Differences Inequalities of General \(L_p\)-Mixed Brightness Integrals

Jinsheng Guo\(^*\)

Abstract—Lutwak introduced the mixed brightness for convex bodies. After, Li and Zhu put forward mixed-brightness integrals. Recently, Yan and Wang defined the general \(L_p\)-mixed brightness integrals. In this article, we establish the Brunn-Minkowski, new cycle and Aleksandrov-Fenchel type inequalities for the differences of general \(L_p\)-mixed brightness integrals.

Index Terms—general \(L_p\)-mixed brightness integrals, Brunn-Minkowski type inequality, new cycle type inequality, Aleksandrov-Fenchel type inequality.

I. INTRODUCTION

Let \(K^n\) denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space \(R^n\). For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in \(R^n\), we write \(K^o\) and \(K^os\), respectively. Let \(S^{n-1}\) denote the unit sphere and \(V(M)\) denote the \(n\)-dimensional volume of the body \(M\). For the centered unit ball \(B\), write \(V(B) = \omega_n\).

The projection bodies were introduced by Minkowski at the turn of the previous century. For each \(M \in K^n\), the projection body, \(PM, \) of \(M\) is an origin-symmetric convex body whose support function is defined by (see [3], [18])

\[ h(\Pi M, u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| dS(M, v), \]

for all \(u \in S^{n-1}\). Here \(S(M, \cdot)\) denotes the surface area measure of \(M\).

Lutwak first introduced the notion of the mixed brightness of convex bodies in [15]. After, associated with the notion of the projection bodies and the mixed brightness, Li and Zhu [12] introduced the notion of mixed brightness integrals and given the \(L_p\)-mixed brightness integrals, moreover, they also established analogous to the Fenchel-Aleksandrov inequality and isoperimetric inequality of the mixed brightness integrals for the mixed volumes. For the mixed brightness integrals, Zhao [24] established the greatest upper bound for the product of the mixed brightness integrals of a convex body and its polar dual. After, Zhou, Wang and Feng [27] obtained some Brunn-Minkowski type inequalities for the mixed brightness integrals. Recently, Li et al. firstly introduced the notion of mixed complex brightness integrals [10] and dual mixed complex brightness integrals [11], they extended the classical concepts of mixed brightness integrals in real vector space to complex cases.

In 2005, Ludwig ([13]) combined with a function \(\varphi_\tau : \mathbb{R} \rightarrow [0, +\infty)\) by \(\varphi_\tau(t) = |t| + \tau t, \tau \in [-1, 1]\), introduced general \(L_p\)-projection bodies as follows: For \(M \in K^n_0\), \(p \geq 1\) and \(\tau \in [-1, 1]\), the general \(L_p\)-projection body \(\Pi^\tau_pM \in K^n_o\) is defined by

\[ h^p(\Pi^\tau_pM, u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p dS_p(M, v), \]

where

\[ \alpha_{n,p}(\tau) = \frac{2 \alpha_{np}}{(1 + \tau)^p + (1 - \tau)^p}. \]

The normalization is chosen such that \(\Pi^0_p B = B\). Obviously, \(\Pi^0_p M = \Pi_p M\).

Recently, using the general \(L_p\)-projection bodies, Yan and Wang [22] defined the general \(L_p\)-mixed brightness integrals as follows: For \(M_1, \ldots, M_n \in K^n_0\), \(p \geq 1\) and \(\tau \in [-1, 1]\), the general \(L_p\)-mixed brightness integrals, \(D^{(\tau)}_p(M_1, \ldots, M_n)\), of \(M_1, \ldots, M_n\) is defined by

\[ D^{(\tau)}_p(M_1, \ldots, M_n) = \frac{1}{n} \int_{S^{n-1}} \delta^{(\tau)}_p(M_1, u) \cdots \delta^{(\tau)}_p(M_n, u) dS(u), \quad (1.1) \]

where \(\delta^{(\tau)}_p(M, u) = \frac{1}{2} h^p(\Pi^\tau_p M, u)\) denotes the half general \(L_p\)-brightness of \(M \in K^n_0\) in the direction \(u\). Convex bodies \(M_1, \ldots, M_n\) are said to have similar general \(L_p\)-brightness if there exist constants \(\lambda_1, \ldots, \lambda_n > 0\) such that, for all \(u \in S^{n-1}\),

\[ \lambda_1 \delta^{(\tau)}_p(M_1, u) = \lambda_2 \delta^{(\tau)}_p(M_2, u) = \cdots = \lambda_n \delta^{(\tau)}_p(M_n, u). \]

Obviously, for \(\tau = 0\) and \(p = 1\), (1.1) is just the mixed brightness integrals \(D(M_1, \ldots, M_n)\).

Let \(M_i = \cdots = M_{i-1} = M, M_{i+1} = \cdots = M_n\), \(N (i = 0, 1, \ldots, n) \) in (1.1), we write \(D^{(\tau)}_{p,i}(M, N) = D^{(\tau)}_{p,i}(M, \ldots, M, N, \ldots, N)\). More general, if allow \(i\) is any real, for \(M, N \in K^n_0\), \(p \geq 1\), and \(\tau \in [-1, 1]\), the general \(L_p\)-mixed brightness integrals, \(D^{(\tau)}_{p,i}(M, N)\), of \(M\) and \(N\) is defined by

\[ D^{(\tau)}_{p,i}(M, N) = \frac{1}{n} \int_{S^{n-1}} \delta^{(\tau)}_p(M, u)^{n-i} \delta^{(\tau)}_p(N, u)^i. \quad (1.2) \]

For \(N = B\) in (1.2), we write \(D^{(\tau)}_{p,i}(M, B) = \frac{1}{2} D^{(\tau)}_{p,i}(M)\) and notice that \(\delta^{(\tau)}_p(B, u) = \frac{1}{2} h(\Pi^\tau_p B, u) = \frac{1}{2}\), for all \(u \in S^{n-1}\), which together with (1.2) yields

\[ D^{(\tau)}_{p,i}(M) = \frac{1}{2^{n-1}} \int_{S^{n-1}} \delta^{(\tau)}_p(M, u)^{n-i} dS(u), \quad (1.3) \]

where \(D^{(\tau)}_{p,i}(M)\) is called the \(i\)-th general \(L_p\)-mixed brightness integrals of \(M\).

* Jinsheng Guo is corresponding author with the School of Mathematics and Statistics, Hexi University, Zhangye, 734000, China, e-mail: guojinsheng1979@163.com.
For $N = M$ in (1.2), write $D_p^{(r)}(M, M) = D_p^{(r)}(M)$, which is called the general $L_p$-brightness integrals of $M$. Clearly,

$$D_p^{(r)}(M) = \frac{1}{n} \int_{S^{n-1}} \delta_p^{(r)}(M, u)^n dS(u), \quad (1.4)$$

From (1.2) and (1.4), we easily obtain

$$D_p^{(r)}(M, N) = D_p^{(r)}(M), D_p^{(r)}(M, N) = D_p^{(r)}(N). \quad (1.5)$$

For general $L_p$-mixed brightness integrals, Yan and Wang [22] also established the following cycle, Brunn-Minkowski and Aleksandrov-Fenchel type inequalities.

**Theorem 1A.** If $M, N \in K^n_o, p \geq 1, \tau \in [-1, 1], \text{ and } i \in \mathbb{R}$, then for $i < n - p$,

$$D_p^{(r)}(\lambda \circ M + \mu \circ N)^{p/i} \leq \lambda D_p^{(r)}(M)^{p/i} + \mu D_p^{(r)}(N)^{p/i}, \quad (1.6)$$

for $i > n - p$ and $i \neq n$, then

$$D_p^{(r)}(\lambda \circ M + \mu \circ N)^{p/i} \geq \lambda D_p^{(r)}(M)^{p/i} + \mu D_p^{(r)}(N)^{p/i}, \quad (1.7)$$

in each case, equality holds if and only if $M$ and $N$ have similar general $L_p$-brightness. For $i = n - p$, equality always holds in (1.6) and (1.7).

**Theorem 1B.** If $M, N \in K^n_o, p \geq 1, \tau \in [-1, 1], \text{ and } i, j, k \in \mathbb{R}$ such that $i < j < k$, then

$$D_p^{(r)}(M, N)^{k-i} \leq D_p^{(r)}(M, N)^{k-j} D_p^{(r)}(M, N)^{j-i}, \quad (1.8)$$

with equality if and only if $M$ and $N$ have similar general $L_p$-brightness.

**Theorem 1C.** If $M_1, \ldots, M_n \in K^n_o, p \geq 1, \tau \in [-1, 1] \text{ and } 1 \leq m \leq n$, then

$$D_p^{(r)}(M_1, \ldots, M_n)^m \leq \prod_{i=1}^m D_p^{(r)}(M_1, \ldots, M_{n-m}, M_{n-m+1}, \ldots, M_{n-1}), \quad (1.9)$$

with equality if and only if $M_{n-m+1}, \ldots, M_n$ are all of similar general $L_p$-brightness.

The general $L_p$-mixed brightness integrals belong to a new and rapidly evolving asymmetric $L_p$ Brunn-Minkowski theory that has its origins in the work of Ludwig et al. (see [14], [5], [6], [41]). For the further recent research of asymmetric $L_p$ Brunn-Minkowski theory, also see [22], [9], [19], [20], [21], [23].

In 2004, Leng [8] defined the volume differences function of convex bodies $D$ and $K$, where $D \subseteq K$, by

$$D_p(K, D) = V(K) - V(D).$$

Meanwhile, Leng [8] established the following Brunn-Minkowski type inequality for volume difference functions.

**Theorem 1D.** If $K, L$ and $D$ are compact domains, $D \subseteq K$, $D' \subseteq L$, $D'$ is a homothetic copy of $D$, then

$$(V(K + L) - V(D + D'))^\frac{1}{2} \geq (V(K) - V(D))^{\frac{1}{2}} + (V(L) - V(D'))^{\frac{1}{2}},$$

equality holds if and only if $K$ and $L$ are homothetic and $(V(K), V(D)) = \mu(V(L), V(D'))$, where $\mu$ is a constant. Here “$+$” is Minkowski sum.

Since these seminal paper, inequalities for differences of geometric functionals have become the focus of increased attention (see [16], [25], [26]).

The aim of this paper is to establish the new differences inequalities for general $L_p$-mixed brightness integrals. First, we establish the following Brunn Minkowski type inequality for differences of general $L_p$-mixed brightness integrals.

**Theorem 1.1.** Let $M, N, K, L \in K^n_o, \tau \in [-1, 1]$ and $\delta_p^{(r)}(K, \cdot) < \delta_p^{(r)}(M, \cdot)$, $\delta_p^{(r)}(L, \cdot) < \delta_p^{(r)}(N, \cdot)$, $M$ and $N$ have similar general $L_p$-brightness, if $p \geq 1$ and $i < n-p$,

$$[D_p^{(r)}(M + p) - D_p^{(r)}(N)]^{p/i}, \quad (1.10)$$

$$\geq [D_p^{(r)}(M) - C_p^{(r)}(K)]^{p/i} + [D_p^{(r)}(N) - D_p^{(r)}(L)]^{p/i};$$

if $i > n - p$ and $i \neq n$, then

$$[D_p^{(r)}(M + p) - D_p^{(r)}(N)]^{p/i}, \quad (1.11)$$

$$\leq [D_p^{(r)}(M) - C_p^{(r)}(K)]^{p/i} + [D_p^{(r)}(N) - D_p^{(r)}(L)]^{p/i},$$

in each case, with equality if and only if $K$ and $L$ have similar general $L_p$-brightness and there exists constant $\lambda$ such that $(D_p^{(r)}(M), D_p^{(r)}(N), D_p^{(r)}(L)) = \lambda(D_p^{(r)}(N), D_p^{(r)}(L))$.

Next, we give the following new cycle type inequalities for the differences of general $L_p$-mixed brightness integrals.

**Theorem 1.2.** Let $M, N, K, L \in K^n_o, \tau \in [-1, 1]$ and $\delta_p^{(r)}(K, \cdot) < \delta_p^{(r)}(M, \cdot)$, $\delta_p^{(r)}(L, \cdot) < \delta_p^{(r)}(N, \cdot)$, $M$ and $N$ have similar general $L_p$-brightness, if $p \geq 1, i, j, k \in \mathbb{R}$ and $0 \leq i < j < k$,

$$[D_p^{(r)}(M, N) - D_p^{(r)}(K, L)]^{k-i}, \quad (1.12)$$

$$\geq [D_p^{(r)}(M, N) - D_p^{(r)}(K, L)]^{k-j} [D_p^{(r)}(M, N) - D_p^{(r)}(K, L)]^{j-i},$$

with equality if and only if $K$ and $L$ have similar general $L_p$-brightness and there exists constant $\lambda$ such that $(D_p^{(r)}(M, N), D_p^{(r)}(K, L), D_p^{(r)}(K, L)) = \lambda(D_p^{(r)}(M, N), D_p^{(r)}(K, L))$.

Finally, we also establish the following Aleksandrov-Fenchel type inequalities for differences of general $L_p$-mixed brightness integrals.

**Theorem 1.3.** Let $p \geq 1, \tau \in [-1, 1], M_i, N_i \in K^n_o (i = 1, \ldots, n)$, if $\delta_p^{(r)}(N_i, \cdot) < \delta_p^{(r)}(M_i, \cdot) (i = 1, \ldots, n)$ and the bodies $M_{n-m+1}, \ldots, M_n$ are all of similar general $L_p$-brightness, then for every $1 < m \leq n$,

$$[D_p^{(r)}(M_1, \ldots, M_n) - C_p^{(r)}(N_1, \ldots, N_n)]^m \quad (1.13)$$

$$\geq \prod_{i=1}^m [D_p^{(r)}(M_1, \ldots, M_{n-m}, M_{n-m+1}, \ldots, M_{n-1}),$$

$$-D_p^{(r)}(N_1, \ldots, N_{n-m}, N_{n-m+1}, \ldots, N_{n-1})],$$

with equality if and only if $N_{n-m+1}, \ldots, N_n$ are all of similar general $L_p$-brightness and there exists constant $\lambda$ such that

$$[D_p^{(r)}(M_1, \ldots, M_{n-m}, M_{n-m+1}, \ldots, M_{n-1}) - C_p^{(r)}(N_1, \ldots, N_{n-m}, N_{n-m+1}, \ldots, N_{n-1})] = \lambda[D_p^{(r)}(N_1, \ldots, N_{n-m}, N_{n-m+1}, \ldots, N_{n-1})].$$
II. PRELIMINARIES

A. Support function

For $M \in K_\mathbb{R}^n$, its support function, (see [3], [18]) $h(M, \cdot) : \mathbb{R}^n \to \mathbb{R}$, is defined by

$$h(M, x) = \max \{ x \cdot y : y \in M \}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of $x$ and $y$.

B. $L_p$-Blaschke combination

For $M, N \in K_\mathbb{R}^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the $L_p$-Blaschke combination, $\lambda \circ M \equiv_p \mu \circ N \in K_\mathbb{R}^n$, of $M$ and $N$ is defined by (see [3], [18]),

$$dS_p(\lambda \circ M \equiv_p \mu \circ N, \cdot) = \lambda dS_p(M, \cdot) + \mu dS_p(N, \cdot),$$

where $\lambda \circ M \equiv_p \mu \circ N$ denotes the Blaschke scalar multiplication. If $p = 1$, then $\lambda \circ M \equiv_p \mu \circ N$ is classical Blaschke combination.

III. RESULTS AND PROOFS

In this part, we will give the proofs of Theorems 1.1-1.3. First, in order to prove Theorem 1.1, the following lemma is required.

**Lemma 3.1 (Bellman’s inequality [1]).** Let $a = \{a_1, \ldots, a_n\}$ and $b = \{b_1, \ldots, b_n\}$ be two series of positive real numbers. If $a_i > \sum_{i=2}^{n} a_i^p > 0$, $b_i > \sum_{i=2}^{n} b_i^p > 0$, then for $p > 1$,

$$\left( a_i^p - \sum_{i=2}^{n} a_i^p \right)^\frac{1}{p} + \left( b_i^p - \sum_{i=2}^{n} b_i^p \right)^\frac{1}{p} \leq \left( (a_1 + b_1)^p - \sum_{i=2}^{n} (a_i + b_i)^p \right)^\frac{1}{p},$$

(3.1)

for $p < 0$ or $0 < p < 1$,

$$\left( a_i^p - \sum_{i=2}^{n} a_i^p \right)^\frac{1}{p} + \left( b_i^p - \sum_{i=2}^{n} b_i^p \right)^\frac{1}{p} \geq \left( (a_1 + b_1)^p - \sum_{i=2}^{n} (a_i + b_i)^p \right)^\frac{1}{p},$$

(3.2)

with equality if and only if $a = cb$, where $c$ is a constant.

**Proof of Theorem 1.1.** For $M, N, K, L \in K_\mathbb{R}^n$, $p \geq 1$ and $\tau \in (-1, 1)$, if $i < n - p$ and $\lambda, \mu = 1$, by (1.6), then

$$D_{p,\tau}(K) \equiv_p L \leq D_{p,\tau}(K) \equiv_p + D_{p,\tau}(L) \equiv_p,$$

(3.3)

with equality if and only if $K$ and $L$ have similar general $L_p$-brightness. Since $M$ and $N$ have similar general $L_p$-brightness, thus according to the equality condition of inequality (1.6), we have

$$D_{p,\tau}(M \equiv_p N) \equiv_p = D_{p,\tau}(M) \equiv_p + D_{p,\tau}(N) \equiv_p.$$  

(3.4)

Since $\delta_{p,\tau}(K, \cdot) < \delta_{p,\tau}(M, \cdot)$, $\delta_{p,\tau}(L, \cdot) < \delta_{p,\tau}(N, \cdot)$, by (1.3), we obtain

$$D_{p,\tau}(K) \equiv_p > D_{p,\tau}(M), \quad D_{p,\tau}(N) \equiv_p > D_{p,\tau}(L).$$

From these, notice that $n - i > p$ ($i < n - p$), i.e. $\frac{n-i}{p}$, and from (3.1), (3.3) and (3.4), we obtain

$$\left[ \left( D_{p,\tau}(M) \equiv_p + D_{p,\tau}(N) \equiv_p \right) \right]^{\frac{n-i}{p}} \geq \left( D_{p,\tau}(K) \equiv_p + D_{p,\tau}(L) \equiv_p \right)^{\frac{n-i}{p}} + \left( D_{p,\tau}(M) \equiv_p + D_{p,\tau}(N) \equiv_p \right)^{\frac{n-i}{p}} + \left( D_{p,\tau}(K) \equiv_p + D_{p,\tau}(L) \equiv_p \right)^{\frac{n-i}{p}},$$

with equality if and only if $K$ and $L$ have similar general $L_p$-brightness. Since $M$ and $N$ have similar general $L_p$-brightness, thus according to the equality condition of inequality (1.8), we have

$$D_{p,\tau}(M \equiv_p N) \equiv_p = D_{p,\tau}(M, N) \equiv_p - D_{p,\tau}(M, K, L) \equiv_p,$$

(3.7)

Hence, by (3.6) and (3.7), we get

$$D_{p,\tau}(M, N) - D_{p,\tau}(K, L) \equiv_p,$$

(3.8)

$$D_{p,\tau}(M, N) \equiv_p \geq D_{p,\tau}(K, L) \equiv_p + D_{p,\tau}(M) \equiv_p + D_{p,\tau}(N) \equiv_p - D_{p,\tau}(M, K, L) \equiv_p,$$

with equality if and only if $K$ and $L$ have similar general $L_p$-brightness.

Notice that

$$\delta_{p,\tau}(K, \cdot) < \delta_{p,\tau}(M, \cdot),$$

and

$$\delta_{p,\tau}(L, \cdot) < \delta_{p,\tau}(N, \cdot).$$
Let $\tau$ be constant. We see that equality holds in (1.12) if and only if $\tau = 1$, thus according to (3.5) and (3.8), we have

$$(D_p(\tau)(M, N) - D_p(\tau)(K, L))^{n-i} \geq \left(\frac{D_p(\tau)(M, N) - D_p(\tau)(K, L)}{D_p(\tau)(M, N) - D_p(\tau)(K, L)}\right)^{n-i}.$$

This is just (1.12).

By the equality conditions of inequalities (3.8) and (3.5), we see that equality holds in (1.12) if and only if $K$ and $L$ have similar general $L_p$-brightness and there exists constant $\lambda$ such that $(D_p(\tau)(M, N), D_p(\tau)(K, L)) = \lambda(D_p(\tau)(M, N), D_p(\tau)(K, L))$.

Furthermore, let $p = 1$ and $\tau = 0$ in Theorem 1.2, the following result is obtained.

**Corollary 3.2.** Let $M, N, K, L \in \mathcal{K}_p^0$, $\delta(K, L) \leq \delta(M, N)$, $\delta(L, N) \leq \delta(M, K)$, and $M$ and $N$ have similar brightness, if $i, j, k \in \mathbb{R}$ and $0 \leq i < j < k$, then

$$(D_p(\tau)(M, N) - D_p(\tau)(K, L))^{k-i}$$

with equality if and only if $K$ and $L$ have similar brightness and exists constant $\lambda$ such that $(D_p(\tau)(M, N), D_p(\tau)(K, L)) = \lambda(D_p(\tau)(M, N), D_p(\tau)(K, L))$.

In particular, if $N = L = B$ in Theorem 1.2, by (1.3) and (1.12), the following result is obvious.

**Corollary 3.3.** Let $M, N, K, L \in \mathcal{K}_p^0$, $\tau \in [-1, 1]$ and $\delta_p^\tau(M, N) \leq \delta_p^\tau(K, L)$, $\delta_p^\tau(L, N) \leq \delta_p^\tau(M, K)$, $M$ and $N$ have constant general $L_p$-brightness, if $p > 0$, $i, j, k \in \mathbb{R}$ and $0 \leq i < j < k$, then

$$(D_p(\tau)(M, N) - D_p(\tau)(K, L))^{k-i}$$

with equality if and only if $K$ and $L$ have constant general $L_p$-brightness.

Specially, if $i = 0$, $k = n$ in (1.12), by (1.5), we have the following cycle Minkowski type inequality for the differences of general $L_p$-mixed brightness integrals.

**Corollary 3.4.** Let $M, N, K, L \in \mathcal{K}_p^0$, $\tau \in [-1, 1]$, $\delta_p^\tau(K, L) \leq \delta_p^\tau(M, N)$, $\delta_p^\tau(L, N) \leq \delta_p^\tau(M, K)$, $M$ and $N$ have similar general $L_p$-brightness, if $p > 0$, $i \in \mathbb{R}$ and $0 < i < n$, then

$$(D_p(\tau)(M, N) - D_p(\tau)(L, K))^n$$

with equality if and only if $K$ and $L$ have similar general $L_p$-brightness and exists constant $\lambda$ such that $(D_p(\tau)(M, N), D_p(\tau)(L, K)) = \lambda(D_p(\tau)(M, N), D_p(\tau)(L, K))$.

Finally, according to the following Lemma, we prove the proof of Theorem 1.3.

**Lemma 3.3** ([7]). If $c_i > 0$, $b_i > 0$, $c_i > b_i$, $i = 1, \ldots, n$, then

$$\left(\prod_{i=1}^n (c_i - b_i)\right)^\frac{1}{n} \leq \left(\prod_{i=1}^n c_i\right)^\frac{1}{n} - \left(\prod_{i=1}^n b_i\right)^\frac{1}{n}.$$  \hfill (3.9)

with equality if and only if $\frac{c_i}{b_i} = \frac{c_j}{b_j} = \cdots = \frac{c_n}{b_n}$.

**Proof of Theorem 1.3.** From (1.9), we have

$$D_p(\tau)(N_1, \ldots, N_m)$$

with equality if and only if $N_{n-m+1}, \ldots, N_n$ are all of similar general $L_p$-brightness. Since the body $M_{n-m+1}, \ldots, M_n$ are all of similar general $L_p$-brightness, by (1.9), then

$$D_p(\tau)(M_1, \ldots, M_n) = \prod_{i=1}^m D_p(\tau)(M_1, \ldots, M_{n-m}, M_{n-m-i+1}, \ldots, M_{n-i}).$$

**Notice** that if $\delta_p^\tau(N_i, \cdot) < \delta_p^\tau(M_i, \cdot)$ ($i = 1, \ldots, n$), by (1.1), we get

$$D_p(\tau)(M_1, \ldots, M_n) > D_p(\tau)(N_1, \ldots, N_n).$$

Taking $M_{n-m+1} = \cdots = M_n = M_{n-i+1}, N_{n-m+1} = \cdots = N_n = N_{n-i+1}$ in (3.12), we obtain

$$D_p(\tau)(M_1, \ldots, M_{n-m}, M_{n-m-i+1}, \ldots, M_{n-i}) > D_p(\tau)(N_1, \ldots, N_{n-m}, N_{n-m-i+1}, \ldots, N_{n-i+1}).$$

From (3.10), (3.11) and (3.12), we obtain

$$D_p(\tau)(M_1, \ldots, M_n) - D_p(\tau)(N_1, \ldots, N_n) \geq \left(\prod_{i=1}^m D_p(\tau)(M_1, \ldots, M_{n-m}, M_{n-m-i+1}, \ldots, M_{n-i+1})\right)^\frac{1}{n}$$

with equality if and only if $N_{n-m+1}, \ldots, N_n$ are all of similar general $L_p$-brightness.

By (3.14), (3.13) and (3.9), we obtain

$$[D_p(\tau)(M_1, \ldots, M_n) - D_p(\tau)(N_1, \ldots, N_n)]^m$$

with equality if and only if $N_{n-m+1}, \ldots, N_n$ are all of similar general $L_p$-brightness and there exists constant $\lambda$ such that

$$[D_p(\tau)(M_1, \ldots, M_{n-m}, M_{n-m-i+1}, \ldots, M_{n-i+1}) - D_p(\tau)(N_1, \ldots, N_{n-m}, N_{n-m-i+1}, \ldots, N_{n-i+1})]^n \geq \prod_{i=1}^m D_p(\tau)(M_1, \ldots, M_{n-m}, M_{n-m-i+1}, \ldots, M_{n-i+1}).$$

By the equality conditions of inequalities (3.14) and (3.9), we see that equality holds in (1.13) if and only if $N_{n-m+1}, \ldots, N_n$ are all of similar general $L_p$-brightness and there exists constant $\lambda$ such that

$$[D_p(\tau)(M_1, \ldots, M_{n-m}, M_{n-m-i+1}, \ldots, M_{n-i+1}) - D_p(\tau)(N_1, \ldots, N_{n-m}, N_{n-m-i+1}, \ldots, N_{n-i+1})]^n \geq \prod_{i=1}^m D_p(\tau)(M_1, \ldots, M_{n-m}, M_{n-m-i+1}, \ldots, M_{n-i+1}).$$

Obviously, let $p = 1$ and $\tau = 0$ in Theorem 1.3, the following result is obtained.

**Corollary 3.5.** Let $M_i, N_i \in \mathcal{K}_p^0$ ($i = 1, \ldots, n$), if $\delta(N_i, \cdot) < \delta(M_i, \cdot)$ and the bodies $M_{n-m+1}, \ldots, M_n$ are all of similar brightness, then for every $1 < m \leq n$,

$$[D(M_1, \ldots, M_n) - D(N_1, \ldots, N_n)]^m.$$
\[ \geq \prod_{i=1}^{m} \{D(M_1, \ldots, M_{n-m}, M_{n-i+1}, \ldots, M_{n-1}) \\
\quad - D(N_1, \ldots, N_{n-m}, N_{n-i+1}, \ldots, N_{n-1})\}, \]

with equality if and only if \( N_{n-m+1}, \ldots, N_n \) are all of similar brightness and there exists constant \( \lambda \) such that
\[ [D(M_1, \ldots, M_{n-m}, M_{n}, \ldots, M_{n}), \ldots, D(M_1, \ldots, M_{n-m}, M_{n-m+1}, \ldots, M_n), \ldots, D(N_1, \ldots, N_{n-m}, N_{n-m+1}, \ldots, N_n)] = \lambda[D(N_1, \ldots, N_{n-m}, N_{n-m+1}, \ldots, N_{n-1})]. \]

**ACKNOWLEDGMENT**

The authors want to express earnest thankfulness for the referees who provided extremely precious and helpful comments and suggestions.

**REFERENCES**


