $f, g: (a, \infty) \to R$ be (left) α-differentiable functions, where $0 < \alpha \le 1$. Let $h(t) = f(g(t))$. The $h(t)$ is α-differentiable for all $t \ne a$ and $g(t) \ne 0$, therefore

$$(T_\alpha^a h)(t) = (T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t).(g(t))^{\alpha-1} \tag{5}$$

If $t = a$, then

$$(T_\alpha^a h)(a) = \lim_{t\to a^+}(T_\alpha^a f)(g(t)) \cdot (T_\alpha^a g)(t).(g(t))^{\alpha-1} \tag{6}$$

Theorem 2.4 ***(Rolle´s Theorem).*** [2]. Let $a > 0$, $\alpha \in (0,1]$ and $f: [a, \infty) \to$ be a given function that satisfies the following:

- f is continuous on $[a, b]$.
- f is α −differentiable on (a, b).
- $f(a) = f(b)$.

Then, there exists $c \in (a, b)$, such that $(T_\alpha f)(c)$=0.

Corollary 2.1. [2]. Let $I \subset [0.\infty)$, $\alpha \in (0,1]$ and $f: I \to R$ be a given function that satisfies

- f is α −differentiable on I.
- $f(a) = f(b) = 0$ for certain $c \in I$.

Then, there exists $c \in (a, b)$, such that $(T_\alpha f)(c)$=0.

Theorem 2.5. ***(Mean Value Theorem).*** [2]. Let $a > 0$, $\alpha \in (0,1]$ and $f: [a, \infty) \to R$ be a given function that satisfies

- f is continuous in $[a, b]$.
- f is α-differentiable on (a, b).

Then, exists $c \in (a, b)$ such that

$$(T_\alpha f)(c) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}} \tag{7}$$

Theorem 2.6. [9]. Let $a > 0$, $\alpha \in (0,1]$ and $f: [a, \infty) \to R$ be a given function that satisfies

- f is continuous in $[a, b]$.
- f is α-differentiable on (a, b).

If $(T_\alpha f)(c) = 0$ for all $t \in (a, b)$, then f is a constant on $[a, b]$.

Corollary 2.2. [9]. Let $a > 0$, $\alpha \in (0,1]$ and $F, G: [a, \infty) \to R$ be functions such that $(T_\alpha F)(t) = (T_\alpha G)(t)$ for all $t \in (a, b)$. Then there exists a constant C such that

$$F(t) = G(t) + C \tag{8}$$

The following definition is the α-fractional integral of a function f starting from $a \ge 0$.

Definition 2.3. $I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} \cdot dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$, [2].

With the above definition, it was shown that

Theorem 2.5. [2]. $T_\alpha^a I_\alpha^a(f)(t) = f(t)$, for $t \ge a$, where f is any continuous function in the domain of I_α.

Lemma 2.1. Let $f:(a,b)\to R$ be differentiable and $\alpha \in (0,1]$. Then, for all $a>0$ we have, [3],

$$I_\alpha^a T_\alpha^a(f)(t) = f(t) - f(a) \tag{9}$$

III. Conformable Fractional Derivative of Complex-Valued of a Real Variabl

In this section, some new results on conformable fractional derivative of complex-valued functions are presented.

Definition 3.1. Let $f(t)=u(t)+iv(t)$ where $u(t)$ and $v(t)$ are real-valued functions of the variable t, for $t\in[0,\infty)$. Then the conformable derivative of f of order α is defined by

$$(T_\alpha f)(t) = (T_\alpha u)(t) + i(T_\alpha v)(t) \tag{10}$$

for all $t>0$, $0<\alpha\le 1$. If f is α-differentiable in some $(0,a)$, $a>0$, and $\lim_{t\to 0^+}(T_\alpha f)(t)$ exist, then it is defined as

$$(T_\alpha f)(0) = \lim_{t\to 0^+}(T_\alpha f)(t) \tag{11}$$

As a consequence of the above definition, the following useful theorem is obtained:

Theorem 3.1. If a complex-valued function $f:[0,\infty)\to R$ is α -differentiable at $t_0>0$, $0<\alpha\le 1$, then f is continuous at t_0.
Proof. This result follows from the similar theorem for real conformable fractional derivative in [2].

Remark 3.1. The conformable fractional derivative is linear on C. In other words, if $c\in C$, f and g are α-differentiable complex-valued functions at point $t>0$, then $cf$and $f+g$ are also α-differentiable complex-valued functions at point $t>0$, hence we have:

$$T_\alpha(cf)(t) = cT_\alpha(f)(t) \tag{12}$$

$$T_\alpha(f+g)(t) = T_\alpha(f)(t) + T_\alpha(g)(t) \tag{13}$$

Remark 3.2. As in case of real conformable fractional derivative, if f is differentiable, then

$$(T_\alpha f)(t) = t^{1-\alpha}\frac{df}{dt}(t) \tag{14}$$

Example 3.1. Another expected rule that is often used

$$T_\alpha\left(e^{z_0\frac{t^\alpha}{\alpha}}\right) = e^{z_0\frac{t^\alpha}{\alpha}}$$

where $z_0 = x_0 + iy_0$. To prove this, we write

$$\begin{aligned} e^{z_0\frac{t^\alpha}{\alpha}} &= e^{x_0\frac{t^\alpha}{\alpha}}\cdot e^{iy_0\frac{t^\alpha}{\alpha}} \\ &= e^{x_0\frac{t^\alpha}{\alpha}}\left(cos\left(y_0\frac{t^\alpha}{\alpha}\right) + isin\left(y_0\frac{t^\alpha}{\alpha}\right)\right)\end{aligned}$$

and refer to Definition 3.1 to see that

$$\begin{aligned} T_\alpha\left(e^{z_0\frac{t^\alpha}{\alpha}}\right) &= T_\alpha\left(e^{x_0\frac{t^\alpha}{\alpha}}cos\left(y_0\frac{t^\alpha}{\alpha}\right)\right) + iT_\alpha\left(e^{x_0\frac{t^\alpha}{\alpha}}sin\left(y_0\frac{t^\alpha}{\alpha}\right)\right) \\ &= \left(x_0e^{x_0\frac{t^\alpha}{\alpha}}cos\left(y_0\frac{t^\alpha}{\alpha}\right) - y_0e^{x_0\frac{t^\alpha}{\alpha}}sin\left(y_0\frac{t^\alpha}{\alpha}\right)\right) \\ &+ i\left(x_0e^{x_0\frac{t^\alpha}{\alpha}}sin\left(y_0\frac{t^\alpha}{\alpha}\right) + y_0e^{x_0\frac{t^\alpha}{\alpha}}cos\left(y_0\frac{t^\alpha}{\alpha}\right)\right) \\ &= x_0e^{z_0\frac{t^\alpha}{\alpha}} + iy_0e^{z_0\frac{t^\alpha}{\alpha}}\end{aligned}$$

or

$$T_\alpha\left(e^{z_0\frac{t^\alpha}{\alpha}}\right) = e^{z_0\frac{t^\alpha}{\alpha}}$$

It is good to point out that not every rule for conformable fractional calculus can be valid for complex-valued functions. The following example illustrates this:

Example 3.2. Let $\alpha>0$. Suppose that $f(t)=u(t)+iv(t)$ is continuous on a closed interval $[a,b]$ such that its component functions $u(t)$ and $u(t)$ are continuous there. Even if $(T_\alpha f)(t)$ exists when $a<t<b$, Mean Value Theorem for Conformable Differentiable Functions is no longer applicable, [2]. To be precise, it is not necessarily true that there is a number c in the open interval (a,b) such that

$$(T_\alpha f)(c) = \frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}$$

To see this, let´s consider a function $f(t)=e^{i\frac{t^\alpha}{\alpha}}$ on the closed interval $\left[0,(2\pi\alpha)^{\frac{1}{\alpha}}\right]0$, for some $\alpha\in(0,1]$. By using that function, it is easy obtain $|(T_\alpha f)(t)| = \left|e^{i\frac{t^\alpha}{\alpha}}\right| = 1$, and this means that the conformable fractional derivative $(T_\alpha f)(t)$ is never zero, while

$$f\left((2\pi\alpha)^{\frac{1}{\alpha}}\right) - f(0) = 0$$

IV. Conformable Fractional Integral of Complex-Valued Functions of a Real Variable

Definition 4.1. Let $\alpha\in(0,1]$ and $0\le a<b$. Let $f(t)=u(t)+iv(t)$ where $u(t)$ and $v(t)$ are real-valued functions of the variable t, for $t\in[a,b]$. The α −fractional integral of f is defined as

$$\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt = \int_a^b \frac{u(t)}{t^{1-\alpha}}\cdot dt + i\int_a^b \frac{v(t)}{t^{1-\alpha}}\cdot dt \tag{15}$$

where there are the individual integrals on the right. So,

$$Re\left(\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt\right) = \int_a^b \frac{Re\big(f(t)\big)}{t^{1-\alpha}}\cdot dt$$

$$Im\left(\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt\right) = \int_a^b \frac{Im\big(f(t)\big)}{t^{1-\alpha}}\cdot dt$$

Remark 4.1.

$$I_{\alpha}^{a}(f)(t)=\int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}\cdot dx$$

$$I_{\alpha}^{a}(u)(t)=\int_{a}^{t}\frac{u(x)}{x^{1-\alpha}}\cdot dx$$

$$I_{\alpha}^{a}(v)(t)=\int_{a}^{t}\frac{v(x)}{x^{1-\alpha}}\cdot dx$$

where the integrals are the usual Riemann improper integral and $\alpha\in(0,1]$.

Remark 4.2. The usual rules of operations with $\alpha-$fractional integrals of real-valued functions carry over to complex-valued functions. We have for example

$$\int_{a}^{b}\frac{Kf(t)}{t^{1-\alpha}}\cdot dt=K\int_{a}^{b}\frac{f(t)}{t^{1-\alpha}}\cdot dt \tag{16}$$

$$\int_{a}^{b}\frac{(f(t)+g(t))}{t^{1-\alpha}}\cdot dt=\int_{a}^{b}\frac{f(t)}{t^{1-\alpha}}\cdot dt+\int_{a}^{b}\frac{g(t)}{t^{1-\alpha}}\cdot dt \tag{17}$$

where $K\in C$, $c\in(a,b)$ and f and g are $\alpha-$ fractional integrable functions on closed interval $[a,b]$.

Theorem 4.1. $T_{\alpha}\left(I_{\alpha}^{a}(f)(t)\right)=f(t)$, for all $t>a$, when f is any continuous complex- valued function in the domain of I_{α}^{a} and $\alpha\in(0,1]$.

Proof. Since f is continuous, then $I_{\alpha}^{a}(f)(t)$ is clearly differentiable, we have

$$\begin{aligned}T_{\alpha}(I_{\alpha}^{a}(f)(t))&=t^{1-\alpha}\frac{d}{dt}(I_{\alpha}^{a}(f)(t))=t^{1-\alpha}\frac{d}{dt}\left(\int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}\cdot dx\right)\\&=t^{1-\alpha}\frac{d}{dt}\left(\int_{a}^{t}\frac{u(x)}{x^{1-\alpha}}\cdot dx+i\int_{a}^{t}\frac{v(x)}{x^{1-\alpha}}\cdot dx\right)\\&=t^{1-\alpha}\left(\frac{d}{dt}\left(\int_{a}^{t}\frac{u(x)}{x^{1-\alpha}}\cdot dx\right)+i\frac{d}{dt}\left(\int_{a}^{t}\frac{v(x)}{x^{1-\alpha}}\cdot dx\right)\right)\\&=t^{1-\alpha}\left(\frac{u(t)}{t^{1-\alpha}}+i\frac{v(t)}{t^{1-\alpha}}\right)=u(t)+iv(t)=f(t)\end{aligned}$$

Now, a conformable version of the classical Second Fundamental Theorem of Calculus in the context of the valued-complex functions of a real variable is presented.

Theorem 4.2. Let $a>0$, $\alpha\in(0,1]$ and f be a continuous complex-valued function on interval $[a,b]$. Let G any complex-valued function with the property $T_{\alpha}(G)(t)=f(t)$ for all $t\in[a,b]$. Then

$$\int_{a}^{b}\frac{f(t)}{t^{1-\alpha}}\cdot dt=G(b)-G(a) \tag{19}$$

To prove Theorem 4.2, we need the following result:

Theorem 4.3. Let $a>0$, $\alpha\in(0,1]$ and f be a continuous real-valued function on interval $[a,b]$. Let G any real-valued function with the property $T_{\alpha}(G)(t)=f(t)$ for all $t\in[a,b]$. Then

$$\int_{a}^{b}\frac{f(t)}{t^{1-\alpha}}\cdot dt=G(b)-G(a) \tag{20}$$

Proof. First, let F a function on $[a,b]$ defined as $F(t)=\int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}\cdot dx$, which can be called $\alpha-$ *fractional integral function of* f.

By using theorem 2.5, $T_{\alpha}(F)(t)=f(t)$ for all $t\in[a,b]$.

Since F and G have the same fractional derivative, then by Corollary 2.2 there exists a real constant C such that $G(t)=F(t)+C$ for all $t\in[a,b]$.

Finally, $G(b)-G(a)$ is computed

$$\begin{aligned}&G(b)-G(a)\\&=(F(b)+C)-(F(a)+C)\\&=\int_{a}^{b}\frac{f(t)}{t^{1-\alpha}}\cdot dt-\int_{a}^{a}\frac{f(t)}{t^{1-\alpha}}\cdot dt\\&=\int_{a}^{b}\frac{f(t)}{t^{1-\alpha}}\cdot dt\end{aligned}$$

Proof of Theorem 4.2. Consider $f(t)=u(t)+iv(t)$ and $G(t)=U(t)+iV(t)$, for all $t\in[a,b]$. As $T_{\alpha}(G)(t)=f(t)$ for all $t\in[a,b]$.

By applying the above theorem, $U(t)$ and $V(t)$ can be written as follows:

$$U(t)=\int_{a}^{t}\frac{u(x)}{x^{1-\alpha}}\cdot dx+C_{1}$$

$$V(t)=\int_{a}^{t}\frac{v(x)}{x^{1-\alpha}}\cdot dx+C_{2}$$

for some real constants C_1 and C_2.

Finally, $G(b)-G(a)$ is computed as follows:

$$\begin{aligned}G(b)-G(a)&=\left(U(b)+iV(b)\right)-\left(U(a)+iV(a)\right)\\&=\left(U(b)-U(a)\right)+i\left(V(b)-V(a)\right)\\&=\int_{a}^{b}\frac{u(t)}{t^{1-\alpha}}\cdot dt+i\int_{a}^{b}\frac{v(t)}{t^{1-\alpha}}\cdot\\&=\int_{a}^{b}\frac{f(t)}{t^{1-\alpha}}\cdot dt\end{aligned}$$

Using all our definitions and results, the $\alpha-$fractional integrals according can be evaluated as follows:

Example 4.1. Evaluate the following integral

$$\int_{1}^{4}\left(\frac{1}{\sqrt{t}}-i\right)^{2}\cdot\frac{1}{\sqrt{t}}\cdot dt$$

Solution. We write the integrand in terms of its real and imaginary parts

$$f(t)=\left(\frac{1}{\sqrt{t}}-i\right)^{2}=\left(\frac{1}{t}-1\right)-i\frac{2}{\sqrt{t}}$$

Here $u(t)=\frac{1}{t}-1$ and $(t)=-\frac{2}{\sqrt{t}}$.

The $\frac{1}{2}-$fractional integrals of $u(t)$ and $v(t)$ can be written as

$$\int_1^4 \frac{u(t)}{\sqrt{t}}\cdot dt = \int_1^4 \left(\frac{1}{\sqrt{t^3}} - \frac{1}{\sqrt{t}}\right)\cdot dt = \left[-2\left(\frac{1}{\sqrt{t}} - \sqrt{t}\right)\right]_{t=1}^{t=4} = 1$$

and

$$\int_1^4 \frac{v(t)}{\sqrt{t}}\cdot dt = -2\int_1^4 \frac{1}{t}\cdot dt = -2[logt]_{t=1}^{t=4} = -log16$$

Therefore, by definition 4.1

$$\int_1^4 \left(\frac{1}{\sqrt{t}} - i\right)^2 \cdot \frac{1}{\sqrt{t}}\cdot dt = \int_1^4 \frac{u(t)}{\sqrt{t}}\cdot dt + i\int_1^4 \frac{v(t)}{\sqrt{t}}\cdot dt$$
$$= 1 - ilog16$$

Example 4.2. Evaluate the following $\frac{1}{3}-$fractional integral

$$\int_0^{\left(\frac{\pi}{3}\right)^3} \frac{e^{i2\sqrt[3]{t}}}{\sqrt[3]{t^2}}\cdot dt$$

Solution. We seek a function f with the property that $\left(T_{\frac{1}{3}}f\right)(t) = \frac{2}{3}ie^{i2\sqrt[3]{t}}$ - We note that $f(t) = \frac{3e^{i2\sqrt[3]{t}}}{2i}$ satisfies this requirement, so

$$\int_0^{\left(\frac{\pi}{3}\right)^3} \frac{e^{i2\sqrt[3]{t}}}{\sqrt[3]{t^2}}\cdot dt = \left[\frac{3e^{i2\sqrt[3]{t}}}{2i}\right]_{t=0}^{t=\left(\frac{\pi}{3}\right)^3} = \frac{3}{2i}\left(cos\frac{\pi}{3} + isin\frac{\pi}{3} - 1\right)$$
$$= \frac{3\sqrt{3}}{4} + i\frac{3}{4}$$

Let´s now introduce the following important property of moduli of $\alpha-$fractional integrals of the complex-valued functions of a real variable.

Theorem 4.4. Let $0 < a < b$ and $f: [a, b] \to C$ be continuous complex-valued function. Then for $\alpha \in (0,1]$

$$\left|\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt\right| \le \int_a^b \frac{|f(t)|}{t^{1-\alpha}}\cdot dt \tag{21}$$

Proof. If $\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt = 0$, the inequality is trivial.

If $\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt \ne 0$, the complex number $z = \int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt$ can be written in the form $\rho e^{i\theta}$, where $\rho = |z|$ and θ is an argument for z. By (17) then we have

$$\rho = ze^{-i\theta} = \int_a^b \frac{e^{-i\theta}f(t)}{t^{1-\alpha}}\cdot dt = \int_a^b \frac{Re\left(e^{-i\theta}f(t)\right)}{t^{1-\alpha}}\cdot dt$$

because

$$\int_a^b \frac{Im\left(e^{-i\theta}f(t)\right)}{t^{1-\alpha}}\cdot dt = 0$$

Since the integral of $e^{-i\theta}f(t)$ is the number real ≥ 0. Therefore, we have:

$$\left|\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt\right| \le \left|\int_a^b \frac{Re\left(e^{-i\theta}f(t)\right)}{t^{1-\alpha}}\cdot dt\right| \le \int_a^b \frac{\left|Re\left(e^{-i\theta}f(t)\right)\right|}{t^{1-\alpha}}\cdot dt \le$$
$$\int_a^b \frac{\left|e^{-i\theta}f(t)\right|}{t^{1-\alpha}}\cdot dt \le \int_a^b \frac{\left|e^{-i\theta}\right||f(t)|}{t^{1-\alpha}}\cdot dt \le \int_a^b \frac{|f(t)|}{t^{1-\alpha}}\cdot dt$$

where we used $Re(w) \le |Re(w)| \le |w|$ for every $w \in C$ and $\left|e^{-i\theta}\right| = 1$.

Remark 4.2. With only minor modification, the above discussion yields inequalities such that

$$\left|\int_a^{+\infty} \frac{f(t)}{t^{1-\alpha}}\cdot dt\right| \le \int_a^{+\infty} \frac{|f(t)|}{t^{1-\alpha}}\cdot dt \tag{22}$$

provided that both improper integrals exist.

Corollary 4.1. Let $0 < a < b$ and $f: [a, b] \to C$ be continuous complex-valued function such that

$$M = \max_{[a,b]}|f|$$

Then for $\alpha \in (0,1]$

$$\left|\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt\right| \le M\left(\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}\right) \tag{23}$$

Proof. From above theorem, we have the following for $\alpha \in (0,1]$:

$$\left|\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt\right| \le \int_a^b \frac{|f(t)|}{t^{1-\alpha}}\cdot dt \le M\int_a^b t^{\alpha-1}\cdot dt = M\left(\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}\right)$$

Remark 4.3. It is easy to propose the following conformable version of the classical Mean Value Theorem for definite real integrals.

Theorem 4.5. Let $a > 0$ and $f: [a, b] \to R$ be a given function that satisfies

- $\int_a^t \frac{f(x)}{x^{1-\alpha}}\cdot dx\ \forall t \in [a, b]$, exists for some $\alpha \in (0,1]$.
- $m \le f(t) \le M\ \forall t \in [a, b]$, for certain real numbers m and M.

Then there exists $\mu \in [m, M]$ such that

$$\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt = \frac{\mu}{\alpha}\left(\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}\right) \tag{24}$$

Proof. By the monotonicity property of the usual definite integrals, [19], we have

$$\frac{m}{\alpha}(b^\alpha - a^\alpha) = \int_a^b \frac{m}{t^{1-\alpha}}\cdot dt \le \int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt \le \int_a^b \frac{M}{t^{1-\alpha}}\cdot dt$$
$$\le \frac{M}{\alpha}(b^\alpha - a^\alpha)$$

Multiplying by $\frac{\alpha}{b^\alpha - a^\alpha}$, then $m \le \mu \le M$, where $\mu = \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \frac{f(t)}{t^{1-\alpha}}\cdot dt$. Hence, (24) is obtained and the proof is completed.

Corollary 4.2. If f is a continuous function on the closed interval $[a,b]$, then there exists $c \in [a,b]$ such that

$$\int_a^b \frac{f(t)}{t^{1-\alpha}} \cdot dt = \frac{f(c)}{\alpha}(b^\alpha - a^\alpha) \tag{25}$$

Proof. Simply apply the classical Maximum and Minimum Value Theorem, [23], to (24).

Finally, the following example we will show that theorem 4.5 cannot be applied to complex-valued functions of a real variable.

Example 4.3. Consider a function $f(t) = e^{i\frac{t^\alpha}{\alpha}}$ on the closed interval $\left[0,(2\pi\alpha)^{\frac{1}{\alpha}}\right]0$, for some $\alpha \in (0,1]$ and observe that

$$\int_0^{(2\pi\alpha)^{\frac{1}{\alpha}}} \frac{f(t)}{t^{1-\alpha}} \cdot dt = \int_0^{(2\pi\alpha)^{\frac{1}{\alpha}}} \frac{e^{i\frac{t^\alpha}{\alpha}}}{t^{1-\alpha}} \cdot dt \left[\frac{e^{i\frac{t^\alpha}{\alpha}}}{i}\right]_0^{(2\pi\alpha)^{\frac{1}{\alpha}}} = 0$$

Since $\left|f(c)\left((2\pi\alpha)^{\frac{1}{\alpha}} - 0\right)\right| = \left|e^{i\frac{c^\alpha}{\alpha}}\right|(2\pi\alpha)^{\frac{1}{\alpha}} = (2\pi\alpha)^{\frac{1}{\alpha}}$ for every real number c, it is clear that there is no number c in the closed $\left[0,(2\pi\alpha)^{\frac{1}{\alpha}}\right]$ such that

$$\int_0^{(2\pi\alpha)^{\frac{1}{\alpha}}} \frac{e^{i\frac{c^\alpha}{\alpha}}}{t^{1-\alpha}} \cdot dt = f(c)\left((2\pi\alpha)^{\frac{1}{\alpha}} - 0\right)$$

V. Complex conformable integral

In this section, using the results above, we will introduce integrals of complex-valued functions of a complex variable in the sense of conformable definition.

First, let us recall some classical notions about classes of curves that are adequate for the study of these integrals, [24].

Remark 5.1.

(i) A continuous mapping $\gamma:[a,b] \to C$ is called a curve in C, with parameter interval $[a,b]$. The point set $\gamma([a,b])$ is denoted γ^*. The points $\gamma(a)$ and $\gamma(b)$ are called the starting point and end point, respectively. The curve γ is called closed if $\gamma(a) = \gamma(b)$.

(ii) A curve $\gamma:[a,b] \to C$ is called smooth if $\gamma \in C^1([a,b])$ and $\gamma'(t) \neq 0\ \forall t \in [a,b]$.

(iii) A contour, or piecewise curve, is a curve consisting of a finite number of smooth curves joined end to end.

Remark 5.2. *Operations with curves*

(i) Let $\gamma:[a,b] \to C$ and $\sigma:[c,d] \to C$ be two curves. We will say that the curves γ and σ are equivalent if there exists a differentiable function $\varphi:[c,d] \to [a,b]$ such that
- $\varphi'(u) > 0$
- $\varphi(c) = a, \varphi(d) = b$
- $\gamma \circ \varphi = \sigma$

(ii) Let $\gamma:[a,b] \to C$ be a curve. Then the curve $-\gamma:[a,b] \to C$ such that $-\gamma(t) = \gamma(b+a-t)$, denote the reverse curve of γ.

(iii) Let $\gamma:[a,b] \to C$ and $\sigma:[c,d] \to C$ be two curves, with $\gamma(b) = \sigma(c)$. We can define a new curve called the sum of its legs γ and σ, and is dented by $\gamma + \sigma$, given by

$$(\gamma+\sigma)(t) = \begin{cases} \gamma(t) & a \le t \le b \\ \sigma(c-b+t) & b \le t \le b+d-c \end{cases}$$

In the following results, we consider a contour $\gamma:[a,b] \to C$, with $a > 0$ and $\gamma^* \subset C - (-\infty. 0]$.

Definition 5.1. Let $\gamma:[a,b] \to C$ be a contour and let $f:\gamma^* \to C$ be continuous. We define the contour $\alpha -$ integral of f along γ as soon as we understand the complex number

$$\int_\gamma f(z)\frac{dz}{z^{1-\alpha}} = \int_a^b f(\gamma(t))\frac{(T_\alpha\gamma)(t)}{(\gamma(t))^{1-\alpha}}\frac{dt}{t^{1-\alpha}} \tag{26}$$

Note that, since γ is a contour, $(T_\alpha\gamma)(t)$ is also piecewise continuous on the interval $[a,b]$, and so the existence of integral (26) is ensured.

From the definition (26) and the properties of the integrals of the complex-valued functions of a real variable studied in section 4, the following results can be easily established.

Theorem 5,1. Let $\gamma:[a,b] \to C$ be a contour and let $f,g:\gamma^* \to C$ be continuous. Then

$$\int_\gamma (\beta f + \lambda g)(z)\frac{dz}{z^{1-\alpha}} = \beta\int_\gamma f(z)\frac{dz}{z^{1-\alpha}} + \lambda\int_\gamma g(z)\frac{dz}{z^{1-\alpha}}$$

for any complex constants β and λ.

Theorem 5.2. Let $\gamma:[a,b] \to C$ and $\sigma:[c,d] \to C$ be two equivalent contours and let $f:\gamma^* \to C$ be continuous. Then

$$\int_\gamma f(z)\frac{dz}{z^{1-\alpha}} = \int_\sigma f(z)\frac{dz}{z^{1-\alpha}}$$

Proof. Since $\gamma^* = \sigma^*$, the contour $\alpha -$ integral of f along σ is well-defined. By hypothesis, there is a differentiable function $\varphi:[c,d] \to [a,b]$ such that $\varphi'(u) > 0$ and $\gamma \circ \varphi = \sigma$. So,

$$\int_\gamma f(z)\frac{dz}{z^{1-\alpha}} = \int_a^b f(\gamma(t))\frac{(T_\alpha\gamma)(t)}{(\gamma(t))^{1-\alpha}}\frac{dt}{t^{1-\alpha}} = \left\{\begin{matrix} t = \varphi(s) \\ dt = (T_\alpha\varphi)(s)\frac{ds}{s^{1-\alpha}} \end{matrix}\right\}$$
$$= \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f\left(\gamma(\varphi(s))\right)\frac{(T_\alpha\gamma)(\varphi(s))}{\left(\gamma(\varphi(s))\right)^{1-\alpha}}\frac{(T_\alpha\varphi)(s)}{(\varphi(s))^{1-\alpha}}\frac{ds}{s^{1-\alpha}}$$
$$= \int_c^d f(\sigma(s))\frac{(T_\alpha\sigma)(s)}{(\sigma(s))^{1-\alpha}}\frac{ds}{s^{1-\alpha}} = \int_\sigma f(z)\frac{dz}{z^{1-\alpha}}$$

Theorem 5.3. Let $\gamma:[a,b]\to C$ and $\sigma:[c,d]\to C$ be two contours, with $\gamma(b)=\sigma(c)$, and let $f:\gamma^*\cup\sigma^*\to C$ be continuous. Then

$$\int_{\gamma+\sigma} f(z)\frac{dz}{z^{1-\alpha}}=\int_{\gamma} f(z)\frac{dz}{z^{1-\alpha}}+\int_{\sigma} f(z)\frac{dz}{z^{1-\alpha}}$$

Proof. From Remark 5.1 (iii), we have

$$\int_{\gamma+\sigma} f(z)\frac{dz}{z^{1-\alpha}}=\int_a^{b+d-c} f\big((\gamma+\sigma)(t)\big)\frac{\big(T_\alpha(\gamma+\sigma)\big)(t)}{\big((\gamma+\sigma)(t)\big)^{1-\alpha}}\frac{dt}{t^{1-\alpha}}$$
$$=\int_a^b f\big(\gamma(t)\big)\frac{(T_\alpha\gamma)(t)}{\big(\gamma(t)\big)^{1-\alpha}}\frac{dt}{t^{1-\alpha}}$$
$$+\int_a^{b+d-c} f\big(\sigma(t-b+c)\big)\frac{(T_\alpha\sigma)(t-b+c)}{\big(\sigma(t-b+c)\big)^{1-\alpha}}\frac{dt}{(t-b+c)^{1-\alpha}}$$
$$=\left\{\begin{matrix} s=t-b+c \\ ds=dt\end{matrix}\right\}$$
$$=\int_a^b f(\gamma)\frac{(T_\alpha\gamma)(t)}{\big(\gamma(t)\big)^{1-\alpha}}\frac{dt}{t^{1-\alpha}}+\int_c^d f\big(\sigma(s)\big)\frac{(T_\alpha\sigma)(s)}{\big(\sigma(s)\big)^{1-\alpha}}\frac{dt}{s^{1-\alpha}}$$
$$=\int_{\gamma} f(z)\frac{dz}{z^{1-\alpha}}+\int_{\sigma} f(z)\frac{dz}{z^{1-\alpha}}$$

Theorem 5.4. Let $-\gamma$ denote the reverse contour of γ. Then

$$\int_{-\gamma} f(z)\frac{dz}{z^{1-\alpha}}=-\int_{\gamma} f(z)\frac{dz}{z^{1-\alpha}}$$

Proof. From Remark 5.1 (ii), we have

$$\int_{-\gamma} f(z)\frac{dz}{z^{1-\alpha}}=\int_a^b f\big(-\gamma(t)\big)\frac{(T_\alpha-\gamma)(t)}{\big(-\gamma(t)\big)^{1-\alpha}}\frac{dt}{t^{1-\alpha}}$$
$$=\int_a^b f\big(\gamma(b+a-t)\big)\frac{(T_\alpha\gamma)(b+a.t)}{\big(\gamma(b+a-t)\big)^{1-\alpha}}\frac{dt}{(b+a.-t)^{1-\alpha}}$$
$$=\left\{\begin{matrix} s=b+a-t \\ ds=dt\end{matrix}\right\}$$
$$=\int_b^a f\big(\gamma(s)\big)\frac{(T_\alpha\gamma)(s)}{\big(\gamma(s)\big)^{1-\alpha}}\frac{dt}{s^{1-\alpha}}=-\int_{\gamma} f(z)\frac{dz}{z^{1-\alpha}}$$

VI. Conclusions

New interesting results for fractional formulations of complex-valued functions of a real variable in the sense of conformable derivatives and integrals have been successfully proposed in this research article. Thus, a future work is opened to construct the theory of conformable integration by studying the functions of a complex variable due to importance of this research study and its applications in the field of natural sciences or engineering.

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