Domination and Independence Parameters in the Total Graph of Zn with respect to Nil Ideal

Arijit Mishra, Kuntala Patra

***Abstract*—For any non-reduced ring $\mathbb{Z}_n$, the total graph of $\mathbb{Z}_n$ with respect to nil ideal, denoted by $T(\Gamma_N(\mathbb{Z}_n))$, is a simple, undirected graph having vertex set $\mathbb{Z}_n$ and any two distinct vertices x and y of $T(\Gamma_N(\mathbb{Z}_n))$ are adjacent if and only if $x + y \in N(\mathbb{Z}_n)$, where $N(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : x^2 = 0\}$ denotes the nil ideal of $\mathbb{Z}_n$. In this paper, we attempt to find the domination and independence numbers, domatic number and independence domination numbers of $T(\Gamma_N(\mathbb{Z}_n))$ and $\overline{T(\Gamma_N(\mathbb{Z}_n))}$, the complement of the total graph of $\mathbb{Z}_n$. We also obtain the number of $\gamma-$sets and independent sets of these graphs.**

***Index Terms*—Total Graph, Nil Ideal, Domination Number, Independence Number.**

I. Introduction

In the year 2008, the total graph of a commutative ring R, denoted by $T(\Gamma(R))$, was first introduced by Anderson and Badawi [4] as a simple undirected graph with vertex set R and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$, where $Z(R)$ denotes the set of all the zero-divisors of R. One can find extensive literature on total graphs and its variants in [2, 4, 5, 6, 8, 10, 11, 12].

Back in the year 2003, P. W. Chen [9] introduced a kind of graph structure of a commutative ring R having vertex set R and two distinct vertices x and y are adjacent if and only if $xy \in N(R)$, where $N(R)$ denotes the set of all the nil elements of the ring R. This concept was later modified by Ai-Hua Li and Qi-Sheng Li [3] who defined it as an undirected simple graph $\Gamma_N(R)$ with vertex set $Z_N(R)^* = \{x \in R^* \mid xy \in N(R) \text{ for some } y \in R^* = R - \{0\}\}$ and two distinct vertices x and y are adjacent if and only if $xy \in N(R)$ or $yx \in N(R)$.

The total graph of the non-reduced commutative ring $\mathbb{Z}_n$, denoted by $T(\Gamma_N(\mathbb{Z}_n))$, is a simple undirected graph with all the elements of $\mathbb{Z}_n$ as vertices and two distinct vertices x and y are adjacent if and only if $x + y \in N(\mathbb{Z}_n)$, where $N(\mathbb{Z}_n)$ denotes the set of all the nil elements of $\mathbb{Z}_n$, i.e. $N(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : x^2 = 0\}$.

II. Preliminaries

Let $G = (V, E)$ be any graph having vertex set V and edge set E. For any vertex v, the *open neighbourhood* of v is $N(v) = \{u \in V : uv \in E\}$, while the *closed neighbourhood* of v is $N[v] = \{v\} \cup N(v)$. A non-empty subset S of the vertex set V of a graph is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* γ of a graph G is defined to be the minimum cardinality of a dominating set in G and the corresponding dominating set is called a $\gamma-$set of G. The partition of the vertex set V(G) of a graph G into dominating sets is called a *domatic partition* of G. The maximum number of such partitions is called the *domatic number* of G and is denoted by $d(G)$. A graph G is said to be *domatically full* if $d(G) = \delta(G) + 1$. A graph G is called *excellent* if for every vertex v of G, there exist a $\gamma-$set containing v. A dominating set S of the vertex set V of a graph G is said to be a *perfect dominating set* if every vertex in $V - S$ is adjacent to exactly one vertex in S. The minimum cardinality of a perfect dominating set is called the *perfect domination number* of the graph and is denoted by $\gamma_p(G)$. The maximum number of sets into which the vertex set of a graph G can be partitioned in such a way that each partition is a perfect dominating set is called the *perfect domatic number* of the graph and is denoted by $d_p(G)$. A set of vertices in a graph is said to be *independent* if no two vertices in that set are adjacent. The maximum cardinality of an independent set of a graph G is called the *independence number* of the graph G and is denoted by $\beta_0(G)$. A dominating set S of a graph G is said to be an *independent dominating set* if no two vertices of S are adjacent. The *independence domination number*, denoted by $i(G)$, is the minimum cardinality of an independent dominating set. The maximum number of partitions of the vertex set of a graph G into minimum dominating sets is called the *independent domatic number* of G and is denoted by $d_{ind}(G)$ A non-empty subset S of the vertex set $V(G)$ of a graph G is said to be a *clique dominating set* of G if S is a dominating set and the induced subgraph $< S >$ of S is complete. The minimum cardinality among all the clique dominating sets of G, denoted by $\gamma_{cl}(G)$, is called the *clique domination number* of G. A graph is said to be *well-covered* if every maximal independent set has the same size. Alternatively, a graph G is said to be *well-covered* if $i(G) = \beta_0(G)$.

A ring is said to be *non-reduced* if it contains at least one non-zero nil element, otherwise it is said to be *reduced.*

III. Graphical structure of $T(\Gamma_N(\mathbb{Z}_n))$

In this section, we obtain the basic graphical structure of the total graph $T(\Gamma_N(\mathbb{Z}_n))$ of $\mathbb{Z}_n$ with respect to its nil ideal $N(\mathbb{Z}_n)$. This new graph structure is a part of one of our previous papers (not yet communicated) where we have defined $T(\Gamma_N(\mathbb{Z}_n))$ as an undirected simple graph of the non-reduced ring $\mathbb{Z}_n$ having vertex set $\mathbb{Z}_n$ and any two distinct vertices x and y are adjacent if and only if $x+y \in N(\mathbb{Z}_n)$, where $N(\mathbb{Z}_n) = \{x \in \mathbb{Z}_n : x^2 = 0(mod n)\}$. Here $T_{N(\mathbb{Z}_n)}$ and $T_{\overline{N(\mathbb{Z}_n)}}$ denote the induced subgraphs of $T(\Gamma_N(\mathbb{Z}_n))$ whose vertex sets are $N(\mathbb{Z}_n)$ and $\overline{N(\mathbb{Z}_n)}$ respectively, where $\overline{N(\mathbb{Z}_n)} = \mathbb{Z}_n - N(\mathbb{Z}_n)$. Also, throughout this section, we use the following notations: $\alpha = |N(\mathbb{Z}_n)|$ and $\beta = |\mathbb{Z}_n - N(\mathbb{Z}_n)|$.

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Theorem 3.1 :
(i) Let $R = \mathbb{Z}_n$ be non-reduced and $N(\mathbb{Z}_n)$ be the set of all the nil elements of $\mathbb{Z}_n$. Then $T_{N(\mathbb{Z}_n)}$ is a complete subgraph of $T(\Gamma_N(\mathbb{Z}_n))$ and $T_{N(\mathbb{Z}_n)}$ is disjoint from $T_{\overline{N(\mathbb{Z}_n)}}$.
(ii) Let $R = \mathbb{Z}_n$ and let n_1 be the smallest non-zero nil element of $\mathbb{Z}_n$. Then
(1) If $|R|$ is odd, then $T_{\overline{N(R)}} = (\frac{n_1-1}{2})K_{\frac{n}{n_1},\frac{n}{n_1}}$.
(2) If $|R|$ is even, then $T_{\overline{N(R)}} = K_{\frac{n}{n_1}} \cup (\frac{n_1}{2}-1)K_{\frac{n}{n_1},\frac{n}{n_1}}$.
(iii) Let $R = \mathbb{Z}_n$ and n_1 be the smallest non-zero nil element of $\mathbb{Z}_n$. Then
(1) If $|R|$ is odd, then $T(\Gamma_N(R)) = K_{\frac{n}{n_1}} \cup (\frac{n_1-1}{2})K_{\frac{n}{n_1},\frac{n}{n_1}}$.
(2) If $|R|$ is even, then $T(\Gamma_N(R)) = 2K_{\frac{n}{n_1}} \cup (\frac{n_1}{2} - 1)K_{\frac{n}{n_1},\frac{n}{n_1}}$.

Proof:
(i) Since the ring $R = \mathbb{Z}_n$ is commutative, so $N(\mathbb{Z}_n)$ is an ideal of $\mathbb{Z}_n$. Thus $\forall\ x, y \in N(\mathbb{Z}_n)$, $x + y \in N(\mathbb{Z}_n)$. Consequently all the vertices of $T_{N(\mathbb{Z}_n)}$ are adjacent to each other and therefore $T_{N(\mathbb{Z}_n)}$ is a complete subgraph of $T(\Gamma_N(\mathbb{Z}_n))$. The second part of the result is clear from the definition.

(ii) (1) Let $|R|$ be odd. Then each $x_i \in N(\mathbb{Z}_n)$, $\forall\ i = 1, 2, ..., \frac{n}{n_1}$ is adjacent to each other since $N(R)$ is an ideal of R. Thus the set of vertices $N(R)$ forms the clique $K_{\frac{n}{n_1}}$. Again, for $u_i \in R\ \forall\ u_i = 1, 2, ..., \frac{n_1-1}{2}$ such that $2u_i \notin N(R)$, the elements of the cosets $u_i + N(R)$ are adjacent to the elements of the cosets $(ln_1 - u_i) + N(R)$, for $l \in \mathbb{Z}^+$ since $(u_i + r_1) + (ln_1 - u_i + r_2)$, for some $r_1, r_2 \in N(R)$, gives $(u_i + ln_1 - u_i) + (r_1 + r_2) = ln_1 + (r_1 + r_2) \in N(R)$. However for each i, the elements of the cosets $u_i + N(R)$ are not adjacent to each other since $(u_i + r_1) + (u_i + r_2) = (u_i + u_i) + (r_1 + r_2) \notin N(R)$, since $u_i + u_i = 2u_i \notin N(R)$. Thus, $(u_i + N(R)) \cup ((ln_1 - u_i) + N(R))$ is the complete bipartite graph $K_{\frac{n}{n_1},\frac{n}{n_1}}$. Also, for some $y_i \in R$ such that $y_i \neq u_i$, if the elements of the cosets $y_i + N(R)$ are adjacent to the elements of the cosets $u_i + N(R)$, then $u_i + y_i \in N(R)$ and thus, $y_i + N(R) = (ln_1 - u_i) + N(R)$. Hence, $(u_i + N(R)) \cup ((ln_1 - u_i) + N(R))$ forms $\frac{n_1-1}{2}$ disjoint complete bipartite graphs $K_{\frac{n}{n_1},\frac{n}{n_1}}$.

(2) Let $|R|$ be even. Here again, the vertices of the set $N(R)$ form the clique $K_{\frac{n}{n_1}}$, since $N(R)$ is an ideal of R. Also, for the smallest non-zero nil element n_1 of R, the vertices of the coset $\frac{n_1}{2} + N(R)$ having cardinality $\frac{n}{n_1}$ are adjacent to each other since for some $r_1, r_2 \in N(R)$, $(\frac{n_1}{2} + r_1) + (\frac{n_1}{2} + r_2) = (\frac{n_1}{2} + \frac{n_1}{2}) + (r_1 + r_2) = n_1 + (r_1 + r_2) \in N(R)$ since $N(R)$ is an ideal of R. Thus the vertices of the coset $\frac{n_1}{2} + N(R)$ form the clique $K_{\frac{n}{n_1}}$.

Again, for $u_i \in R\ \forall\ u_i = 1, 2, ..., \frac{n_1}{2} - 1$, such that $u_i + u_i = 2u_i \notin N(R)$, the elements of the cosets $u_i + N(R)$ having cardinality $\frac{n}{n_1}$ are not adjacent to each other, but are adjacent to the elements of the cosets $(ln_1 - u_i) + N(R)$. Thus, $(u_i + N(R)) \cup ((ln_1 - u_i) + N(R))$ forms the complete bipartite graph $K_{\frac{n}{n_1},\frac{n}{n_1}}$. For any $y_i \in R$ such that $y_i \neq u_i$, if the elements of the cosets $y_i + N(R)$ are adjacent to the elements of the cosets $u_i + N(R)$, then $u_i + y_i \in N(R)$ and thus, $y_i + N(R) = (ln_1 - u_i) + N(R)$. Consequently, for each $i = 1, 2, ..., \frac{n_1}{2} - 1$, $(u_i + N(R)) \cup ((ln_1 - u_i) + N(R))$ forms disjoint complete bipartite graphs $K_{\frac{n}{n_1},\frac{n}{n_1}}$. Thus, we can write, $T_{\overline{N(R)}} = K_{\frac{n}{n_1}} \cup (\frac{n_1}{2} - 1)K_{\frac{n}{n_1},\frac{n}{n_1}}$.

(iii) Since $T_{N(R)}$ is the complete graph $K_{\frac{n}{n_1}}$, the result easily follows by (ii). $\square$

Example 3.2: Let us consider the non-reduced ring $\mathbb{Z}_{16}$. The nil elements of $\mathbb{Z}_{16}$ are $\{0, 4, 8, 12\}$. Fig 1 shows that the total graph of $\mathbb{Z}_{16}$ with respect to its nil ideal is a disjoint union of two K_4's and $(\frac{4}{2} - 1)\ K_{4,4}$.

Similarly, the nil elements of the non-reduced ring $\mathbb{Z}_9$ are $N(\mathbb{Z}_9) = \{0, 3, 6\}$. The total graph of $\mathbb{Z}_9$ with respect to $N(\mathbb{Z}_9)$, as shown in Fig 2, is a disjoint union of a K_3 and $(\frac{3-1}{2})\ K_{3,3}$.

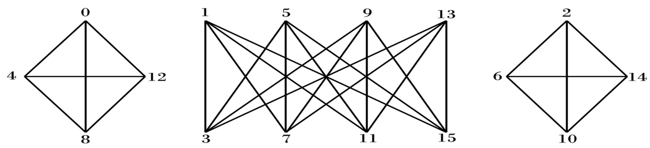

Fig 1 : $T(\Gamma_N(Z_{16}))$

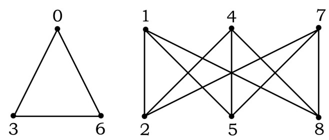

Fig 2 : $T(\Gamma_N(Z_9))$

Corollary 3.3: From theorem 3.1 (iii), it is obvious that for any $x \in V(T(\Gamma_N(\mathbb{Z}_n)))$,
$$deg(x) = \begin{cases} \frac{n}{n_1}, \text{ if } x \in K_{\frac{n}{n_1},\frac{n}{n_1}}. \\ \frac{n}{n_1} - 1, \text{ if } x \in K_{\frac{n}{n_1}}. \end{cases} \qquad \square$$
Having obtained the structure of $T(\Gamma_N(\mathbb{Z}_n))$, we now characterize the domination parameters of $T(\Gamma_N(\mathbb{Z}_n))$ in the following section.

IV. Domination properties of $T(\Gamma_N(\mathbb{Z}_n))$

Theorem 4.1: For any non-reduced $\mathbb{Z}_n$, if $|N(\mathbb{Z}_n)| = \alpha = \frac{n}{n_1}$, where n_1 is the smallest non-zero nil element of $\mathbb{Z}_n$, then $\gamma(T(\Gamma_N(\mathbb{Z}_n))) = n_1$.

Proof : Let us consider two cases here:
Case 1: Let n be even. Then $T(\Gamma_N(\mathbb{Z}_n)) = 2K_{\frac{n}{n_1}} \cup (\frac{n_1}{2} - 1)K_{\frac{n}{n_1},\frac{n}{n_1}}$.
Clearly, the $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ of minimum cardinality contains $1 + 1 + (\frac{n_1}{2} - 1).2$, i.e. n_1 vertices.
Case 2: Let n be odd. Then $T(\Gamma_N(\mathbb{Z}_n)) = K_{\frac{n}{n_1}} \cup (\frac{n_1-1}{2})K_{\frac{n}{n_1},\frac{n}{n_1}}$.
The $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ of minimum cardinality contains $1 + (\frac{n_1-1}{2}).2$, i.e. n_1 vertices. Therefore, in both the cases, $\gamma(T(\Gamma_N(\mathbb{Z}_n))) = n_1$. $\square$

Having obtained the number of vertices in a minimum dominating set of $T(\Gamma_N(\mathbb{Z}_n))$, we now proceed to find out the number of such minimum dominating sets.

Theorem 4.2: For any non-reduced $\mathbb{Z}_n$, the total number of $\gamma-$sets of $T(\Gamma_N(\mathbb{Z}_n)) = (\alpha)^{\frac{n}{\alpha}}$, where $\alpha = |N(\mathbb{Z}_n)|$.

Proof : For any value of n such that n is not square-free, any $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ contains $(\frac{n}{\alpha})$ vertices. Also, each vertex x_i in any $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ has α choices. So the total number of $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ is $\underbrace{{}^{\alpha}C_1 \times {}^{\alpha}C_1 \times ... \times {}^{\alpha}C_1}_{(\frac{n}{\alpha})}$, i.e. $(\alpha)^{\frac{n}{\alpha}}$. $\square$

Alternatively, the total number of $\gamma-$sets of $T(\Gamma_N(\mathbb{Z}_n)) = (\frac{n}{n_1})^{n_1}$ since $\alpha = \frac{n}{n_1}$. $\square$

Example 4.2.1: In Fig.2, we have $n = 9$. Here $n_1 = 3$ which is the smallest non-zero nil element of $\mathbb{Z}_9$. The different $\gamma-$sets of this graph are $\{0,1,2\}$, $\{0,1,5\}$, $\{0,1,8\}$, $\{0,4,2\}$, $\{0,4,5\}$, $\{0,4,8\}$, $\{0,7,2\}$, $\{0,7,5\}$, $\{0,7,8\}$, $\{3,1,2\}$, $\{3,1,5\}$, $\{3,1,8\}$, $\{3,4,2\}$, $\{3,4,5\}$, $\{3,4,8\}$, $\{3,7,2\}$, $\{3,7,5\}$, $\{3,7,8\}$, $\{6,1,2\}$, $\{6,1,5\}$, $\{6,1,8\}$, $\{6,4,2\}$, $\{6,4,5\}$, $\{6,4,8\}$, $\{6,7,2\}$, $\{6,7,5\}$ and $\{6,7,8\}$.

Clearly each of these $\gamma-$sets contains $3(= n_1)$ vertices and there are $27(= (\frac{n}{n_1})^{n_1})$ $\gamma-$sets in all. □

The following theorem characterizes all the $\gamma-$sets of $T(\Gamma_N(\mathbb{Z}_n))$ for any non-reduced $\mathbb{Z}_n$.

Theorem 4.3: Let $S = \{x_1, x_2, x_3, ..., x_{n_1}\} \subset V(T(\Gamma_N(\mathbb{Z}_n)))$, where n_1 is the smallest non-zero nil element of $\mathbb{Z}_n$. Then S is a $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ if and only if $x_i + N(\mathbb{Z}_n)$ form distinct cosets of $\frac{\mathbb{Z}_n}{N(\mathbb{Z}_n)}$ for each $x_i \in S$.

Proof: Let $S = \{x_1, x_2, x_3, ..., x_{n_1}\}$ be a $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$. If possible, suppose that there exist distinct x_i, $x_j \in S$, such that $x_i + N(\mathbb{Z}_n) = x_j + N(\mathbb{Z}_n)$. Then $x_i \equiv x_j (mod n_1)$. Thus each vertex adjacent to x_i is also adjacent to x_j and so S is not a $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$, a contradiction. Therefore each $x_i + N(\mathbb{Z}_n)$ forms distinct cosets of $\frac{\mathbb{Z}_n}{N(\mathbb{Z}_n)}$, for each $x_i \in S$.

Conversely, let $x_i + N(\mathbb{Z}_n)$ form distinct cosets for each $x_i \in S$. Each $x_i \in S$ is adjacent to each element of the coset $(ln_1 - x_i) + N(\mathbb{Z}_n)$, where $l = 1, 2, ..., \frac{n}{n_1}$. Also, $|(ln_1 - x_i) + N(\mathbb{Z}_n)| = n_1$ and $|N[S]| = n$. Clearly, S is a dominating set, where $| S |= n_1$. By theorem 4.1, S is a $\gamma-$set. □

Corollary 4.4: For any non-reduced $\mathbb{Z}_n$, if $| \frac{\mathbb{Z}_n}{N(\mathbb{Z}_n)} |= \mu$, then $\gamma(T(\Gamma_N(\mathbb{Z}_n))) = \mu$. □

Corollary 4.5: From theorem 4.1, since $\gamma(T(\Gamma_N(\mathbb{Z}_n))) = n_1$, where n_1 is the smallest non-zero nil element of $\mathbb{Z}_n$, so $| \frac{\mathbb{Z}_n}{N(\mathbb{Z}_n)} |= n_1$. □

Theorem 4.6: For any non-reduced $\mathbb{Z}_n$, $\gamma_p(T(\Gamma_N(\mathbb{Z}_n))) = n_1$.

Proof : Since the $\gamma-$set $S = \{0, 1, ..., n_1 - 1\}$ contains vertices such that each vertex in $\mathbb{Z}_n \setminus S$ is adjacent to exactly one vertex in S, so the set S is a perfect dominating set. Also S, being a $\gamma-$set, has minimum cardinality. Thus, $\gamma_p(T(\Gamma_N(\mathbb{Z}_n))) = n_1$. □

Theorem 4.7: For any non-reduced $\mathbb{Z}_n$,

(i) $T(\Gamma_N(\mathbb{Z}_n))$ is an excellent graph.

(ii) $d(T(\Gamma_N(\mathbb{Z}_n))) = \frac{n}{n_1}$.

(iii) $T(\Gamma_N(\mathbb{Z}_n))$ is domatically full $\forall$ $n \in \mathbb{N}$.

(iv) For any non-reduced $\mathbb{Z}_n$, $d_p(T(\Gamma_N(\mathbb{Z}_n))) = \frac{n}{n_1}$.

Proof: (i) Since each vertex of $\mathbb{Z}_n$ is a part of a $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n)$, so the result follows immediately.

(ii) Since each $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ contains n_1 vertices, therefore $d(T(\Gamma_N(\mathbb{Z}_n))) = \frac{n}{n_1}$.

(iii) From (ii) and Corollary 3.3, since $d(T(\Gamma_N(\mathbb{Z}_n))) = \frac{n}{n_1}$ and $\delta(T(\Gamma_N(\mathbb{Z}_n))) = \frac{n}{n_1} - 1$, so $d(T(\Gamma_N(\mathbb{Z}_n))) = \delta(T(\Gamma_N(\mathbb{Z}_n))) + 1$. Hence the result follows.

(iv) The result is obvious since $d_p(T(\Gamma_N(\mathbb{Z}_n))) = \frac{|V(T(\Gamma_N(\mathbb{Z}_n)))|}{\gamma_p(T(\Gamma_N(\mathbb{Z}_n)))} = \frac{n}{n_1}$. □

From Example 4.2.1, the three $\gamma-$sets $\{0,1,2\}$, $\{3,4,5\}$ and $\{6,7,8\}$ form disjoint partitions the vertex set of the ring $\mathbb{Z}_9$. So $d(T(\Gamma_N(\mathbb{Z}_9))) = 3 = \frac{9}{3} = \frac{n}{n_1}$. Also, since each vertex of $\mathbb{Z}_9$ is a part of a $\gamma-$set, so $T(\Gamma_N(\mathbb{Z}_9))$ is an excellent graph.

Theorem 4.8:

(i) For each $x_i \in \gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ such that $x_i \notin N(\mathbb{Z}_n)$, but $2x_i \in N(\mathbb{Z}_n)$, the vertices of the coset $x_i + N(\mathbb{Z}_n)$ form disjoint complete graphs.

(ii) For some $\gamma-$set $S = \{x_1, x_2, ..., x_{n_1}\}$ of $T(\Gamma_N(\mathbb{Z}_n))$ and $r \in N(\mathbb{Z}_n)$, the vertices in the coset $r + S$ are adjacent if and only if the vertices in S are adjacent.

(iii) For each $x_i \in \gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ such that neither x_i nor $2x_i \in N(\mathbb{Z}_n)$, the vertices of the cosets $x_i + N(\mathbb{Z}_n)$ and $(ln_1 - x_i) + N(\mathbb{Z}_n)$ form disjoint complete bipartite graphs, for $l \in \mathbb{Z}^+$, where n_1 is the smallest non-zero nil element.

Proof : (i) For $x_i \notin N(\mathbb{Z}_n)$ such that $2x_i \in N(\mathbb{Z}_n)$ and for some $r_1, r_2 \in N(\mathbb{Z}_n)$, $(x_i + r_1) + (x_i + r_2) = 2x_i + (r_1 + r_2) \in N(\mathbb{Z}_n)$. Thus the vertices of the cosets $x_i + N(\mathbb{Z}_n)$ form a complete graph. Also, for some $y_i (\neq x_i) \notin N(\mathbb{Z}_n)$ such that $2y_i \in N(\mathbb{Z}_n)$, $(x_i + r_1) + (y_i + r_2) = (x_i + y_i) + (r_1 + r_2) \notin N(\mathbb{Z}_n)$. So the vertices of the cosets $x_i + N(\mathbb{Z}_n)$ and $y_i + N(\mathbb{Z}_n)$ are disjoint. Consequently for each i, the cosets $x_i + N(\mathbb{Z}_n)$ form disjoint complete graphs.

(ii) For some $x_i, x_j \in S$, let the vertices $r + x_i$ and $r + x_j$ in the coset $r + S$ be adjacent. Then $(x_i + r) + (x_j + r) \in N(\mathbb{Z}_n)$ $\Rightarrow (x_i + x_j) + (r + r) \in N(\mathbb{Z}_n) \Rightarrow x_i + x_j \in N(\mathbb{Z}_n)$. Thus x_i is adjacent to x_2 in S. Conversely, let x_i be adjacent to x_j in S. Then $x_i + x_j \in N(\mathbb{Z}_n) \Rightarrow (x_i + r) + (x_j + r) \in N(\mathbb{Z}_n)$. Thus the result follows.

(iii) For each $x_i \notin N(\mathbb{Z}_n)$ such that $2x_i \notin N(\mathbb{Z}_n)$, the vertices of the cosets $x_i + N(\mathbb{Z}_n)$ are adjacent to every vertex of the cosets $(ln_1 - x_i) + N(\mathbb{Z}_n)$, since $(x_i + r_1) + (ln_1 - x_i) + r_2 = (x_i + ln_1 - x_i) + (r_1 + r_2) = ln_1 + (r_1 + r_2) \in N(\mathbb{Z}_n)$, for some $r_1, r_2 \in N(\mathbb{Z}_n)$. Also, since $(x_i + r_1) + (x_i + r_2) = 2x_i + (r_1 + r_2) \notin N(\mathbb{Z}_n)$, so the vertices of the coset $x_i + N(\mathbb{Z}_n)$ are not adjacent to each other. Consequently, $\{x_i + N(\mathbb{Z}_n)\} \cup \{(ln_1 - x_i) + N(\mathbb{Z}_n)\}$ form disjoint complete bipartite graphs. □

Theorem 4.9: For any non-reduced $\mathbb{Z}_n$, let S be any $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ and let x_1 and x_2 be any two distinct vertices of $T(\Gamma_N(\mathbb{Z}_n))$ such that $x_2 \in x_1 + N(\mathbb{Z}_n)$. If a vertex $u \in x_1 + S$ is adjacent to a vertex $v \in x_2 + S$, then each vertex of the coset $x_1 + S$ is adjacent to a vertex in the coset $x_2 + S$.

Proof: Let $u \in x_1 + S$ be adjacent to $v \in x_2 + S$. Then $u + v \in N(\mathbb{Z}_n)$. Let $u + v = r$, for some $r \in N(\mathbb{Z}_n)$. Then $r = u + v \in x_1 + x_2 + S$. Now let $u' \in x_1 + S$ and $v' \in V(T(\Gamma_N(\mathbb{Z}_n)))$ such that $u' + v' \in N(\mathbb{Z}_n)$.

$\Rightarrow u' + v' = ln_1$, for some $l \in \mathbb{Z}^+$ and n_1 is the smallest non-zero nil element of $\mathbb{Z}_n$.

$\Rightarrow x_1 + s' + v' = ln_1$, for some $s' \in S$

$\Rightarrow v' = ln_1 - x_1 - s'$

$\Rightarrow v' = l_1 n_1 + l_2 n_1 - x_1 - s'$, for some $l_1, l_2 \in \mathbb{Z}^+$ such that $l = l_1 + l_2$

$\Rightarrow v' = (x_1 + l_1 n_1) + (l_2 n_1 - 2x_1 - s') \in x_2 + S$.

Thus each vertex of the coset $x_1 + S$ is adjacent to a vertex of the coset $x_2 + S$. □

We now find out the independence number and its variants associated to the graph $T(\Gamma_N(\mathbb{Z}_n))$ in the following section.

V. Independence parameters of $T(\Gamma_N(\mathbb{Z}_n))$

Theorem 5.1: Let $\mathbb{Z}_n$ be non-reduced and let n_1 be the smallest non-zero nil element of $\mathbb{Z}_n$. Then
$$\beta_0(T(\Gamma_N(\mathbb{Z}_n))) = \begin{cases} 2 + \frac{n}{2} - \frac{n}{n_1}, \text{ if } n \text{ is even} \\ 1 + \frac{n}{2} - \frac{n}{2n_1}, \text{ if } n \text{ is odd} \end{cases}$$

Proof : As in theorem 3.1 (iii) ,when n is even, since

$T(\Gamma_N(\mathbb{Z}_n)) = 2K_{\frac{n}{n_1}} \cup (\frac{n_1}{2}-1)K_{\frac{n}{n_1},\frac{n}{n_1}}$, so the independent set of $T(\Gamma_N(\mathbb{Z}_n))$ of maximum cardinality contains $1+1+(\frac{n_1}{2}-1).\frac{n}{n_1}$, i.e. $2+\frac{n}{2}-\frac{n}{n_1}$ vertices.

Therefore $\beta_0(T(\Gamma_N(\mathbb{Z}_n))) = 2+\frac{n}{2}-\frac{n}{n_1}$.

Similarly, when n be odd, the independent set of $T(\Gamma_N(\mathbb{Z}_n))$ of maximum cardinality contains $1+(\frac{n_1-1}{2}).\frac{n}{n_1}$, i.e. $1+\frac{n}{2}-\frac{n}{2n_1}$ vertices. Therefore, $\beta_0(T(\Gamma_N(\mathbb{Z}_n))) = 1+\frac{n}{2}-\frac{n}{2n_1}$. □

Unlike the number of $\gamma-$sets of $T(\Gamma_N(\mathbb{Z}_n))$, the number of maximum independent sets of $T(\Gamma_N(\mathbb{Z}_n))$ differs in accordance with the parity of n, as evident from the following theorem.

Theorem 5.2: Let $\mathbb{Z}_n$ be non-reduced and n_1 be the smallest non-zero nil element of $\mathbb{Z}_n$. Then the total number of maximum independent sets of $T(\Gamma_N(\mathbb{Z}_n))$ is given by

$$\begin{cases} (\frac{n}{n_1})^2.(\sqrt{2})^{n_1-2}, \text{ if } n \text{ is even} \\ \frac{n}{n_1}.(\sqrt{2})^{n_1-1}, \text{ if } n \text{ is odd} \end{cases}$$

Proof : Let n be even. Since $T(\Gamma_N(\mathbb{Z}_n)) = 2K_{\frac{n}{n_1}} \cup (\frac{n_1}{2}-1)K_{\frac{n}{n_1},\frac{n}{n_1}}$, so each vertex x_i in the maximum independent set of $T(\Gamma_N(\mathbb{Z}_n))$ has $\frac{n}{n_1}$ choices if $x_i \in K_{\frac{n}{n_1}}$, and 2 choices if $x_i \in K_{\frac{n}{n_1},\frac{n}{n_1}}$. So the total number of maximum independent sets $= {}^{\frac{n}{n_1}}C_1 \times {}^{\frac{n}{n_1}}C_1 \times \underbrace{{}^2C_1 \times {}^2C_1 \times ... \times {}^2C_1}_{\frac{n_1}{2}-1}$

$= (\frac{n}{n_1})^2.(\sqrt{2})^{n_1-2}$.

Again, let n be odd. Then $T(\Gamma_N(\mathbb{Z}_n)) = K_{\frac{n}{n_1}} \cup (\frac{n_1-1}{2})K_{\frac{n}{n_1},\frac{n}{n_1}}$, so each vertex x_i in the maximum independent set of $T(\Gamma_N(\mathbb{Z}_n))$ has $\frac{n}{n_1}$ choices, if $x_i \in K_{\frac{n}{n_1}}$, and 2 choices, if $x_i \in K_{\frac{n}{n_1},\frac{n}{n_1}}$. So the total number of maximum independent sets $= {}^{\frac{n}{n_1}}C_1 \times \underbrace{{}^2C_1 \times {}^2C_1 \times ... \times {}^2C_1}_{\frac{n_1-1}{2}} =$

$\frac{n}{n_1}.(\sqrt{2})^{n_1-1}$. □

The following theorem characterizes all the maximum independent sets of $T(\Gamma_N(\mathbb{Z}_n))$ for any non-reduced $\mathbb{Z}_n$.

Theorem 5.3: Let $S \subset V(T(\Gamma_N(\mathbb{Z}_n))))$. Then S is a β_0-set of $T(\Gamma_N(\mathbb{Z}_n))$ if and only if for each $x_i \in S$ such that $x_i \notin N(\mathbb{Z}_n), \frac{n_1}{2}+N(\mathbb{Z}_n)$, the coset $x_i+N(\mathbb{Z}_n) \in S$.

Proof: Let n be even.

Let $S \subset V(T(\Gamma_N(\mathbb{Z}_n))))$ be a β_0-set of $T(\Gamma_N(\mathbb{Z}_n))$. For any $x_i \in N(\mathbb{Z}_n)$ such that $x_i \in S$, the vertex x_i is adjacent to the vertices of the coset $x_i+N(\mathbb{Z}_n)$ since $x_i+(x_i+r_1) = 2x_i+r_1 \in N(\mathbb{Z}_n)$, for some $r_1 \in N(\mathbb{Z}_n)$. So for $x_i \in S$, the coset $x_i+N(\mathbb{Z}_n) \notin S$. Again, for $x_i = \frac{n_1}{2} \in S$, where n_1 is the smallest non-zero nil element, the vertex $\frac{n_1}{2}$ is adjacent to the vertices of the coset $\frac{n_1}{2}+N(\mathbb{Z}_n)$, since $\frac{n_1}{2}+(\frac{n_1}{2}+r_1) = n_1+r_1 \in N(\mathbb{Z}_n)$, for some $r_1 \in N(\mathbb{Z}_n)$. So for $\frac{n_1}{2} \in S$, the coset $\frac{n_1}{2}+N(\mathbb{Z}_n) \notin S$. Again, for any $x_i \in S$ such that $1 \leq x_i \leq (\frac{n_1}{2}-1)$, the elements of the coset $x_i+N(\mathbb{Z}_n) \in S$, since $x_i+(x_i+r_1) = 2x_i+r_1 \notin N(\mathbb{Z}_n)$. Similarly, for any $(ln_1-x_i) \in S$, $l \in \mathbb{Z}^+$, the elements of the coset $(ln_1-x_i)+N(\mathbb{Z}_n) \in S$. Conversely, since $x_i+N(\mathbb{Z}_n) \in S$, so $|S| = (\frac{n_1}{2}-1).\frac{n}{n_1}+1+1 = \frac{n}{2}-\frac{n}{n_1}+2$ and from theorem 5.1, S is a β_0-set.

The case for odd values of n can be proven similarly. □

Having obtained the domination and independence properties associated to $T(\Gamma_N(\mathbb{Z}_n))$, the following set of results are immediate.

Theorem 5.4: If n_1 denotes the smallest non-zero nil element of $\mathbb{Z}_n$, then $i(T(\Gamma_N(\mathbb{Z}_n))) =$

$$\begin{cases} 2+\frac{n}{2}-\frac{n}{n_1}, \text{ if } n \text{ is even} \\ 1+\frac{n}{2}-\frac{n}{2n_1}, \text{ if } n \text{ is odd} \end{cases}$$

Proof : Let n be even. Then the independent set S of $T(\Gamma_N(\mathbb{Z}_n))$ with $|S| = 2+\frac{n}{2}-\frac{n}{n_1}$, obtained by choosing one vertex from each $K_{\frac{n}{n_1}}$ and $\frac{n}{n_1}$ vertices from each $K_{\frac{n}{n_1},\frac{n}{n_1}}$ is also a dominating set. Hence S is an independent dominating set. Also, for any $x \in S$, the set $S-\{x\}$ is not dominating. Thus, S is a minimal independent dominating set. Therefore, for any even n, $i(T(\Gamma_N(\mathbb{Z}_n))) = 2+\frac{n}{2}-\frac{n}{n_1}$. Similarly, when n is odd, $i(T(\Gamma_N(\mathbb{Z}_n))) = 1+\frac{n}{2}-\frac{n}{2n_1}$. □

Theorem 5.5: For any non-reduced $\mathbb{Z}_n$, $T(\Gamma_N(\mathbb{Z}_n))$ is a well-covered graph.

Proof : From theorem 5.1 and theorem 5.4, since $i(T(\Gamma_N(\mathbb{Z}_n))) = \beta_0(T(\Gamma_N(\mathbb{Z}_n)))$, the result follows. □

Next, we discuss a couple of results for the case when $\mathbb{Z}_n$ is reduced.

Theorem 5.6: Let $\mathbb{Z}_n$ be reduced and let n be even. Then the following results hold:

(i) $\gamma(T(\Gamma_N(\mathbb{Z}_n))) = \frac{1}{2}(n+2)$.

(ii) Total number of $\gamma-$sets of $T(\Gamma_N(\mathbb{Z}_n)) = (\sqrt{2})^{n-2}$.

(iii) $\beta_0(T(\Gamma_N(\mathbb{Z}_n))) = \frac{1}{2}(n+2)$.

(iv) Total number of β_0-sets of $T(\Gamma_N(\mathbb{Z}_n)) = (\sqrt{2})^{n-2}$.

(v) $T(\Gamma_N(\mathbb{Z}_n))$ is an excellent graph.

Proof: Since the ring $\mathbb{Z}_n$ is reduced, thus $N(\mathbb{Z}_n) = \{0\}$. So each $x_i \in T(\Gamma_N(\mathbb{Z}_n))$ is adjacent to a unique $-x_i$. Also, the vertices 0 and $\frac{n}{2}$, being their own inverses, are isolated. Consequently $T(\Gamma_N(\mathbb{Z}_n)) = 2K_1 \cup (\frac{n-2}{2})K_2$.

(i) It follows clearly that the $\gamma-$set of $T(\Gamma_N(\mathbb{Z}_n))$ contains $1+1+(\frac{n-2}{2}).1$, *i.e.* $\frac{1}{2}(n+2)$ vertices. Therefore $\gamma(T(\Gamma_N(\mathbb{Z}_n))) = \frac{1}{2}(n+2)$.

(ii) The total number of $\gamma-$sets of $T(\Gamma_N(\mathbb{Z}_n))$ are ${}^1C_1 \times {}^1C_1 \times \underbrace{{}^2C_1 \times {}^2C_1 \times ... \times {}^2C_1}_{\frac{n-2}{2}}$

$= (2)^{\frac{n-2}{2}} = (\sqrt{2})^{n-2}$.

The results (iii) and (iv) are obvious since for disjoint complete graphs, $\gamma-$sets and maximum independent sets are identical.

The result (v) is trivial. □

In a similar manner, one can also prove the following result.

Theorem 5.7: Let $\mathbb{Z}_n$ be reduced and let n be odd. Then the following results hold:

(i) $\gamma(T(\Gamma_N(\mathbb{Z}_n))) = \frac{1}{2}(n+1)$.

(ii) Total number of $\gamma-$sets of $T(\Gamma_N(\mathbb{Z}_n)) = (\sqrt{2})^{n-1}$.

(iii) $\beta_0(T(\Gamma_N(\mathbb{Z}_n))) = \frac{1}{2}(n+1)$.

(iv) Total number of β_0-sets of $T(\Gamma_N(\mathbb{Z}_n)) = (\sqrt{2})^{n-1}$.

(v) $T(\Gamma_N(\mathbb{Z}_n))$ is an excellent graph. □

Example 5.7.1: Figure 3 and Figure 4 represent the total graphs of the reduced rings $\mathbb{Z}_6$ and $\mathbb{Z}_5$ respectively. The $\gamma-$sets and maximum independent sets of $T(\Gamma_N(\mathbb{Z}_6))$ are $\{0,1,2,3\},\{0,1,4,3\},\{0,5,2,3\}$ and $\{0,5,4,3\}$, while the $\gamma-$sets and maximum independent sets of $T(\Gamma_N(\mathbb{Z}_5))$ are $\{0,1,2\},\{0,1,3\},\{0,4,2\}$ and $\{0,4,3\}$.

VI. DOMINATION PARAMETERS OF $\overline{T(\Gamma_N(\mathbb{Z}_n))}$

The graph $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is the complement of the graph $T(\Gamma_N(\mathbb{Z}_n))$ with vertex set $\mathbb{Z}_n$ and any two distinct vertices x and y are adjacent if and only if $x+y \in \mathbb{Z}_n \setminus N(\mathbb{Z}_n)$.

In this section, we discuss the domination and independence parameters of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$.

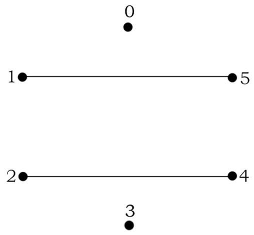

Fig 3. $T(\Gamma_N(Z_6))$

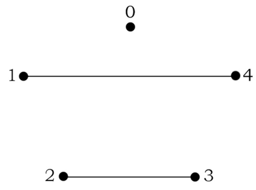

Fig 4. $T(\Gamma_N(Z_5))$

We begin this section with the following lemma.

Lemma 6.1: Let $G = \overline{T(\Gamma_N(\mathbb{Z}_n))}$. Then $\Delta(G) = n - \frac{n}{n_1}$ and $\delta = n - \frac{n}{n_1} - 1$, where Δ and δ denote the maximum and minimum degrees of G respectively.

Proof: The graph $\overline{T(\Gamma_N(\mathbb{Z}_n))}$, being simple and complement of $T(\Gamma_N(\mathbb{Z}_n))$, the result follows immediately from Corollary 3.3. $\square$

Theorem 6.2: (i) Let $R = \mathbb{Z}_n$ be non-reduced, then $\gamma(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 2$ and $\gamma_{cl}(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 2$.

(ii) If $\mathbb{Z}_n$ is reduced, then $\gamma(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 1$ and $\gamma_{cl}(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 1$.

(iii) For any non-reduced $\mathbb{Z}_n$, $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is an excellent graph.

Proof: (i) For $x \in N(\mathbb{Z}_n)$ and $y \in \mathbb{Z}_n \setminus N(\mathbb{Z}_n)$, $\{x, y\}$ is a dominating set of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$. Consequently, $\gamma(\overline{T(\Gamma_N(\mathbb{Z}_n))}) \leq 2$. Also, no vertex of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ has degree $n-1$, for otherwise $T(\Gamma_N(\mathbb{Z}_n))$ would contain an isolated vertex. This is possible only when $\alpha = 1$. But since we consider the ring to be non-reduced, so $\alpha \geq 2$. Consequently, $\gamma(\overline{T(\Gamma_N(\mathbb{Z}_n))}) > 1$. Therefore, $\gamma(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 2$.

Again, since the dominating set $\{x, y\}$, where $x \in N(\mathbb{Z}_n)$ and $y \in \mathbb{Z}_n \setminus N(\mathbb{Z}_n)$, induces the clique K_2 in $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ and since the exclusion of any vertex from this set removes the dominating character of this set, so $\{x, y\}$ is a clique dominating set of minimum cardinality. Therefore, $\gamma_{cl}(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 2$.

(ii) If $\mathbb{Z}_n$ is reduced, then $N(\mathbb{Z}_n) = \{0\}$. Clearly, 0 is adjacent to x, $\forall\ x \in \mathbb{Z}_n \setminus \{0\}$. Thus, $deg(0) = n - 1$ and hence, $\gamma(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 1$.

Again, since the dominating set $\{0\}$ induces the clique K_1, therefore $\gamma_{cl}(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 1$

(iii) For any non-reduced $\mathbb{Z}_n$, since the set $\{x, y\}$ is a $\gamma-$set of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$, for each $x \in N(\mathbb{Z}_n)$ and $y \in \mathbb{Z}_n \setminus N(\mathbb{Z}_n)$, it follows that there exists a $\gamma-$set for each vertex in the graph $\overline{T(\Gamma_N(\mathbb{Z}_n))}$. Hence $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is an excellent graph. $\square$

Remark 6.2.1: From Theorem 6.2 (i) and (ii), the following boundedness condition follows trivially:

For any $\mathbb{Z}_n$, $1 \leq \gamma(\overline{T(\Gamma_N(\mathbb{Z}_n))}) \leq 2$. $\square$

Theorem 6.3: The total number of $\gamma-$sets of $\overline{T(\Gamma_N(\mathbb{Z}_n))} = \frac{n^2}{n_1}(1 - \frac{1}{n_1})$, where n_1 is the smallest non-zero nil element of $\mathbb{Z}_n$.

Proof: By theorem 6.2 (i), $\{x, y\}$ is a $\gamma-$set of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$, where $x \in N(\mathbb{Z}_n)$ and $y \in \mathbb{Z}_n \setminus N(\mathbb{Z}_n)$. Since $|N(\mathbb{Z}_n)| = \frac{n}{n_1}$, so there are ${}^{\frac{n}{n_1}}C_1 \times {}^{(n-\frac{n}{n_1})}C_1$ choices, *i.e.* $\frac{n^2}{n_1}(1 - \frac{1}{n_1})$ choices. Thus the total number of $\gamma-$sets of $\overline{T(\Gamma_N(\mathbb{Z}_n))} = \frac{n^2}{n_1}(1 - \frac{1}{n_1})$. $\square$

As proven in theorem 4.3, one can prove the following result.

Theorem 6.4: For any non-reduced $\mathbb{Z}_n$, let $G = \overline{T(\Gamma_N(\mathbb{Z}_n))}$. Then $\{x, y\}$ is a $\gamma-$set of G if and only if the cosets $x + N(\mathbb{Z}_n)$ and $y + N(\mathbb{Z}_n)$ are distinct. $\square$

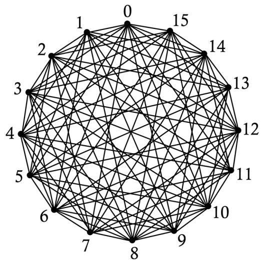

Fig 5. $\overline{T(\Gamma_N(Z_{16}))}$

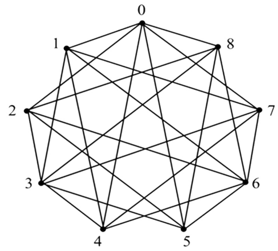

Fig 6. $\overline{T(\Gamma_N(Z_9))}$

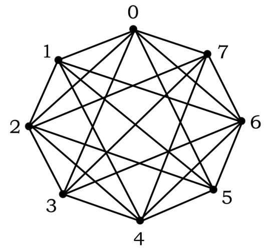

Fig 7. $\overline{T(\Gamma_N(Z_8))}$

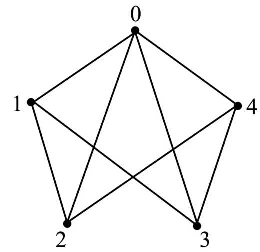

Fig 8. $\overline{T(\Gamma_N(Z_5))}$

Theorem 6.5: For any non-reduced $\mathbb{Z}_n$, $\beta_0(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = \frac{n}{n_1}$.

Proof: Let $S \subset V(\overline{T(\Gamma_N(\mathbb{Z}_n))})$ be a β_0-set of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$. Let $y \in S$. We consider the following two cases:

Case 1: $y \in \mathbb{Z}_n \setminus N(\mathbb{Z}_n)$.

Subcase 1(a): Let $y \notin \frac{n_1}{2} + N(\mathbb{Z}_n)$. Then y is not adjacent to z_i for each $z_i = ln_1 - y$, $l = 1, 2, ..., \frac{n}{n_1}$. Also, for any distinct $l_1, l_2 \in [1, \frac{n}{n_1}]$, $(l_1n_1 - y) + (l_2n_1 - y) = n_1(l_1 + l_2) - 2y \notin N(\mathbb{Z}_n)$. So for each $l = 1, 2, ..., \frac{n}{n_1}$, the vertices $(ln_1 - y)$ are adjacent. Thus $|S| = 2$.

Subcase 1(b): Let $y \in \frac{n_1}{2} + N(\mathbb{Z}_n)$. Since the vertices of the coset $\frac{n_1}{2} + N(\mathbb{Z}_n)$ of cardinality $\frac{n}{n_1}$ are independent, so $|S| = \frac{n}{n_1}$.

Case 2: $y \in N(\mathbb{Z}_n)$. Since the vertices of $N(\mathbb{Z}_n)$ are independent, so $|S| = \frac{n}{n_1}$.

For any non-reduced $\mathbb{Z}_n$, since $\frac{n}{n_1} \geq 2$, clearly from the above cases, $\beta_0(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = \frac{n}{n_1}$. $\square$

Theorem 6.6: For any non-reduced $\mathbb{Z}_n$, the total number of maximum independent sets of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ are given by

$$\begin{cases} 1, \text{ if } n \text{ is odd} \\ 2, \text{ if } n \text{ is even and } \frac{n}{n_1} > 2 \\ 2 + (\frac{n}{n_1})^2(\frac{n_1}{2} - 1), \text{ if } n \text{ is even and } \frac{n}{n_1} = 2 \end{cases}$$

Proof: Let n be odd. Then $N(\mathbb{Z}_n)$ is the only maximum independent set of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$.

Let n be even and $\frac{n}{n_1} > 2$. Then $N(\mathbb{Z}_n)$ and $\frac{n_1}{2} + N(\mathbb{Z}_n)$ are the only two maximum independent sets of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$.

Let n be even and $\frac{n}{n_1} = 2$. Here again, $N(\mathbb{Z}_n)$ and $\frac{n_1}{2} + N(\mathbb{Z}_n)$ are maximum independent sets of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$. Also, for each $x_i \in V(\overline{T(\Gamma_N(\mathbb{Z}_n))})$ such that $1 \leq x_i \leq \frac{n_1}{2} - 1$, the vertices of the coset $x_i + N(\mathbb{Z}_n)$ are not adjacent to the vertices of the coset $(n_1 - x_i) + N(\mathbb{Z}_n)$. So the total number of maximum independent sets of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ are $1 + 1 + (\frac{n}{n_1})^2(\frac{n_1}{2} - 1)$, *i.e.* $2 + (\frac{n}{n_1})^2(\frac{n_1}{2} - 1)$. $\square$

Theorem 6.7: For any non-reduced $\mathbb{Z}_n$,
(i) $i(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = \frac{n}{n_1}$.
(ii) $d_{ind}(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = n_1$.

Proof : (i) Since the vertices of the independent set $N(\mathbb{Z}_n)$ are also dominating and since for some $n \in N(\mathbb{Z}_n)$, the set $N(\mathbb{Z}_n) \setminus \{n\}$ is not dominating, so $N(\mathbb{Z}_n)$ is the independent dominating set of minimum cardinality. Therefore, $i(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = \frac{n}{n_1}$.

(ii) The proof is obvious since $d_{ind}(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = \frac{|V(\overline{T(\Gamma_N(\mathbb{Z}_n))})|}{i(\overline{T(\Gamma_N(\mathbb{Z}_n))})} = \frac{n}{\frac{n}{n_1}} = n_1$. □

Corollary 6.8: For any non-reduced $\mathbb{Z}_n$, the graph $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is well-covered.

Proof: From theorem 6.5 and theorem 6.7 (i), since $\beta_0(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = i(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = \frac{n}{n_1}$, the result follows immediately. □

VII. Conclusion and Future Scope

Our emphasis throughout this paper has been on establishing properties associated to the variants of domination and independence numbers of $T(\Gamma_N(\mathbb{Z}_n))$ and $\overline{T(\Gamma_N(\mathbb{Z}_n))}$. For any non-reduced $\mathbb{Z}_n$, the following table shows a small comparison between the two types of graphs.

TABLE I
A brief comparison between $T(\Gamma_N(\mathbb{Z}_n))$ and $\overline{T(\Gamma_N(\mathbb{Z}_n))}$

	$T(\Gamma_N(\mathbb{Z}_n))$	$\overline{T(\Gamma_N(\mathbb{Z}_n))}$
1	$\gamma(T(\Gamma_N(\mathbb{Z}_n))) = n_1$	$\gamma(\overline{T(\Gamma_N(\mathbb{Z}_n))}) = 2$
2	Number of $\gamma-$sets = $(\frac{n}{n_1})^{n_1}$	Number of $\gamma-$sets = $\frac{n^2}{n_1}(1 - \frac{1}{n_1})$
3	$T(\Gamma_N(\mathbb{Z}_n))$ is excellent	$\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is excellent
4	$T(\Gamma_N(\mathbb{Z}_n))$ is well-covered	$\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is well-covered
5	β_0 depends on the parity of n	β_0 is always $\frac{n}{n_1}$

Another interesting result that we found out is that for any reduced $\mathbb{Z}_n$, the domination number of $T(\Gamma_N(\mathbb{Z}_n))$ depends on the parity of n, while that of $\overline{T(\Gamma_N(\mathbb{Z}_n))}$ is always constant (one).

Our future goal is to introduce and study two new graph structures with all these $\gamma-$sets and maximum independent sets as vertices respectively (specifically for non-reduced $\mathbb{Z}_n$), define a new adjacency condition and then compare the structures and properties associated to these two graphs.

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