Color Energy of Generalised Complements of a Graph

Swati Nayak, Sabitha D'Souza*, Gowtham H. J. and Pradeep G. Bhat,

***Abstract*—The color energy of a graph is defined as sum of absolute color eigenvalues of graph, denoted by $E_c(G)$. Let $G_c = (V, E)$ be a color graph and $P = \{V_1, V_2, \dots, V_k\}$ be a partition of V of order $k \geq 1$. The k -color complement $\{G_c\}_k^P$ of G_c is defined as follows:**

For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add the edges which are not in G_c such that end vertices have different colors. For each set V_r in the partition P, remove the edges of G_c inside V_r, and add the edges of $\overline{G_c}$ (the complement of G_c) joining the vertices of V_r. The graph $\{G_c\}_{k(i)}^P$ thus obtained is called the $k(i)-$ color complement of G_c with respect to the partition P of V. In this paper we characterize generalized color complements of some graphs. We also compute color energy of generalised complements of star, complete, complete bipartite, crown, cocktail party, double star and friendship graphs.

***Index Terms*—$k-$color complement, $k(i)-$color complement, color energy, color spectrum.**

I. INTRODUCTION

Let $G = (V, E)$ be a graph of order n. The complement of a graph G, denoted as $\overline{G}$ has the same vertex set as that of G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. If G is isomorphic to $\overline{G}$ then G is said to be self-complementary graph. For all notations and terminologies we refer [1], [2].

The concept of energy of a graph was introduced by I. Gutman in the year 1978 as the sum of absolute eigenvalues of graph G. More on energy of graphs we refer to [3], [4], [5], [6], [7], [8], [9], [13], [14], [15]. In 1998 E. Sampathkumar et.al defined generalised complements of a graph i.e, $k-$complement G_k^P of a graph G and $k(i)-$complement $G_{k(i)}^P$ of a graph G respectively [10],[11].

Let $G = (V, E)$ be a graph and $P = \{V_1, V_2, \dots, V_k\}$ be a partition of V of order $k \geq 1$. The $k-$ complement G_k^P of G [10] is defined as follows: For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add the edges which are not in G. The graph G is $k-$ self complementary $(k-s.c)$ with respect to P if $G_k^P \cong G$. Further, G is $k-$co$-$self complementary $(k-co-s.c.)$ if $G_k^P \cong \overline{G}$.

For each set V_r in the partition P, remove the edges of G inside V_r, and add the edges of $\overline{G}$ joining the vertices of V_r. The graph $G_{k(i)}^P$ thus obtained is called the $k(i)-$complement [11] of G with respect to the partition P of V. The graph G is $k(i)-$self complementary $(k(i)-s.c)$ if $G_{k(i)}^P \cong G$ for some partition P of order k. Further, G is $k(i)-$co$-$self complementary $(k(i)-co-s.c.)$ if $G_k^P \cong \overline{G}$.

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A coloring of graph G is a coloring of its vertices such that no two adjacent vertices receive the same color. The minimum number of colors needed for coloring a graph G is called chromatic number, denoted by $\chi(G)$.

The color matrix is as follows. If $c(v_i)$ is the color of v_i, then

$$a_{ij} = \begin{cases} 1, & if\ v_i \sim v_j \ \ with\ c(v_i) \neq c(v_j), \\ -1, & if\ v_i \nsim v_j \ \ with\ c(v_i) = c(v_j), \\ 0, & otherwise \end{cases}$$

The matrix thus obtained is the L-matrix of the colored graph G denoted by $A_c(G)$. Color energy of colored graph is the sum of the absolute colored eigenvalues.

i.e.,

$$E_c(G) = \sum_{i=1}^{n} \lambda_i.$$

In 2013, the concept of color energy of a graph was introduced by Chandrashekar Adiga et.al [12].

Let $G = (V, E)$ be a colored graph. Then the *complement of colored graph* G denoted by $\overline{G_c}$ has the same coloring of G with the following properties:

- v_i and v_j are adjacent in $\overline{G_c}$, if v_i and v_j are non-adjacent in G with $c(v_i) \neq c(v_j)$.
- v_i and v_j are non-adjacent in $\overline{G_c}$, if v_i and v_j are non-adjacent in G with $c(v_i) = c(v_j)$.
- v_i and v_j are non-adjacent in $\overline{G_c}$, if v_i and v_j are adjacent in G.

The paper is organised as follows. In section II, we introduce the concept of generalised color complements of a graph. The preliminary results are shown in section III. The characterisation of generalised color complements is presented in section IV. The color energy of generalised complements of few families of graph is obtained in section V.

II. GENERALISED COLOR COMPLEMENTS OF A GRAPH

In this section we introduce a concept of generalised color complements of a graph.

Definition 1. *Let $G_c = (V, E)$ be a color graph and $P = \{V_1, V_2, \dots, V_k\}$ be a partition of V of order $k \geq 1$.*
The k -color complement $\{G_c\}_k^P$ of G_c is defined as follows:

For all V_i and V_j in P, $i \neq j$, remove the edges between V_i and V_j and add the edges which are not in G_c such that end vertices have different colors.

- The graph G_c is $k-$ self color complementary $(k-s.c.c)$ with respect to P if $\{G_c\}_k^P \cong G_c$.
- Further, G_c is $k-$co$-$self color complementary $(k-co-s.c.c)$ if $\{G_c\}_k^P \cong \overline{G}_c$.

Example 2. *Let $\{V_1, V_2\}$ be two partition of vertex set of Cycle graph C_4.*

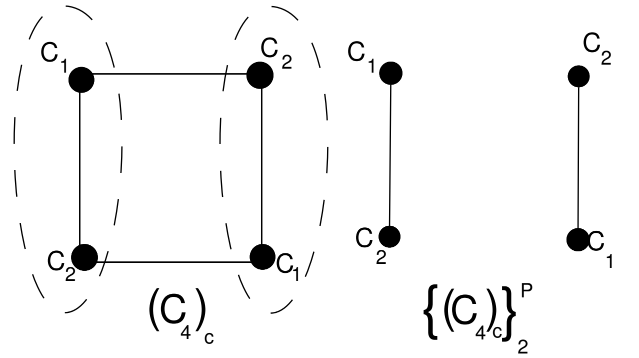

Fig. 1. Colored cycle C_4 and its $2-$ complement.

Definition 3. *For each set V_r in the partition P, remove the edges of G_c inside V_r, and add the edges of $\overline{G_c}$ (the complement of G_c) joining the vertices of V_r. The graph $\{G_c\}^P_{k(i)}$ thus obtained is called the $k(i)-$ color complement of G_c with respect to the partition P of V.*

- The graph G_c is $k(i)-$ self color complementary ($k(i)-s.c.c$) if $\{G_c\}^P_{k(i)} \cong G_c$ for some partition P of order k.
- Further, G_c is $k(i)-$co$-$self color complementary ($k(i)-co-s.c.c$) if $\{G_c\}^P_{k(i)} \cong \overline{G}_c$.

Example 4.

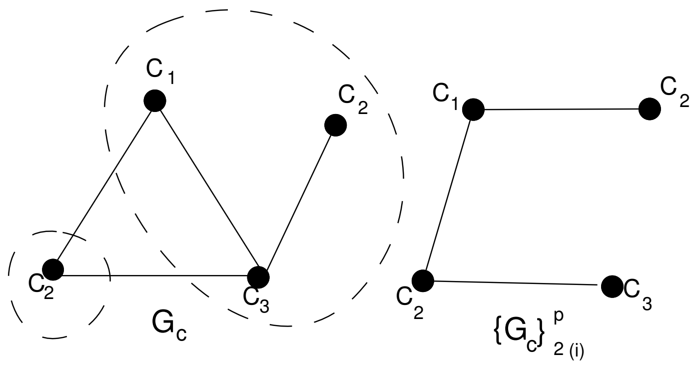

Fig. 2. Graph G_c and $\{G_c\}^P_{2(i)}$.

Definition 5. *In a generalised color complement of graph G, the indegree of vertex v denoted by $i_d(v)$ is defined as number of edges incident to the vertex v inside the partite. Outdegree of vertex v denoted by $o_d(v)$ is defined as number of edges incident to the vertex v outside the partite.*

III. Preliminaries

Now we present few results on k and k(i) self complementary graphs which are extensively used to prove our main results.

Proposition 6. *[10] The $k-$complement and $k(i)-$ complement of G are related as follows:*

1) $\overline{G^P_k} \cong G^P_{k(i)}$.
2) $\overline{G^P_{k(i)}} \cong G^P_k$.

Proposition 7. *[5] Let A=* $\begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$ *be a symmetric 2×2 block matrix. Then the spectrum of A is the union of the spectra of $A_0 + A_1$ and $A_0 - A_1$.*

Proposition 8. *[5] Let M, N, P, Q be matrices, and let M be invertible.*

$$\text{Let } S= \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$$

Then $\det S = \det M. \det[Q - PM^{-1}N]$. If M and P commute, then $\det S = \det\,[MQ - PN]$.

IV. Characterisation of Generalised Color Complements of Graph

The following Proposition highlights the isomorphism between the generalised color complements of a graph.

Proposition 9. *For any G_c,*

1) $\overline{(G_c)^P_k} \cong (\overline{G_c})^P_k$.
2) $\overline{(G_c)^P_{k(i)}} \cong (\overline{G_c})^P_{k(i)}$.

Proof:

1) Consider two vertices u and v in G_c. Then $u \sim v$ in $\overline{(G_c)^P_k} \leftrightarrow u \nsim v$ in $(G_c)^P_k$.
$\leftrightarrow u$ and v are in the same set in P, and are non-adjacent in G_c, or they are in different sets in P, and $u \sim v$ in G_c if $c(u) \neq c(v)$.
$\leftrightarrow$ u and v are in same set in P, and $u \sim v$ in $\overline{G_c}$ if $c(u) \neq c(v)$, or they are in different sets in P, and $u \nsim v$ in $\overline{G_c}$.
$\leftrightarrow u \sim v$ in $(\overline{G_c})^P_k$ if $c(u) \neq c(v)$.
2) Let $u \sim v$ in $\overline{(G_c)^P_{k(i)}}$.
$\leftrightarrow u \nsim v$ in $(G_c)^P_{k(i)}$.
$\leftrightarrow$ u and v are in same set in P, and $u \sim v$ in G_c if $c(u) \neq c(v)$, or they are in different sets in P, and $u \nsim v$ in G_c.
$\leftrightarrow u$ and v are in same set in P, $u \nsim v$ in $\overline{G_c}$, or they are in different sets in P, and $u \sim v$ in $\overline{G_c}$ if $c(u) \neq c(v)$.
$\leftrightarrow u \sim v$ in $(\overline{G_c})^P_{k(i)}$ if $c(u) \neq c(v)$.

■

As a consequence of Proposition 9, we have

Corollary 10. *For any graph G_c,*

1) $(G_c)^P_k \cong G_c$ *if and only if* $(\overline{G_c})^P_k \cong \overline{G_c}$.
2) $(G_c)^P_{k(i)} \cong G_c$ *if and only if* $(\overline{G_c})^P_{k(i)} \cong \overline{G_c}$.

Proposition 11. 1) $\overline{(G_c)^P_k} \cong (G_c)^P_{k(i)}$.

2) $\overline{(G_c)^P_{k(i)}} \cong (G_c)^P_k$.

Corollary 12. *For any graph G_c,*

1) $\overline{(G_c)^P_k} \cong (\overline{G_c})^P_k \cong (G_c)^P_{k(i)}$.
2) $\overline{(G_c)^P_{k(i)}} \cong (\overline{G_c})^P_{k(i)} \cong (G_c)^P_k$.

Corollary 13. 1) $(G_c)^P_k \cong G_c \leftrightarrow (G_c)^P_{k(i)} \cong \overline{G_c}$.

2) $(G_c)^P_{k(i)} \cong G_c \leftrightarrow (G_c)^P_k \cong \overline{G_c}$.

Example 14. *Let $\{V_1, V_2\}$ be two partition of the vertex set of the graph $(P_4)_c$.*

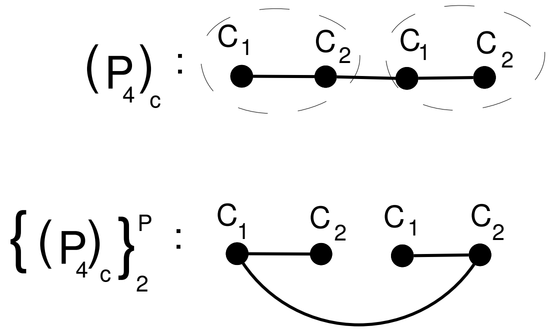

Fig. 3. Colored path P_4 and its $2-$complement.

Since $(P_4)_c \cong \{(P_4)_c\}_2^P$ with respect to the partition $\{V_1, V_2\}$, $(P_4)_c$ is $2-$ self color complementary graph.

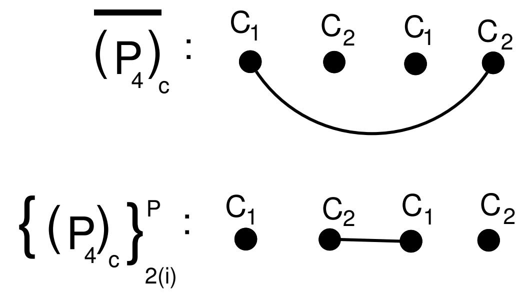

Fig. 4. Complement and $2(i)-$complement of colored path P_4.

Since $\overline{(P_4)_c} \cong \{(P_4)_c\}_{2(i)}^P$ with respect to the partition $\{V_1, V_2\}$, $(P_4)_c$ is $2(i)-$co$-$self color complementary graph.

Theorem 15. *If a $G_c(p, q)$ is $k - s.c.c$, then* $\frac{1}{4}[(k-1)(2p-k)-2q_c'] \leq q \leq \frac{1}{4}[2p(p-k)+k(k-1)+2q_c']$ *where q_c' is number of pairs of non- adjacent vertices with same color between the partites.*

Proof: Let $|V_i| = P_i, 1 \leq i \leq k$. Let q_c' be the number of pair of non-adjacent vertices of same color between the partites in graph G_c. Then total number of edges between V_i and V_j in P, $i \neq j$ in both G_c and $(G_c)_k^P$ is $X = \sum_{i<j} P_i P_j - q_c'$.

Since $G_c \cong (G_c)_k^P$, half the edges of X are in G_c. Hence $q \leq \binom{p}{2} - \frac{1}{2}(\sum_{i<j} P_i P_j - q_c')$. Clearly $\sum_{i<j} P_i P_j$ is minimum when $P_i = 1$ for $i = 1, 2, \ldots, k-1$. Thus we have

$$
\begin{aligned}
q &\leq \binom{P}{2} - \frac{1}{2}\left[\binom{k-1}{2} + (k-1)(p-k+1) - q_c'\right] \\
&\leq \frac{p(p-1)}{2} - \frac{1}{2}\left[\frac{(k-1)(k-2)}{2} + (k-1)(p-k+1) - q_c'\right] \\
&\leq \frac{1}{4}[2p(p-k) + k(k-1) + 2q_c'].
\end{aligned}
$$

To get the lower bound, G_c being $k - s.c.c$ has at least $\sum_{i<j} P_i P_j - q_c'$ edges. So

$$
\begin{aligned}
q &\geq \frac{1}{2}(\sum_{i<j} P_i P_j - q_c') \\
&\geq \frac{1}{2}\left[\binom{k-1}{2} + (k-1)(p-k+1) - q_c'\right] \\
&\geq \frac{1}{2}\left[\frac{(k-1)(k-2)}{2} + (k-1)(p-k+1) - q_c'\right] \\
&\geq \frac{1}{4}[(k-1)(2p-k) - 2q_c'].
\end{aligned}
$$

■

Corollary 16. *If $G_c(p, q)$ is $k(i)-$co$-$s.c.c., then* $\frac{1}{4}[(k-1)(2p-k)-2q_c'] \leq q \leq \frac{1}{4}[2p(p-k)+k(k-1)+2q_c']$.

Remark 17.
1) *Complete bipartite graph $K_{l,m}$ of order n, Star graph $K_{1,n-1}$ and Crown graph S_n^0 are not $k-$self color and $k(i)-$co$-$self color complementary for any n.*
2) *Cycle graph $C_n, n \leq 10$, is $2-$self color and $2(i)-$co$-$self color complementary for $n = 7, 8$.*
3) *Path graphs $P_n, n \leq 10$, is 2-self color and $2(i)-$co$-$self color complementary except for $n = 2, 3, 10$.*
4) *Every Complete bipartite graph $K_{l,m}$ is $k(i)-$self color and $k-$co$-$self color complementary with respect to the partition of same color class.*
5) *All even Cycle graph C_n and Path graph P_n are $2(i)-$self color and $2-$co$-$self color complementary for $n \leq 10$.*

Theorem 18. *$(W_{1,n})_c$ is not $k-$ self color complementary except for $n = 7$ and $k = 3$.*

Proof: Let $(W_{1,n})_c$ be color wheel graph with O as apex vertex and $F = \{f_i / i = 1, 2, \ldots, n\}$ as set of peripheral vertices. Let $P = \{V_1, V_2, \ldots, V_k\}$ be a partition of vertex set of wheel. We observe that the apex vertex O remains either as apex vertex or peripheral vertex in $\{(W_{1,n})_c\}_k^P$. But O cannot be apex vertex in $\{(W_{1,n})_c\}_k^P$. Suppose O belongs to the set V_1. Then, there should be at least 3 peripheral vertices in V_1 so that $deg(O) = 3$ and $n - 3$ peripheral vertices must be in other sets of P. Let u, v and w be peripheral vertices in V_1 so that $u \sim v, w$ and $v \nsim w$. Then $i_d(O) = i_d(u) = 3$ and $i_d(v) = i_d(w) = 2$.

Let us take following two cases. Let us assign numbers $1, 2, 3, 4$ for colors.

Case 1: If n is even, then $\chi(W_{1,n})_c = 3$.
Let $V_1 = \{(1, O), (2, v), (3, u), (2, w)\}$.
Then in $k-$ complement of $(W_{1,n})_c$,
$o_d(u) = o_d(v) = o_d(w) = \frac{n-4}{2}$ and $o_d(O) = 0$.
So that $deg(u) = \frac{n+2}{2}$, $deg(v) = deg(w) = \frac{n}{2}$ and $deg(O) = 3$.
This contradicts the fact that degree of apex vertex of wheel graph is n and that of peripheral vertices is 3. Hence this case fails.

Case 2: If n is odd, then $\chi(W_{1,n})_c = 4$.
Subcase 1: Let $V_1 = \{(1, O), (2, v), (3, u), (2, w)\}$.
Then $o_d(u) = o_d(v) = o_d(w) = \frac{n-3}{2}$ and $o_d(O) = 0$.

Hence $deg(u) = \dfrac{n+3}{2}, deg(v) = deg(w) = \dfrac{n+1}{2}$ and $deg(O) = 3$.

Subcase 2: Let $V_1 = \{(1,O),(3,v),(2,u),(4,w)\}$.

Then $o_d(u) = \dfrac{n-3}{2}, o_d(v) = n-4$, $o_d(w) = \dfrac{n-5}{2}$ and $o_d(O) = 0$.

Therefore $deg(u) = \dfrac{n-3}{2}, deg(v) = n-2, deg(w) = \dfrac{n-1}{2}$ and $deg(O) = 3$.

These two subcases are discarded as the criteria fails.

Subcase 3: Let $V_1 = \{(1,O),(2,v),(4,u),(3,w)\}$.

Then $o_d(u) = n-3, o_d(v) = o_d(w) = \dfrac{n-5}{2}$ and $o_d(O) = 0$.

Thus $deg(u) = n, deg(v) = deg(w) = \dfrac{n-1}{2}$ and $deg(O) = 3$.

Therefore, u is a apex vertex and v, w and O are peripheral vertices for $n = 7$. i.e, $deg(v) = deg(w) = 3$ for $n = 7$.

Possible partitions are :

$\{\{O,u,v,w\},\{x,y,z,m\}\}$
$\{\{O,u,v,w\},\{y,z,m\},\{x\}\}$
$\{\{O,u,v,w\},\{x,y,z\},\{m\}\}$
$\{\{O,u,v,w\},\{x,y,m\},\{z\}\}$
$\{\{O,u,v,w\},\{x,z,m\},\{y\}\}$
$\{\{O,u,v,w\},\{x,y\},\{z,m\}\}$
$\{\{O,u,v,w\},\{x,z\},\{y,m\}\}$
$\{\{O,u,v,w\},\{x,m\},\{y,z\}\}$

Out of these eight partitions, we see that $(W_{1,7})_c$ is isomorphic to $\{(W_{1,7})_c\}_k^P$ for $\{\{O,u,v,w\},\{x,y\},\{z,m\}\}$ only.

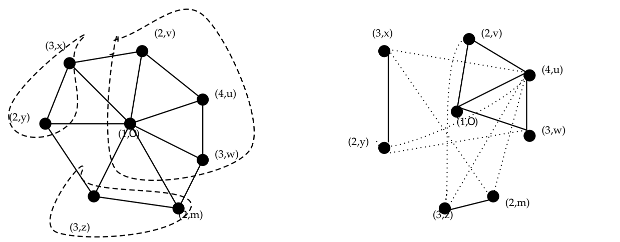

Fig. 5. Color wheel graph $(W_{1,7})_c$ and its 3-color complement $\{(W_{1,7})_c\}_3^P$

Hence $(W_{1,n})_c$ is not $k-$ self color complement except $(W_{1,7})_c$ for $k = 3$. ■

Corollary 19. *$(W_{1,7})_c$ is $3(i)-$ co$-$ self color complementary graph.*

Proof: Proof follows from Theorem 18 and Corollary 13. ■

Theorem 20. *$(W_{1,n})_c$ is $2(i)-$self color complementary if and only if $n = 8$.*

Proof: Let us take the following two cases.

Case 1: Let $P = \{V_1, V_2\}$ be two partition of $(W_{1,n})_c$ such that $< V_1 >= O$ and $< V_2 >= \{f_i/i = 1,2,\ldots,n\}$, where f_i denotes the peripheral vertex. Then the degree of apex vertex in $\{(W_{1,n})_c\}_{2(i)}^P$ will be n as all the edges between V_1 and V_2 remain unaltered so that $\{(W_{1,n})_c\}_{2(i)}^P = K_1 + \overline{(C_n)_c}$. This is possible only when $n = 8$ as C_8 is self color complementary.

Case 2: Let V_1 be the set of apex and some peripheral vertices and V_2 be the set of remaining vertices. Then $\{(W_{1,n})_c\}_{2(i)}^P$ results into disconnected graphs.

Hence $(W_{1,n})_c$ is $2(i)-$self color complementary if and only if $n = 8$ with respect to partition $\{V_1, V_2\}$ such that V_1 has apex vertex and V_2 has remaining vertices. ■

Theorem 21. *$(W_{1,n})_c$ is $2-$ co$-$ self color complementary if and only if $n = 8$.*

Proof: Proof follows from Theorem 20 and Corollary 13. ■

Remark 22. *If G_k^P has q edges and $(G_c)_k^P$ has q' edges, then $q = q' + q'_c$ with respect to same partition, where q'_c is the number of pair of vertices of same color class.*

V. Color energy of generalised complements of some standard Graphs

In this section we consider the colored graph with respect to minimum number of colors. We determine the color energy and color spectrum of generalised complements of some families of graphs.

Let $G = (V,E)$ be a graph. Consider the coloring of G with minimum number of colors, i.e, $\chi(G)$ colors. In such a case, we denote $A_c(G)$ by $A\chi(G)$ and $E_c(G)$ by $E\chi(G)$.

Theorem 23. *Let $\{K_{1,n-1}\}_c$ be colored star graph with partition $P = \{V_1, V_2, \ldots, V_k\}$ where $V_1 = \{v_1, v_2, \ldots, v_m\}$, v_1 be central vertex. Then characteristic polynomial of $k-$ complement of $\{K_{1,n-1}\}_c$ is $(\lambda-1)^{n-3}[\lambda^3 + (n-3)\lambda^2 - (n+m-3)\lambda - (m-1)(n-m-1)]$.*

Proof: Let $\{K_{1,n-1}\}_c$ be a colored star graph on n vertices. Since $\chi(\{K_{1,n-1}\}_c) = 2$, adjacency matrix of $k-$ color complement of star graph is

$$A_\chi(\{K_{1,n-1}\}_c)_k^P = \left[\begin{array}{c|c|c} 0 & J_{1\times m-1} & O_{1\times n-m} \\ \hline J_{m-1\times 1} & -B_{m-1} & -J_{m-1\times n-m} \\ \hline O_{n-m\times 1} & -J_{n-m\times m-1} & -B_{n-m} \end{array}\right]_{n\times n},$$

where O is the matrix of all zero's, J is the matrix of all $1's$ and B is the adjacency matrix of complete subgraph.

$$|\lambda I - A_\chi(\{K_{1,n-1}\}_c)_k^P| = \left|\begin{array}{c|c|c} \lambda & -J_{1\times m-1} & O_{1\times n-m} \\ \hline -J_{m-1\times 1} & (\lambda I + B)_{m-1} & J_{m-1\times n-m} \\ \hline O_{n-m\times 1} & J_{n-m\times m-1} & (\lambda I + B)_{n-m} \end{array}\right|$$

Step 1: For rows $i = 3,4,\ldots,n$, by performing row operation $R_i \longrightarrow R_i - R_{i-1}$, we get $(\lambda-1)^{n-3}\det(C)$.

Step 2: In $\det(C)$, by performing $C_i' = C_i + C_{i+1} + \ldots + C_n$, $i = 2,3,\ldots,n-1$, $\det(C)$ reduces to order 3.

Thus

$$\det(C) = \begin{vmatrix} \lambda & -(m-1) & 0 \\ -1 & \lambda+n-2 & n-m \\ 1 & 0 & \lambda-1 \end{vmatrix}.$$

Step 3: By simplifying, we get

$|\lambda I - A_\chi(\{K_{1,n-1}\}_c)_k^P| = (\lambda-1)^{n-3}[\lambda^3 + (n-3)\lambda^2 - (n+m-3)\lambda - (m-1)(n-m-1)]$. ■

Theorem 24. *Let $\{K_{1,n-1}\}_c$ be colored star graph with partition $P = \{V_1, V_2, \ldots, V_k\}$ where $V_1 = \{v_1, v_2, \ldots, v_m\}$, v_1 being central vertex. Then characteristic polynomial of $k(i)-$ complement of $\{K_{1,n-1}\}_c$ is $(\lambda-1)^{n-3}[\lambda^3 + (n-3)\lambda^2 - (2n-m-2)\lambda - (m-2)(n-m)]$.*

Proof: Adjacency matrix of $(\{K_{1,n-1}\}_c)^P_{k(i)}$ is $A_\chi(\{K_{1,n-1}\}_c)^P_{k(i)} =$

$$\begin{bmatrix} 0 & O_{1\times m-1} & J_{1\times n-m} \\ O_{m-1\times 1} & -B_{m-1} & -J_{m-1\times n-m} \\ J_{n-m\times 1} & -J_{n-m\times m-1} & -B_{n-m} \end{bmatrix}_{n\times n}$$

Consider

$|\lambda I - A_\chi(\{K_{1,n-1}\}_c)^P_{k(i)}| =$

$$\begin{vmatrix} \lambda & O_{1\times m-1} & -J_{1\times n-m} \\ O_{m-1\times 1} & (\lambda I+B)_{m-1} & J_{m-1\times n-m} \\ -J_{n-m\times 1} & J_{n-m\times m-1} & (\lambda I+B)_{n-m} \end{vmatrix}$$

Repeating the step 1 and step 2 of Theorem 23, we obtain the result. ∎

Theorem 25. *If $\{K_{1,n-1}\}_c$ is colored star graph with k partition $P = \{V_1, V_2, \ldots, V_k\}$, where V_1 consists only central vertex, then*

1) $E_\chi(\{K_{1,n-1}\}_c)^P_k = 2(n-2)$.
2) $E_\chi(\{K_{1,n-1}\}_c)^P_{k(i)} = 2(n-1)$.

Proof: Since $\{K_{1,n-1}\}_c$ is $k-$co self and $k(i)-$ self color complement with respect to the above partition, spectrum and energy of $(\{K_{1,n-1}\}_c)^P_k$ and $(\{K_{1,n-1}\}_c)^P_{k(i)}$ respectively are

$Spec_\chi(\{K_{1,n-1}\}_c)^P_k = \begin{pmatrix} 0 & -(n-2) & 1 \\ 1 & 1 & n-2 \end{pmatrix}$ and

$E_\chi(\{K_{1,n-1}\}_c)^P_k = 2(n-2)$.

$Spec_\chi(\{K_{1,n-1}\}_c)^P_{k(i)} = \begin{pmatrix} -(n-1) & 1 \\ 1 & n-1 \end{pmatrix}$ and

$E_\chi(\{K_{1,n-1}\}_c)^P_{k(i)} = 2(n-1)$.

Proof is similar to the theorems [3.8, 3.9] of [12]. ∎

Observation 26. 1) $E_\chi(\{K_n\}_c)^P_k = E(\{K_n\})^P_k$.
2) $E_\chi(\{K_n\}_c)^P_{k(i)} = E(\{K_n\})^P_{k(i)}$.

Theorem 27. *Let $P = \{V_1, V_2\}$ be a partition of colored complete bipartite graph $\{K_{l,m}\}_c$ such that $\langle V_1 \cup V_2\rangle$ is union of color complete bipartite subgraphs. Then characteristic polynomial of 2 and $2(i)-$ color complement of $\{K_{l,m}\}_c$ is $(\lambda-1)^{n-4}[\lambda^4 + \lambda^3(n-4) + \lambda^2(ad - 3n + bc + 6) + \lambda(3n - 2ad - 2bc - abd - abc - acd - bcd - 4) + (ad - n + bc + abc + abd + acd + bcd - 3abcd + 1)]$.*

Proof: 2 and $2(i)-$ color complements of complete bipartite graph is union of colored complete bipartite subgraph i.e, $\{K_{a,b}\}_c \cup \{K_{c,d}\}_c$.

Since the chromatic number of complete bipartite graph is 2, adjacency matrix of 2−color complement of complete bipartite graph is $A_\chi(\{K_{l,m}\}_c)^P_2 =$

$$\begin{bmatrix} -B_a & J_{a\times b} & -J_{a\times c} & 0_{a\times d} \\ J_{b\times a} & -B_b & 0_{b\times c} & -J_{b\times d} \\ -J_{c\times a} & 0_{c\times b} & -B_c & J_{c\times d} \\ 0_{d\times a} & -J_{d\times b} & J_{d\times c} & -B_d \end{bmatrix}_{n\times n},$$

where J is the matrix of all $1's$, B is the adjacency matrix of complete subgraph and 0 is the matrix of all zeroes.

$|\lambda I - A_\chi(\{K_{l,m}\}_c)^P_2| =$

$$\begin{vmatrix} (\lambda I+B)_a & -J_{a\times b} & J_{a\times c} & 0_{a\times d} \\ -J_{b\times a} & (\lambda I+B)_b & 0_{b\times c} & J_{b\times d} \\ J_{c\times a} & 0_{c\times b} & (\lambda I+B)_c & -J_{c\times d} \\ 0_{d\times a} & J_{d\times b} & -J_{d\times c} & (\lambda I+B)_d \end{vmatrix}$$

Step 1: Applying the row operations $R_i \to R_i - R_{i+1}$, where $i \neq a, b, c, d$ result in new determinant $(\lambda-1)^{n-4}\det(C)$.
Step 2: On applying column operations $C_i \to C_i + C_{i-1} + \ldots + C_1$, where $i = a, a-1, \ldots, 2$, $C_j \to C_j + C_{j-1} + \ldots + C_{a+1}$, where $j = b, b-1, \ldots, a+2$, $C_r \to C_r + C_{r-1} + \ldots + C_{b+1}$, where $r = c, c-1, \ldots, b+2$ and $C_s \to C_s + C_{s-1} + \ldots + C_{c+1}$, $s = d, d-1, \ldots, c+2$ in $\det(C)$, we get a new determinant say $\det(D)$.
Step 3: Expanding $\det(D)$ along the rows except at a^{th}, b^{th}, c^{th} and d^{th} rows result into $\det(E)$, which is given by

$$\det(E) = \begin{vmatrix} \lambda+a-1 & -b & c & 0 \\ -a & \lambda+b-1 & 0 & d \\ a & 0 & \lambda+c-1 & -d \\ 0 & b & -c & \lambda+d-1 \end{vmatrix}$$

Step 4: From expansion of $\det(E)$ and by back substitution, we obtain the result. ∎

Corollary 28. *Characteristic polynomial of 2 and $2(i)-$color complement of $\{K_{l,m}\}_c$ with respect to partition of same color classes are $[\lambda+l-1][\lambda+m-1](\lambda-1)^{l+m-2}$ and $[\lambda+l+m-1](\lambda-1)^{l+m-1}$ respectively and the respective color energies are $E_\chi(\{K_{l,m}\}_c)^P_k = 2(l+m-2)$ and $E_\chi(\{K_{l,m}\}_c)^P_{k(i)} = 2(l+m-1)$.*

Proof: Since $\{K_{l,m}\}_c$ with respect to the above partition is $2-$co$-$self and $2(i)-$self color complementary, proof follows from theorems [3.10, 3.11] in [12]. ∎

Corollary 29. *Characteristic polynomial of 2 and $2(i)-$ color complement of $\{K_{l,m}\}_c$ with partition $P = \{V_1, V_2\}$, where $|V_1| = 1$ is*

1) $(\lambda-1)^{l+m-3}[\lambda^3 + (l+m-3)\lambda^2 - (2l+m-3)\lambda + (l+m-lm-1)]$.
2) $(\lambda-1)^{l+m-3}[\lambda^3 + (l+m-3)\lambda^2 - \{l(l-2) - 3(m-1)\}\lambda + \{m(3-2l) + (l-1)\}]$.

Theorem 30. *Let $(S^0_n)_c$ be colored crown graph with vertex set $V = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and partition $P = \{V_1, V_2\}$, where $V_1 = \{u_iv_i/1 \le i \le r, |u_i| = a, |v_i| = b\}$ and $V_2 = \{u_iv_i/i = r+1 \text{ to } n, |u_i| = n-a, |v_i| = n-b\}$. Then*

1) $E_\chi(\{S^0_n\}_c)^P_2 = 2[a(n-a) + n - 2 + \sqrt{n^2 - 3a(n-a)}]$.
2) $E_\chi\{(S^0_n)_c\}^P_{2(i)} = 3n - 4 - 2a + \sqrt{n^2 + 12a(n-a)}$.

Proof: 1) Adjacency matrix of 2−color complement of colored crown graph is $A_\chi(\{S^0_n\}_c)^P_2 =$

$$\begin{bmatrix} -B_a & -J_{a\times n-a} & B_{a\times b} & O_{a\times n-b} \\ -J_{n-a\times a} & -B_{n-a} & O_{n-a\times b} & B_{n-a\times n-b} \\ B_{b\times a} & O_{b\times n-a} & -B_b & -J_{b\times n-b} \\ O_{n-b\times a} & B_{n-b\times n-a} & -J_{n-b\times b} & -B_{n-b} \end{bmatrix},$$

where B is the adjacency matrix of complete subgraph and O is the matrix of all zeroes.

$|\lambda I - A_\chi(\{S^0_n\}_c)^P_2| =$

$$\begin{vmatrix} (\lambda I+B) & J & -B & O \\ J & (\lambda I+B) & O & -B \\ -B & O & (\lambda I+B) & J \\ O & -B & J & (\lambda I+B) \end{vmatrix}$$

Above determinant is of the form $\begin{vmatrix} P & Q \\ Q & P \end{vmatrix}$.

Hence, spectrum is the union of spectra of $|P+Q|$ and $|P-Q|$.

$|P+Q| = \left|\begin{array}{c|c} \lambda I & J \\ \hline J & \lambda I \end{array}\right|_{n\times n}$.

$|P-Q| = \left|\begin{array}{c|c} (\lambda-2)I+2J & J \\ \hline J & (\lambda-2)I+2J \end{array}\right|_{n\times n}$.

Applying the row and column operations respectively $R_i \to R_i - R_{i+1}, i = 1,2,\ldots,a-1,a+1,\ldots,n-1$, $C_i \to C_i + C_{i-1} + C_{i-2} + \ldots + C_1$, where $i = n, n-1, \ldots, 2$ on $|P+Q|$ and $|P-Q|$, we obtain $\lambda^{n-2}[\lambda^2 - a(n-a)]$ and $(\lambda-2)^{n-2}[\lambda^2 + \lambda(2n-4) + 3a(n-a) - 4n + 4]$.

Hence, characteristic polynomial of 2−complement of colored crown graph is $\lambda^{n-2}(\lambda-2)^{n-2}[\lambda^2 - a(n-a)][\lambda^2 + \lambda(2n-4) + 3a(n-a) - 4n + 4]$.

Its color spectrum is

$\begin{pmatrix} 0 & 2 & a(n-a) & -a(n-a) & P & Q \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}$,

where $P = (2-n) + \sqrt{n^2 - 3a(n-a)}$,

$Q = (2-n) - \sqrt{n^2 - 3a(n-a)}$ and

$E_\chi(\{S_n^0\}_c)_2^P = 2[a(n-a) + n - 2 + \sqrt{n^2 - 3a(n-a)}]$.

2) Adjacency matrix of $2(i)$−color complement of colored crown graph is $A_\chi(\{S_n^0\}_c)_{2(i)}^P =$

$$\left[\begin{array}{c|c|c|c} -B_a & -J_{a\times n-a} & I_{a\times b} & J_{a\times n-b} \\ \hline -J_{n-a\times a} & -B_{n-a} & J_{n-a\times b} & I_{n-a\times n-b} \\ \hline I_{b\times a} & J_{b\times n-a} & -B_b & -J_{b\times n-b} \\ \hline J_{n-b\times a} & I_{n-b\times n-a} & -J_{n-b\times b} & -B_{n-b} \end{array}\right]$$

$|\lambda I - A_\chi(\{S_n^0\}_c)_{2(i)}^P| =$

$$\left|\begin{array}{c|c|c|c} (\lambda I + B & J & -I & -J \\ \hline J & (\lambda I + B) & -J & -I \\ \hline -I & -J & (\lambda I + B) & J \\ \hline -J & -I & J & (\lambda I + B) \end{array}\right|$$

Above matrix is of the form $\left|\begin{array}{c|c} P & Q \\ \hline Q & P \end{array}\right|$.

Therefore, spectrum is the union of $|P+Q|$ and $|P-Q|$. Then by operating row and column operations on $|P+Q|$ and $|P-Q|$ as in *(1)*, we get the required characteristic polynomial of $2(i)$−color complement of $(S_n^0)_c$.

i.e, $\lambda^{n-2}(\lambda-2)^{n-2}(\lambda+a-2)(\lambda+n-a-2)[\lambda^2 + n\lambda - 3a(n-a)]$.

Its color spectrum is

$\begin{pmatrix} 0 & 2 & 2-a & 2-n+a & M & N \\ n-2 & n-2 & 1 & 1 & 1 & 1 \end{pmatrix}$,

where $M = -\dfrac{n}{2} + \dfrac{\sqrt{n^2 + 12a(n-a)}}{2}$,

$N = -\dfrac{n}{2} - \dfrac{\sqrt{n^2 + 12a(n-a)}}{2}$

and its color energy is $3n - 2a - 4 + \sqrt{n^2 + 12a(n-a)}$. ■

Corollary 31. *Let $P = \{V_1, V_2\}$ be a partition of $(S_n^0)_c$ with vertices of same color class. Then*

$Spec_\chi\{(S_n^O)_c\}_2^P = \begin{pmatrix} 0 & 2-n & 2 & -n \\ n-1 & 1 & n-1 & 1 \end{pmatrix}$,

$Spec_\chi\{(S_n^O)_c\}_{2(i)}^P = \begin{pmatrix} 0 & 2(1-n) & 2 \\ n & 1 & n-1 \end{pmatrix}$

and $E_\chi\{(S_n^O)_c\}_2^P = E_\chi\{(S_n^O)_c\}_{2(i)}^P = 4(n-1)$.

Proof: Since $(S_n^0)_c$ is 2−co self and $2(i)$−self color complementary with respect to partition of same color class, we get color energy of 2 and $2(i)$ complement of $(S_n^0)_c$. Proof follows from theorems [3.13, 3.14] of [12]. ■

Theorem 32. *Let $(K_{n\times 2})_c$ be a colored cocktail party graph with partition $P = \{V_1, V_2, \ldots, V_k\}$ of same color class. Then*

1) $E_\chi\{(K_{n\times 2})_c\}_k^P = 2n$.

2) $E_\chi\{(K_{n\times 2})_c\}_{k(i)}^P = 6(n-1)$.

Proof: Colored cocktail party graph is k−co-self and $k(i)$− self color complementary with respect to partition of same color class. As the proof is similar to theorems [3.17, 3.18] in [12], we omit it. ■

Definition 33. *A double star $S\{l,m\}_c$ is the graph consisting of union of two stars $K_{1,l}$ and $K_{1,m}$ together with the line joining their centres.*

Theorem 34. *Let $P = \{V_1, V_2\}$ be a partition of colored double star $S\{l,m\}_c$ such that $\langle V_1\rangle = K_{1,l}$ and $\langle V_2\rangle = K_{1,m}$. Then*

$E_\chi(S\{l,m\}_c)_2^P = (n-3) + \sqrt{n^2 + 2n - 7}$.

Proof: Since chromatic number of double star is 2, adjacency matrix of 2−color complement of double star is

$$A_\chi(S\{l,m\}_c)_2^P = \left[\begin{array}{c|c|c} O_2 & C_{2\times l} & -C_{2\times m} \\ \hline C_{l\times 2} & -B_l & J_{l\times m} \\ \hline -C_{m\times 2} & J_{m\times l} & -B_m \end{array}\right]_{n\times n},$$

where B is the adjacency matrix of complete subgraph and $C = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ -1 & -1 & \ldots & -1 \end{pmatrix}$.

$|\lambda I - A_\chi(S\{l,m\}_c)_2^P| =$

$$\left|\begin{array}{c|c|c} \lambda I_2 & -C_{2\times l} & C_{2\times m} \\ \hline -C_{l\times 2} & (\lambda I + B)_l & -J_{l\times m} \\ \hline C_{m\times 2} & -J_{m\times l} & (\lambda I + B)_m \end{array}\right|$$

Step 1: By using row operations $R_1 \to R_1 + R_2$ and $R_i \to R_i - R_{i+1}$, where $i = 3, 4, l-1, l+1, \ldots, m-1$ and simplifying further, we get $\lambda(\lambda-1)^{l+m-2}\det(D)$.

Step 2: On applying the column operation $C_i \to C_i + C_{i+1} + \ldots + C_n$, where $i = 3, 4, \ldots, n-1$ on $\det(D)$, we obtain $\det(E)$.

Step 3: Then by reducing $\det(E)$ along the rows from $R_3, \ldots, R_{l-1}, R_{l+1}, \ldots, R_{m-1}$, we obtain

$Spec_\chi(S\{l,m\}_c)_2^P = \begin{pmatrix} 0 & 1 & X & Y \\ 1 & n-3 & 1 & 1 \end{pmatrix}$,

where $X = \dfrac{-(n-3) + \sqrt{n^2 + 2n - 7}}{2}$,

$Y = \dfrac{-(n-3) - \sqrt{n^2 + 2n - 7}}{2}$ and

$E_\chi(S\{l,m\}_c)_2^P = (n-3) + \sqrt{n^2 + 2n - 7}$.

■

Theorem 35. *Color characteristic polynomial of $2(i)$−complement of double star with partition $P = \{V_1, V_2\}$, such that $\langle V_1\rangle = K_{1,l}$ and $\langle V_2\rangle = K_{1,m}$ is $(\lambda-1)^{n-4}[\lambda^4 + (n-4)\lambda^3 + (lm - 2(n-2))\lambda^2 - 2(lm-1)\lambda + (n-3)]$.*

Proof: Adjacency matrix of $2(i)$− complement of colored double star is

$$A_\chi(S\{l,m\}_c)_{2(i)}^P = \left[\begin{array}{c|c|c} B_{2\times 2} & C_{2\times l} & D_{2\times m} \\ \hline C_{l\times 2} & -B_{l\times l} & O_{l\times m} \\ \hline D_{m\times 2} & O_{m\times l} & -B_{m\times m} \end{array}\right],$$

where $C = \begin{pmatrix} 0 & \ldots & 0 \\ -1 & \ldots & -1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & \ldots & -1 \\ 0 & \ldots & 0 \end{pmatrix}$.

$|\lambda I - A_\chi(S\{l,m\}_c)_{2(i)}^P| =$

$$\left|\begin{array}{c|c|c} (\lambda I - B)_{2\times 2} & -C_{2\times l} & -D_{2\times m} \\ \hline -C_{l\times 2} & (\lambda I + B)_{l\times l} & O_{l\times m} \\ \hline -D_{m\times 2} & O_{m\times l} & (\lambda I + B)_{m\times m} \end{array}\right|$$

Step 1: Reducing the determinant by using row operation $R_i = R_i - R_{i+1}$ for $i = 3, 4, \ldots, l-1, l+1, \ldots, m-1$ results in $(\lambda-1)^{l+m-2}\det(E)$.
Step 2: On applying the column operation $C_i = C_i + C_{i+1} + \ldots + C_n$, where $i = 1, 2, \ldots, n-1$, we obtain $(\lambda-1)^{l+m-2}\det(F)$.
Step 3: On expanding $\det(F)$ along the rows from 3^{rd} row to $(l-1)^{th}$ row and then from $(l+1)^{th}$ row till $(m-1)^{th}$ row, we get $(\lambda-1)^{l+m-2}\det(G)$.
i.e, $\det(G) =$

$$\begin{vmatrix} \lambda-1+m & m-1 & m & m \\ \lambda-1+l & \lambda+l & l & 0 \\ \lambda+l & \lambda+l & \lambda+l-1 & 0 \\ m+\lambda & \lambda+m-1 & \lambda+m-1 & \lambda+m-1 \end{vmatrix}$$

Step 4: On expanding $\det(G)$, we get required characteristic polynomial.
i.e, $(\lambda-1)^{l+m-2}[\lambda^4 + (l+m-2)\lambda^3 + \{(l-2)m-2l\}\lambda^2 - 2(lm-1)\lambda + (l+m-1)]$.
i.e, $(\lambda-1)^{n-4}[\lambda^4 + (n-4)\lambda^3 + \{lm - 2(n-2)\}\lambda^2 - 2(lm-1)\lambda + (n-3)]$.

■

Definition 36. *Friendship graph is constructed by joining n copies of cycle graph C_3 to the common vertex and is denoted by F_n.*

Theorem 37. *If $P = \{v_1, v_2, \ldots. v_{n+1}\}$ is the partition of colored friendship graph of order 2n+1 such that central vertex is in V_1 and $\langle V_i \rangle = K_2$ for $i = 2, \ldots n+1$, then*

1) $E_\chi\{(F_n)_c\}_{n+1}^P = 4n-2$.
2) $E_\chi\{(F_n)_c\}_{n+1(i)}^P = 3(n-1) + \sqrt{n^2+6n+1}$.

Proof: 1) Adjacency matrix of 3 complement of colored friendship graph is

$$A_\chi\{(F_n)_c\}_{n+1}^P = \left[\begin{array}{c|c|c|c|c} B_2 & C_2 & \ldots & C_2 & O_{2\times1} \\ \hline C_2 & B_2 & \ldots & C_2 & O_{2\times1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline C_2 & C_2 & \ldots & B_2 & O_{2\times1} \\ \hline O_{1\times2} & O_{1\times2} & \ldots & O_{1\times2} & O_1 \end{array}\right]_{2n+1}$$

where $C = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$.
$|\lambda I - A_\chi\{(F_n)_c\}_{n+1}^P| =$

$$\left|\begin{array}{c|c|c|c|c} \lambda I - B & -C & \ldots & -C & O \\ \hline -C & \lambda I - B & \ldots & -C & O \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -C & -C & \ldots & \lambda I - B & O \\ \hline O & O & \ldots & O & \lambda \end{array}\right|$$

Step 1: Expanding $|\lambda I - A_\chi\{(F_n)_c\}_k^P|$ along the last row, we get $\lambda\det(D)$ of order $2n$.
Step 2: By using row operation $R_i \to R_i + R_{i+1}, i = 1, 3, \ldots, 2n-1$ on $\det(D)$, it will reduce to $(\lambda-1)^n\det(E)$.
Step 3: On applying the column operation $C_i \to C_i - C_{i-1}, i = 2, 4, \ldots, 2n$ on $\det(E)$ and reducing the matrix along n alternative rows starting from first row of $\det(E)$, we obtain $\det(F)$ of order n.
i.e, $|\lambda I - A_\chi\{(F_n)_c\}_{n+1}^P| =$

$$\lambda(\lambda-1)^n \begin{vmatrix} \lambda+1 & 2 & \ldots & 2 & 2 \\ 2 & \lambda+1 & \ldots & 2 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \ldots & 2 & \lambda+1 \end{vmatrix}$$

Thus by expansion of determinant of order 4, we get $\lambda(\lambda-1)^{2n-1}[\lambda + (2n-1)]$ which is the characteristic polynomial of n complement of colored friendship graph.
Its colored spectrum is $\begin{pmatrix} 0 & 1 & 1-2n \\ 1 & 2n-1 & 1 \end{pmatrix}$.
Hence $E_\chi\{(F_n)_c\}_{n+1}^P = 4n-2$
2) Adjacency matrix of $n+1(i)$ complement of colored friendship graph with one of the partite having central vertex and remaining with K_2 each is

$$A_\chi\{(F_n)_c\}_{n+1(i)}^P = \left[\begin{array}{c|c|c|c|c} O_1 & J_{1\times2} & \ldots & J_{1\times2} & J_{1\times2} \\ \hline J_{2\times1} & O_2 & \ldots & -I_2 & -I_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline J_{2\times1} & -I_2 & \ldots & O_2 & -I_2 \\ \hline J_{2\times1} & -I_2 & \ldots & -I_2 & O_2 \end{array}\right]_{2n+1}$$

$$|\lambda I - A_\chi\{(F_n)_c\}_{n+1(i)}^P| = \left|\begin{array}{c|c|c|c|c} \lambda & -J & \ldots & -J & -J \\ \hline -J & \lambda I & \ldots & I & I \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -J & I & \ldots & \lambda I & I \\ \hline -J & I & \ldots & I & \lambda I \end{array}\right|$$

Step 1: On applying the block row operation $R_i \to R_i - R_{i+1}, i = 2, 3, \ldots, n-1$ on $|\lambda I - A_\chi\{(F_n)_c\}_{k(i)}^P|$, we get $(\lambda-1)^{2(n-1)}\det(D)$.
Step 2: By using block operation $C_i \to C_i + C_{i-1} + \ldots + C_2, i = n, n-1, \ldots, 3$ on $\det(D)$ simplifies to $(\lambda-1)^{2(n-1)}\det(E)$.
Step 3: Expanding $\det(E)$ along the block rows starting from 2^{nd} row till n^{th} row, we get $\det(F)$.
i.e., $\det(F) = \begin{vmatrix} \lambda & -nJ_{1\times2} \\ -J_{2\times1} & (n+\lambda-1)I_2 \end{vmatrix}_{3\times3}$
By back substitution we obtain the polynomial $(\lambda-1)^{2(n-1)}[\lambda + (n-1)][\lambda^2 + (n-1)\lambda - 2n]$.
So that $Spec_\chi\{(F_n)_c\}_{n+1(i)}^P =$

$$\begin{pmatrix} 1 & 2(n-1) \\ 1-n & 1 \\ \dfrac{-(n-1)+\sqrt{(n-1)^2+8n}}{2} & 1 \\ \dfrac{-(n-1)-\sqrt{(n-1)^2+8n}}{2} & 1 \end{pmatrix}$$

Thus, $E_\chi\{(F_n)_c\}_{n+1(i)}^P = 3(n-1) + \sqrt{n^2+6n+1}$.

■

Theorem 38. *Let $(F_n)_c$ be the colored friendship graph with partition $P = \{V_1, V_2, V_3\}$ of same color class. Then*

1) $E_\chi\{(F_n)_c\}_3^P = 4n-4$.
2) $E_\chi\{(F_n)_c\}_{3(i)}^P = 4n$.

Proof: 1) Adjacency matrix of 3−complement of colored friendship graph is

$$A_\chi\{(F_n)_c\}_3^P = \left[\begin{array}{c|c|c} O_1 & O_{1\times n} & O_{1\times n} \\ \hline O_{n\times1} & -B_n & B_n \\ \hline O_{n\times1} & B_n & -B_n \end{array}\right]_{2n+1}$$

$$|\lambda I - A_\chi\{(F_n)_c\}_3^P| = \left|\begin{array}{c|c|c} \lambda & O & O \\ \hline O & \lambda I + B & -B \\ \hline O & -B & \lambda I + B \end{array}\right|$$

The result is got by algorithmic approach.
Step 1: We note that

$$|\lambda I - A_\chi\{(F_n)_c\}_3^P| = \lambda\left|\begin{array}{c|c} \lambda I + B & -B \\ \hline -B & \lambda I + B \end{array}\right|.$$

Step 2: Also we can write

$$\left|\begin{array}{c|c} \lambda I+B & -B \\ \hline -B & \lambda I+B \end{array}\right| = \lambda^n |\lambda I+2B|.$$

Step 3: Applying the row operation $R_i \to R_i - R_{i+1}, i = 2, 3, \ldots, n-1$ and column operation $C_i = C_i + C_{i-1} + \ldots + C_1, i = n, n-1, \ldots, 2$ on $|\lambda I + 2B|$ gives $(\lambda-2)^{n-1}[\lambda + 2(n-1)]$.
Therefore, characteristic polynomial of $3-$complement of colored friendship graph is $\lambda^{n+1}(\lambda-2)^{n-1}(\lambda+2n-2)$ and its color spectrum is $\begin{pmatrix} 0 & 2 & 2-2n \\ n+1 & n-1 & 1 \end{pmatrix}$.
Thus, $E_\chi\{(F_n)_c\}_3^p = 4n-4$.
2) Since chromatic number of friendship graph is 3, adjacency matrix of $3(i)-$ complement of friendship graph is

$$A_\chi\{(F_n)_c\}_{3(i)}^p = \left[\begin{array}{c|c|c|c|c} 0_1 & J_{1\times2} & \ldots & J_{1\times2} & J_{1\times2} \\ \hline J_{2\times1} & B_2 & \ldots & -I_2 & -I_2 \\ \hline J_{2\times1} & -I_2 & \ldots & -I_2 & -I_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline J_{2\times1} & -I_2 & \ldots & B_2 & -I_2 \\ \hline J_{2\times1} & -I_2 & \ldots & -I_2 & B_2 \end{array}\right]_{2n+1}$$

Step 1: On applying block row operation $R_i \to R_i - R_{i+1}, i = 2, 3, \ldots, n-1$ and block column operation $C_i \to C_i + C_{i-1} + \ldots + C_2, i = n, n-1, \ldots, 3$ on $|\lambda I - A_\chi\{(F_n)_c\}_{3(i)}^p|$, we get

$$[\det((\lambda-1)I-B)]^{n-1} \left|\begin{array}{c|c} \lambda & -nJ_{1\times2} \\ \hline -J_{2\times1} & [(\lambda+n-1)I-B]_2 \end{array}\right|_3$$

Step 2: Further simplification gives the characteristic polynomial $\lambda^{n-1}(\lambda-2)^n(\lambda+n)^2$.
Hence its color spectrum is

$Spec_\chi\{(F_n)_c\}_{3(i)}^p = \begin{pmatrix} 0 & 2 & -n \\ n-1 & n & 2 \end{pmatrix}$ and its color energy $E_\chi\{(F_n)_c\}_{3(i)}^p = 4n$. ■

Theorem 39. *Let $P = \{V_1, V_2\}$ be partition of colored friendship graph such that central vertex in V_1 and other vertices in V_2. Then*

1) $E_\chi\{(F_n)_c\}_2^P = 4n-4$.
2) $E_\chi\{(F_n)_c\}_{2(i)}^P = 4(n-1) + 2\sqrt{2n}$.

Proof: 1) Adjacency matrix of 2 complement of colored friendship graph is

$$A_\chi\{(F_n)_c\}_2^P = \left[\begin{array}{c|c|c|c|c} B_2 & -I_2 & \ldots & -I_2 & O_{2\times1} \\ \hline -I_2 & B_2 & \ldots & -I_2 & O_{2\times1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline -I_2 & -I_2 & \ldots & B_2 & O_{2\times1} \\ \hline O_{1\times2} & O_{1\times2} & \ldots & O_{1\times2} & O_{1\times1} \end{array}\right]_{2n+1}$$

$$|\lambda I - A_\chi\{(F_n)_c\}_2^P| = \left|\begin{array}{c|c|c|c|c} \lambda I-B & I & \ldots & I & O \\ \hline I & \lambda I-B & \ldots & I & O \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline I & I & \ldots & \lambda I-B & O \\ \hline O & O & \ldots & O & \lambda \end{array}\right|$$

Step 1: We note that $|\lambda I - A_\chi\{(F_n)_c\}_2^P| = \lambda\det(C)$.
Step 2: Employing block row and column operations $R_i \to R_i - R_{i+1}, i = 1, 2, \ldots, n-1$, $C_i \to C_i + C_{i-1} + \ldots + C_1, i = n, n-1, \ldots, 2$ on $\det(C)$ and further simplification gives us the required result.
Therefore, characteristic polynomial of $2-$complement of colored friendship graph is $\lambda^n(\lambda+n)(\lambda+n-2)(\lambda-2)^{n-1}$ and its color spectrum is $\begin{pmatrix} 0 & -n & 2-n & 2 \\ n & 1 & 1 & n-1 \end{pmatrix}$.
Thus, $E_\chi\{(F_n)_c\}_2^P = 4n-4$.
2) Adjacency matrix of $2(i)$ complement of colored friendship graph is

$$A_\chi\{(F_n)_c\}_{2(i)}^P = \left[\begin{array}{c|c|c|c|c} O_2 & J_2 & \ldots & J_2 & J_{2\times1} \\ \hline J_2 & O_2 & \ldots & C_2 & C_{2\times1} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline J_2 & C_2 & \ldots & O_2 & C_{2\times1} \\ \hline J_{1\times2} & C_{1\times2} & \ldots & C_{1\times2} & O_{1\times1} \end{array}\right]_{2n+1}$$

■

Step 1: In $|\lambda I - A_\chi\{(F_n)_c\}_{2(i)}^P|$, applying block row operation $R_i \to R_i - R_{i+1}, i = 2, 3, \ldots, n-1$ and block column operation $C_i \to C_i + C_{i-1} + \ldots + C_2, i = n, n-1, \ldots, 3$ gives $[\det(\lambda I + C)]^{n-1} \left|\begin{array}{c|c} \lambda & -nJ_{1\times2} \\ \hline -J_{2\times1} & (\lambda I-(n-1)C)_2 \end{array}\right|_3$
Step 2: Further simplifying, we obtain $\phi_\chi\{(F_n)_c\}_{2(i)}^P = [\lambda(\lambda-2)]^{n-1}(\lambda^2-2n)(\lambda+2(n-1))$ and its color spectrum

$$Spec_\chi\{(F_n)_c\}_{2(i)}^P = \begin{pmatrix} 0 & n-1 \\ 2 & n-1 \\ \sqrt{2n} & 1 \\ -\sqrt{2n} & 1 \\ -2(n-1) & 1 \end{pmatrix}.$$

Hence $E_\chi\{(F_n)_c\}_{2(i)}^P = 4(n-1) + 2\sqrt{2n}$.

Observation 40. *1)* $(F_n)_c$ *is* $3-$*co*$-$*self and* $3(i)-$*self color complementary with respect to the partition of same color class.*
2) $\{(F_n)_c\}_3^P$ *with respect to the partition of same color class and* $\{(F_n)_c\}_2^P$ *with respect to the partition* $P = \{V_1, V_2\}$ *such that central vertex in* V_1 *and remaining vertices in* V_2 *are non-co spectrum equi-energetic graphs.*

Conclusion: Generalised color complement of graph not only depends on the partition of vertex set but also depends on the assigned colors to the vertices. In this paper, we have defined and characterised generalised color complement of graph. The color spectrum and color energy of generalised complements of families of graphs are derived.

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1) Modification date: 04-12-2020.
2) We have starred (*) for the 2^{nd} author (Sabitha D’Souza) in the first page below the main title.
3) At the foot note in the first page, we have written *Corresponding author : SABITHA D’SOUZA, Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, 576104 India.