Some Fast Algorithms for Exterior Anisotropic Problems in Concave Angle Domains

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***Abstract*—In this paper, some fast algorithms using elliptical arc artificial boundary is designed to solve exterior anisotropic problems in concave angle domains. Some exact nonlocal boundary conditions are derived on the elliptical arc artificial boundary. Based on the above artificial boundary conditions, the Dirichlet-Neumann alternating method is presented. The convergence of this algorithm is examined. Finally, some numerical examples are given to show the effectiveness of our methods.**

***Index Terms*—elliptical arc artificial boundary, Dirichlet -Neumann alternating method, anisotropic problems, concave angle domains, convergence**

I. Introduction

THE problems in unbounded domains are encountered in many fields of scientific and engineering computing. There is a variety of numerical methods to solve such problems. One of the commonly used techniques is the method of artificial boundary conditions [1]-[9]. Based on artificial boundary conditions, the overlapping and non-overlapping domain decomposition methods can be viewed as effective ways to solve problems in unbounded domains. These techniques have been used to solve many linear or nonlinear problems [10]-[21]. Recently, the authors used an overlapping domain decomposition method to solve anisotropic problems in concave angle domains [22]. In this paper, we design a non-overlapping domain decomposition method to solve the above problems.

Let Ω be an exterior concave angle domain with angle ω, and $0 < \omega < 2\pi$. The boundary of domain Ω is decomposed into three disjoint parts: Γ, Γ_0 and Γ_ω (see Fig. 1), i.e. $\partial\Omega = \overline{\Gamma \cup \Gamma_0 \cup \Gamma_\omega}$, $\Gamma_0 \cap \Gamma_\omega = \emptyset$, $\Gamma \cap \Gamma_0 = \emptyset$, $\Gamma \cap \Gamma_\omega = \emptyset$. The boundary Γ is a simple smooth curve part, Γ_0 and Γ_ω are two half lines. We consider the following anisotropic problem:

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_0 \cup \Gamma_\omega, \\ \mathcal{A}\nabla u \cdot n = g, & \text{on } \Gamma, \\ u \text{ is vanish at infinity}, \end{cases} \tag{1}$$

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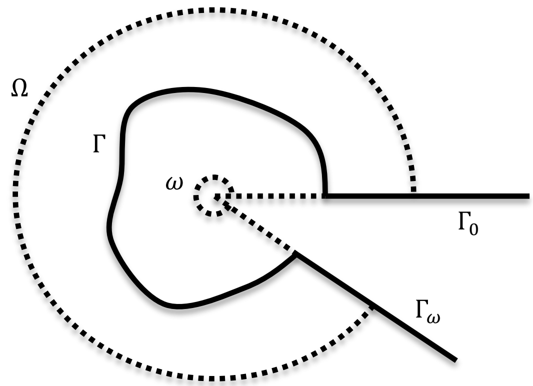

Fig. 1: The illustration of domain Ω

and

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) = f, & \text{in } \Omega, \\ \mathcal{A}\nabla u \cdot n = 0, & \text{on } \Gamma_0 \cup \Gamma_\omega, \\ u = h, & \text{on } \Gamma, \\ u \text{ is bounded at infinity}, \end{cases} \tag{2}$$

where $\mathcal{A} = \begin{pmatrix} k^2 & 0 \\ 0 & 1 \end{pmatrix}$, k is a constant and $0 < k < 1$, u is the unknown function, $f \in L^2(\Omega)$ and $g, h \in L^2(\Gamma)$ are given functions, $\mathrm{supp}(f)$ is compact.

The outline of the paper is as follows. In Section 2, we derive an exact elliptical arc artificial boundary condition for the above anisotropic problem. In Section 3, we construct a Dirichlet-Neumann alternating, and give the convergence of the method. Finally, in Section 4 we present some numerical results to show its accuracy and the effectiveness of our methods.

II. The Exact Artificial Boundary Condition

Let f_0 denote the half distance between the two foci of an ellipse, we introduce an elliptic system of co-ordinates (μ, φ) such that the artificial boundary $\mathcal{B}$ coincides with the elliptical arc $\{(\mu, \varphi) | \mu = \mu_R,\ 0 < \varphi < \omega\}$, where $f_0 = \frac{\sqrt{1-k^2}}{k} R$, $\mu_R = \ln \frac{1+k}{\sqrt{1-k^2}}$. Thus, the Cartesian co-ordinates (ξ, η) are related to the elliptic co-ordinates (μ, φ), that is $\xi = f_0 \cosh\mu \cos\varphi$, $\eta = f_0 \sinh\mu \sin\varphi$. The domain exterior to $\mathcal{B}$, namely the $\{(\mu, \varphi) | \mu > \mu_R,\ 0 < \varphi < \omega\}$ is denoted by D, . Let $\mathcal{B}$ be a circle arc with radius R at the origin, enclosing Γ and $\mathrm{supp}(f)$. We first introduce the following transformation $x = k\xi,\ y = \eta$, then the anisotropic problem (1) become the following Poisson problem:

$$\begin{cases} -\Delta u = f, \quad \text{in } \tilde{\Omega}, \\ u = 0, \quad \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \\ \frac{\partial u}{\partial n} = \tilde{g}, \quad \text{on } \tilde{\Gamma}, \end{cases} \tag{3}$$

where $\tilde{g} = \frac{R}{k\sqrt{J}} g, J = \frac{R^2}{k^2}(\sin^2\varphi + k^2\cos^2\varphi)$. The artificial boundary is an elliptical arc $\tilde{\mathcal{B}} = \{(\xi,\eta) | k^2\xi^2 + \eta^2 = R^2,\ (\xi,\eta) \in \tilde{\Omega}\}$, and the exterior domain to $\tilde{\mathcal{B}}$ is $\tilde{D} = \{(\xi,\eta) | k^2\xi^2 + \eta^2 > R^2,\ (\xi,\eta) \in \tilde{\Omega}\}$.

Assume that $f = 0$ in the domain $\tilde{D}$, then problem (3) confines in $\tilde{D}$ is

$$\begin{cases} -\Delta u = 0, \quad \text{in } \tilde{D}, \\ u = 0, \quad \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \\ u \text{ is vanish at infinite.} \end{cases} \tag{4}$$

By separation of variables, we know that the solution of problem (4) has the form

$$u(\mu,\varphi) = \sum_{n=1}^{+\infty} b_n e^{(\mu_R-\mu)\frac{n\pi}{\omega}} \sin\frac{n\pi\varphi}{\omega}, \tag{5}$$

where

$$b_n = \frac{2}{\omega}\int_0^\omega u(\mu_R,\phi)\sin\frac{n\pi\phi}{\omega}d\phi,\ n = 1,2,\cdots. \tag{6}$$

Thus (5) can be written as

$$u(\mu,\varphi) = \frac{2}{\omega}\sum_{n=1}^{+\infty} e^{(\mu_R-\mu)\frac{n\pi}{\omega}}\int_0^\omega u(\mu_R,\phi)\sin\frac{n\pi\varphi}{\omega}\sin\frac{n\pi\phi}{\omega}d\phi \triangleq H(\mu_R,\mu,\varphi). \tag{7}$$

We differentiate (7) with respect to μ and set $\mu = \mu_R$ to obtain

$$\frac{\partial u}{\partial \mu}|_{\tilde{\mathcal{B}}} = -\frac{2\pi}{\omega^2}\sum_{n=1}^{+\infty} n\int_0^\omega u(\mu_R,\phi)\sin\frac{n\pi\varphi}{\omega}\sin\frac{n\pi\phi}{\omega}d\phi. \tag{8}$$

Since $\frac{\partial u}{\partial n}|_{\tilde{\mathcal{B}}} = -\frac{1}{\sqrt{J}}\frac{\partial u}{\partial \mu}|_{\tilde{\mathcal{B}}}$, we obtain the exact artificial boundary condition on $\tilde{\mathcal{B}}$:

$$\frac{\partial u}{\partial n}|_{\tilde{\mathcal{B}}} = \frac{2\pi}{\omega^2\sqrt{J}}\sum_{n=1}^{+\infty} n\int_0^\omega u(\mu_R,\phi)\sin\frac{n\pi\varphi}{\omega}\sin\frac{n\pi\phi}{\omega}d\phi \triangleq \mathcal{K}u(\mu_R,\varphi). \tag{9}$$

III. Dirichlet-Neumann Alternating Method

Draw a circular arc $\Gamma_1 = \{(\mu,\varphi) | \mu > \mu_1,\ 0 < \varphi < \omega\}$, which enclose Γ such that $\text{dist}(\Gamma,\Gamma_1) > 0$. Then Ω is divided into two non-overlapping subdomains Ω_1 and Ω_2 (see Fig. 2). Let Ω_1 be the bounded domain among Γ, Γ_0, Γ_ω and Γ_1, and Ω_2 be the unbounded domain outside Γ_1, Γ_0 and Γ_ω. Then the problem (1) is decomposed into two subproblems in domains Ω_1 and Ω_2, we proposed the Dirichlet-Neumann alternating method as follows.

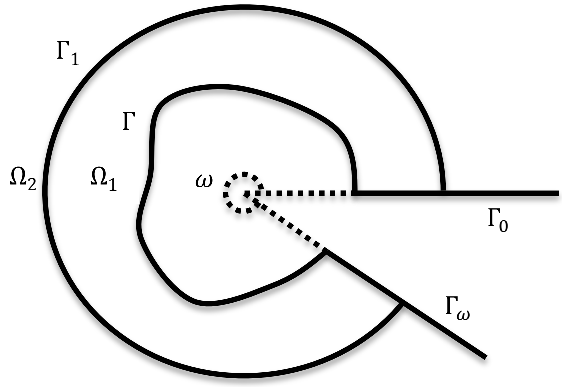

Fig.2: The illustration of domain Ω_1 and Ω_2

Step 1. Pick an initial value $\lambda^{(0)} \in H^{\frac{1}{2}}(\Gamma_1)$, and put $l = 0$.

Step 2. Solve a Dirichlet problem in Ω_2:

$$\begin{cases} -\nabla\cdot(\mathcal{A}u_2^{(l)}) = f, \quad \text{in } \Omega_2, \\ u_2^{(l)} = 0, \quad \text{on } \Gamma_0 \cup \Gamma_\omega, \\ u_2^{(l)} = \lambda^{(l)}, \quad \text{on } \Gamma_1, \\ u_2^{(l)} \text{ is vanish at infinity.} \end{cases} \tag{10}$$

Step 3. Solve a mixed problem in Ω_1:

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u_1^{(l)}) = f, \quad \text{in } \Omega_1, \\ u_1^{(l)} = 0, \quad \text{on } \Gamma_0 \cup \Gamma_\omega, \\ \mathcal{A}\nabla u_1^{(l)}\cdot n = g, \quad \text{on } \Gamma, \\ \mathcal{A}\nabla u_1^{(l)}\cdot n = -\mathcal{A}\nabla u_2^{(l)}\cdot n, \quad \text{on } \Gamma_1. \end{cases} \tag{11}$$

Step 4. Update the boundary value on Γ_1 by

$$\lambda^{(l+1)} = \theta_l u_1^{(l)} + (1-\theta_l)\lambda^{(l)}, \tag{12}$$

Step 5. Set $l = l + 1$, then goto Step 2.

where $u_1^{(l)}$ and $u_2^{(l)}$ are the lth approximate solutions in Ω_1 and Ω_2, respectively. θ_l denotes the lth relaxation factor and $\lambda^{(0)}$ is an arbitrary function in $H^{\frac{1}{2}}(\Gamma_1)$.

In the following, we just consider the convergence and convergence rate of problem (1), we can obtain corresponding result of problem (2) in the same way.

It is difficult to analyze the convergence of the above alternating method in the general domain. However, the analysis is possible for some special curve Γ. Therefore, we only consider the case where the boundaries Γ and Γ_1 both are elliptical arcs, i.e., $\Gamma = \{(x,y) | \frac{x^2}{k^2} + y^2 = R_0^2,\ (x,y) \in \Omega\}$, $\Gamma_1 = \{(x,y) | \frac{x^2}{k^2} + y^2 = R_1^2,\ (x,y) \in \Omega\}$, and $R_1 > R_0$. Let $x = k\xi, y = \eta$, then the mixed problem:

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla u)=f, & \text{in } \Omega_1,\\ u=0, & \text{on } \Gamma_0\cup\Gamma_\omega,\\ \mathcal{A}\nabla u\cdot n=g, & \text{on } \Gamma,\\ \mathcal{A}\nabla u\cdot n=g_1, & \text{on } \Gamma_1,\end{cases} \tag{13}$$

become the following problem:

$$\begin{cases} -\Delta u=f, & \text{in } \tilde{\Omega}_1,\\ u=0, & \text{on } \tilde{\Gamma}_0\cup\tilde{\Gamma}_\omega,\\ \frac{\partial u}{\partial n}=\tilde{g}, & \text{on } \tilde{\Gamma},\\ \frac{\partial u}{\partial n}=\tilde{g}_1, & \text{on } \tilde{\Gamma}_1,\end{cases} \tag{14}$$

where $\tilde{g}=\frac{\sqrt{k^2\sin^2\varphi+\cos^2\varphi}}{k}g$, $\tilde{g}_1=\frac{\sqrt{k^2\sin^2\varphi+\cos^2\varphi}}{k}g_1$, $\tilde{\Gamma}=\{(\xi,\eta)|\xi^2+\eta^2=R_0^2,\ (\xi,\eta)\in\tilde{\Omega}\}$, $\tilde{\Gamma}_1=\{(\xi,\eta)|\xi^2+\eta^2=R_1^2,\ (\xi,\eta)\in\tilde{\Omega}\}$. And on Γ_1 we have

$$\mathcal{K}u(\mu_1,\varphi)=\frac{k}{\sqrt{k^2\sin^2\varphi+\cos^2\varphi}}\frac{2\pi}{\omega^2R_1}\sum_{n=1}^{+\infty}n\int_0^\omega u(\mu_1,\phi)\sin\frac{n\pi\varphi}{\omega}\sin\frac{n\pi\phi}{\omega}\,d\phi,$$

where $f_0=\sqrt{k^2-1}R_1,\ \mu_1=\ln\frac{k+1}{\sqrt{k^2-1}}$.

Let

$$e_2^{(l)}=\lambda-\lambda^{(l)}=\sum_{n=1}^{+\infty}b_n\sin\frac{n\pi\varphi}{\omega},\ \text{on } \Gamma_1,$$

we have

$$\begin{aligned}\mathcal{A}\nabla e_1^{(l)}\cdot n&=-\mathcal{K}e_2^{(l)}\\&=-\frac{k}{\sqrt{k^2\sin^2\varphi+\cos^2\varphi}}\frac{\pi}{\omega R_1}\sum_{n=1}^{+\infty}nb_n\sin\frac{n\pi\varphi}{\omega}.\end{aligned} \tag{15}$$

By the separation of variables, we have

$$e_1^{(l)}=-\sum_{n=1}^{+\infty}b_nH_n(r)\sin\frac{n\pi\varphi}{\omega},$$

where

$$H_n(r)=\frac{R_1^{\frac{n\pi}{\omega}}\left(r^{\frac{n\pi}{\omega}}-R_0^{\frac{2n\pi}{\omega}}r^{-\frac{n\pi}{\omega}}\right)}{R_1^{\frac{2n\pi}{\omega}}+R_0^{\frac{2n\pi}{\omega}}}.$$

Hence

$$\mathcal{K}e_1^{(l)}=-\frac{k}{\sqrt{k^2\sin^2\varphi+\cos^2\varphi}}\frac{\pi}{\omega R_1}\sum_{n=1}^{+\infty}nb_nH_n(R_1)\sin\frac{n\pi\varphi}{\omega}.$$

Then, we have

$$\begin{aligned}\mathcal{A}\nabla e_1^{(l+1)}\cdot n&=-\mathcal{K}\left(\lambda-\lambda^{(l+1)}\right)\\&=\mathcal{K}\left(\theta_ku_1^{(l)}+(1-\theta_k)\lambda^{(l)}-\lambda\right)\\&=-\frac{k}{\sqrt{k^2\sin^2\varphi+\cos^2\varphi}}\frac{\pi}{\omega R_1}\\&\cdot\sum_{n=1}^{+\infty}nb_n(\theta_lH_n(R_1)-1+\theta_l)\sin\frac{n\pi\varphi}{\omega}.\end{aligned}$$

If we let

$$E^{(l)}=\left\|\mathcal{A}\nabla e_1^{(l)}\cdot n\right\|^2_{-\frac{1}{2},\Gamma_1},$$

then

$$E^{(l)}=\frac{k^2}{k^2\sin^2\varphi+\cos^2\varphi}\frac{\pi^2}{\omega^2R_1^2}\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}}n^2b_n^2,$$

and

$$\begin{aligned}E^{(l+1)}&=\frac{k^2}{k^2\sin^2\varphi+\cos^2\varphi}\frac{\pi^2}{\omega^2R_1^2}\\&\cdot\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}}n^2b_n^2(\theta_lH_n(R_1)-1+\theta_l)^2\\&=(1-\theta_l)^2E^{(l)}+\frac{k^2}{k^2\sin^2\varphi+\cos^2\varphi}\frac{\pi^2}{\omega^2R_1^2}\\&\cdot\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}}n^2b_n^2\theta_lH_n(R_1)(\theta_lH_n(R_1)+2\theta_l-2.)\end{aligned}$$

Let

$$\delta=\inf_{n\in Z^+}\frac{2}{2+H_n(R_1)}.$$

A computation shows that $\delta=\frac{2}{3}$.

If we let $\theta_l=0,1,2,\cdots$, satisfy $0<\theta_l\le\delta$, then

$$E^{(l+1)}<(1-\theta_l)^2E^{(l)}.$$

By the trace theorem, we have

$$\left\|e_1^{(l)}\right\|_{1,\Omega_1}\le CE^{(l)}\to 0,\quad l\to+\infty.$$

This means that the Dirichlet-Neumann alternating method is convergent if $0<\theta_l\le\delta$.

We also have

$$\begin{aligned}E^{(l+1)}&=\frac{k^2}{k^2\sin^2\varphi+\cos^2\varphi}\frac{\pi^2}{\omega^2R_1^2}\\&\cdot\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}}n^2b_n^2(\theta_lH_n(R_1)-1+\theta_l)^2\\&=\frac{k^2}{k^2\sin^2\varphi+\cos^2\varphi}\frac{\pi^2}{\omega^2R_1^2}\\&\cdot\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}}n^2b_n^2\left(2\theta_l-1-2\theta_lG_n(R_1)\right)^2\\&=(1-2\theta_l)^2E^{(k)}+\frac{k^2}{k^2\sin^2\varphi+\cos^2\varphi}\frac{\pi^2}{\omega^2R_1^2}\\&\cdot\sum_{n=1}^{+\infty}(1+n^2)^{-\frac{1}{2}}n^2b_n^2\theta_lG_n(R_1)(\theta_lG_n(R_1)-2\theta_l+1),\end{aligned}$$

where

$$G_n(R_1)=\frac{1-H_n(R_1)}{2}.$$

Let

$$\sigma=\sup_{n\in Z^+}\frac{1}{2-G_n(R_1)}.$$

It is easy to get $\sigma = \frac{2}{3}$.

Similar to the above analysis, if we take $\theta_l = 0, 1, 2, \cdots$, satisfy $\sigma \le \theta_l < 1$, the Dirichlet-Neumann alternating method is also convergent.

Therefore, for $0 < \theta_l < 1$, the Dirichlet-Neumann alternating method is convergent.

IV. Numerical Examples

In this section, we give a numerical example to show the effectiveness of Dirichlet-Neumann alternating method. The finite element method with liner elements is used in the computation.

Example 1. We consider problem (1), where $\Omega = \{(r,\theta)|r > 2,\ 0 < \theta < 2\pi\}$, $\Gamma = \{(r,\theta)|r = 2,\ 0 < \theta < 2\pi\}$, $\Gamma_0 = \{(r,\theta)|r > 2,\ \theta = 0\}$, and $\Gamma_\omega = \{(r,\theta)|r > 2, \theta = 2\pi\}$. By using coordinate transformation $x = k\xi$, $y = \eta$, we turn the original problem into the problem as the following

$$\begin{cases} -\Delta u = f, & \text{in } \tilde{\Omega}, \\ u = 0, & \text{on } \tilde{\Gamma}_0 \cup \tilde{\Gamma}_\omega, \\ \frac{\partial u}{\partial n} = \tilde{g}, & \text{on } \tilde{\Gamma}, \end{cases} \quad (16)$$

where $\tilde{\Omega} = \{(\mu,\varphi)|\mu > \mu_0,\ 0 < \varphi < 2\pi\}$, $\tilde{\Gamma} = \{(\mu,\varphi)|\mu = \mu_0,\ 0 < \varphi < 2\pi\}$, $\tilde{\Gamma}_0 = \{(\mu,\varphi)|\mu = \mu_0,\ \varphi = 0\}$, $\tilde{\Gamma}_\omega = \{(\mu,\varphi)|\mu = \mu_0,\ \varphi = 2\pi\}$, $f_0 = \frac{2\sqrt{1-k^2}}{k}$, and $\mu_0 = \ln\frac{1+k}{\sqrt{1-k^2}}$. Let $u(x,y) = \frac{k^2 y}{x^2+k^2y^2}$ be the exact solution of original problem and $g = \frac{\partial u}{\partial n}|_\Gamma$.

u_{1h} is the finite element solution in $\tilde{\Omega}_1$, e and e_h denote the maximal error of all node functions in $\tilde{\Omega}_1$, respectively, i.e.,

$$e(l) = \sup_{P_i \in \tilde{\Omega}_1} |u(P_i) - u_{1h}^l(P_i)|,$$

$$e_h(l) = \sup_{P_i \in \tilde{\Omega}_1} |u_{1h}^{l+1}(P_i) - u_{1h}^l(P_i)|.$$

$q_h(l)$ is the approximation of the convergence rate, i.e.,

$$q_h(l) = \frac{e_h(l-1)}{e_h(l)}.$$

We consider the Dirichlet-Neumann alternating method. Let $\tilde{\Gamma}_1 = \{(\mu,\varphi)|\mu = \mu_0 + t_1,\ 0 < \varphi < 2\pi\}$ be the artificial boundary, and $t_1 = 1$. Figure 3 shows the mesh h of subdomain $\tilde{\Omega}_1$, Table 1 shows the convergence rate for different anisotropic coefficient k (Mesh $h/4$, $\theta = 0.5$). Table 2 shows the relation between convergence rate and mesh ($k = 0.5$, $\theta = 0.5$). Table 3 shows the relation between convergence rate and relaxation factor ($k = 0.5$, Mesh $h/4$, $l = 6$). Figure 4 shows $L^\infty(\tilde{\Omega}_1)$ errors for different mesh. Figure 5 shows the convergence rate for different relaxation factor.

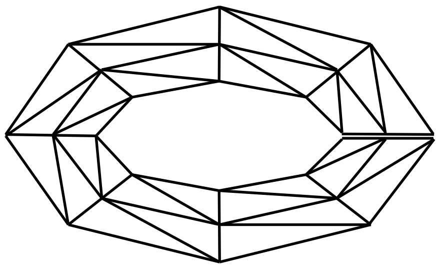

Fig. 3: Mesh h of domain $\tilde{\Omega}_1$.

TABLE 1: THE CONVERGENCE RATE FOR DIFFERENT ANISOTROPIC COEFFICIENT k (MESH $h/4$, $\theta = 0.5$).

k	l	0	1	2	3	4	5
0.8	$e(l)$	0.213	0.034	0.009	0.013	0.012	0.012
	$e_h(l)$		0.247	0.038	0.006	0.001	0.000
	$q_h(l)$			6.566	6.487	6.482	6.444
0.5	$e(l)$	0.164	0.034	0.017	0.022	0.019	0.019
	$e_h(l)$		0.189	0.028	0.004	0.001	0.000
	$q_h(l)$			6.771	6.516	6.502	6.364
0.2	$e(l)$	0.085	0.065	0.057	0.058	0.058	0.058
	$e_h(l)$		0.097	0.014	0.002	0.000	0.000
	$q_h(l)$			7.155	6.577	6.497	6.003

TABLE 2: THE RELATION BETWEEN CONVERGENCE RATE AND MESH ($k = 0.5$, $\theta = 0.5$)

M	l	0	1	2	3	4	5
$h/2$	$e(l)$	0.155	0.075	0.060	0.062	0.062	0.062
	$e_h(l)$		0.180	0.025	0.004	0.001	0.000
	$q_h(l)$			7.297	6.667	6.304	5.843
$h/4$	$e(l)$	0.164	0.034	0.017	0.022	0.019	0.019
	$e_h(l)$		0.189	0.028	0.004	0.001	0.000
	$q_h(l)$			6.771	6.516	6.502	6.364
$h/8$	$e(l)$	0.166	0.025	0.004	0.001	0.001	0.001
	$e_h(l)$		0.191	0.029	0.005	0.001	0.000
	$q_h(l)$			6.633	6.426	6.426	6.418

TABLE 3: THE RELATION BETWEEN CONVERGENCE RATE AND RELAXATION FACTOR θ ($k = 0.5$, MESH $h/4$, $l = 6$)

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
q_h	1.298	1.845	3.153	6.128	5.921	2.592	1.628	1.185

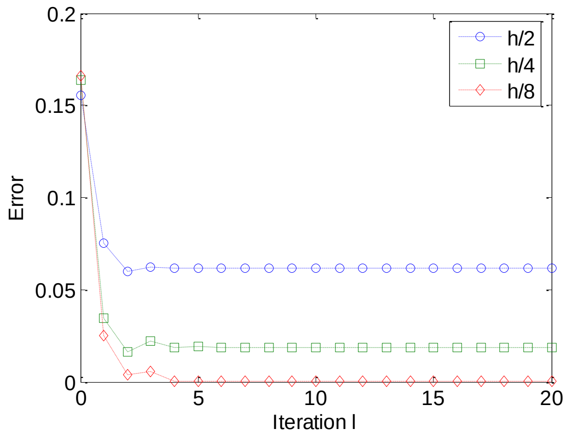

Fig. 4: $L^\infty(\tilde{\Omega}_1)$ errors for different mesh.

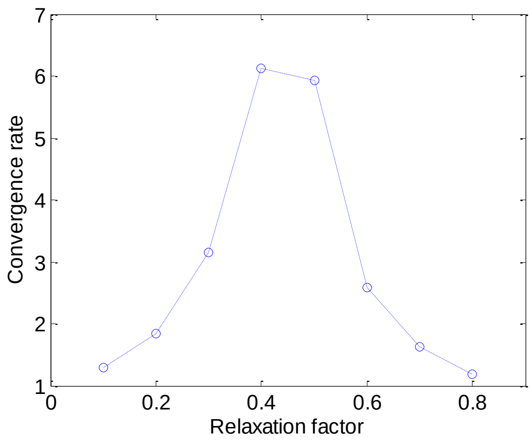

Fig. 5: The convergence rate for different relaxation factor.

The numerical results show that the Dirichlet-Neumann alternating method is feasible and convergent quickly. Its convergence rate is independent of finite element mesh parameter h. The method is convergent for all relaxation factor $\theta \in (0,1)$, and the convergence of the method is the best when the relaxation factor $\theta \in (0.4,0.5)$.

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