Classification of Time Fractional Fin Equation via Lie Symmetry Method

Saeed M. Ali and Mohammed D. Kassim

***Abstract*—In this work, we study the nonlinear fin equation with time fractional. Our study extends the problem considered in [7], [8] by taking the derivative for time variable in fractional order β with $0 < \beta \leq 1$. The symmetry analysis is applied to this problem and reduction of this equation to nonlinear ordinary differential equation of fractional order is established**

***Index Terms*—Lie Symmetry Analysis, fractional-order derivatives, Fin Equation, Reduction.**

I. Introduction

Fins are extended surfaces used, inter alia, to increase the heat exchange from a hot or cold surface to surrounding areas. The study of heat transfer has received much consideration from analysts and engineers, as have tools that increase heat transfer. Among them are fins that have been used for a variety of purposes, such as aircraft engines, compressors and computer processors.

The heat transfer in fins of different shapes and profiles with a variety of boundary conditions can be depicted by mathematical models [1]. These models can be solved by a number of different methods (see [2], [3], [4], [5], [6], [7], [8]). In 2014, Ali et al. [9] considered the fin equation in cylindrical coordinates, which has the form

$$\tfrac{1}{x}\tfrac{\partial}{\partial x}(xJ(u)u_x) + \tfrac{1}{x}\tfrac{\partial}{\partial y}(\tfrac{1}{x}J(u)u_y) - N^2 g(x)u = u_t. \tag{1}$$

They used Lie symmetry analysis to transform this equation into an ordinary differential equation. Pakdemirli and Sahin [7], [8] investigated the equation

$$\frac{\partial}{\partial x}\left(J(\theta)\frac{\partial\theta}{\partial x}\right) - N^2 g(x)\theta = \theta_t, \tag{2}$$

by using the Lie symmetries of the governing partial differential equation. In 2014, Iyiola and Zaman [15] studied tumor models with fractional-order derivatives and compared their results with those derived from classical derivatives with respect to time.

Recently, the study of differential equations of noninteger order has received great attention because the description of some phenomena can be portrayed better and more accurately with this type of equation than with integer-order differential equations. For example, in 2013, Wang et al. [12] studied the time-fractional potential of Burger's equation by using Lie symmetry analysis. In [14], the researchers considered the generalized Kdv equation. They used symmetry analysis to reduce this equation into the Erdèlyi-Kober fractional derivative. In 2017, Gaur and Singh [13] considered the time-fractional generalized Burger's and Korteweg-de Vries equations. These equations were transformed into nonlinear ordinary differential equations of fractional order. For more results in this direction, we refer to [16], [17], [18], [19], [20], [21], [22], [23]. This paper is devoted to studying a nonlinear fin equation of time-fractional order by considering variable heat transfer coefficients and nonlinear thermal conductivity. To this end, we use Lie symmetry approach to reduce the nonlinear partial differential equations with fractional order into nonlinear ordinary differential equations of fractional order. In this paper, we are concerned with the following problem:

$$J(u)u_{xx} + J_u(u_x)^2 - N^2 g(x)u = u_t^{\beta}, \quad 0 < \beta \leq 1. \tag{3}$$

The article is organized as follows. In the next section, we introduce some definitions and notations and prepare some material that will be used that throughout this article. In Section 3, symmetry analysis of the given problem is performed using Lie symmetry, and in Section 4, complete classifications of the solutions of the equation (3) are presented. In Section 5, symmetry generators are listed. The reduction of the problem to fractional differential equations is shown in Section 6. Finally, we conclude some results about this problem.

II. Preliminaries

In this section, we introduce some definitions of fractional calculus and show in detail the Lie symmetry analysis for fractional partial differential equations with two independent variables. There are many definitions of fractional derivatives and fractional integrals, but in this article, the definition of The modified Riemann-Liouville derivative is used to study this problem:

Definition 2.1:[25] The fractional derivative of order β of $g(t)$ is

$$D_t^{\beta} g(t) = \begin{cases} [g^{(n)}(t)]^{(\beta-n)},\ n \leq \beta < n+1, n \geq 1 \\ \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\displaystyle\int_0^t (t-s)^{-\beta}(g(s)-g(0))ds, \\ 0 < \beta \leq 1 \end{cases} \tag{4}$$

Consider a time-fractional partial differential equation in the form

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} = G(x, t, u, u_x, u_{xx}, ...),\ \beta > 0. \tag{5}$$

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For parameter ϵ, we suppose that (5) is invariant, that is,

$$\begin{aligned} t^* &= t + \epsilon\psi(x,t,u) + O(\epsilon^2), \\ x^* &= x + \epsilon\zeta(x,t,u) + O(\epsilon^2), \\ u^* &= u + \epsilon\chi(x,t,u) + O(\epsilon^2), \\ \frac{\partial u^*}{\partial t^*} &= \frac{\partial u^\beta}{\partial t^\beta} + \epsilon\chi^0_\beta(x,t,u) + O(\epsilon^2), \\ \frac{\partial u^*}{\partial x^*} &= \frac{\partial u}{\partial x} + \epsilon\chi^{(x)}(x,t,u) + O(\epsilon^2), \\ \frac{\partial^2 u^*}{\partial x^{*2}} &= \frac{\partial^2 u}{\partial x^2} + \epsilon\chi^{(xx)}(x,t,u) + O(\epsilon^2), \\ &\vdots \end{aligned} \tag{6}$$

where

$$\begin{aligned} \chi^{(x)} &= \chi_x + (\chi_u - \zeta_x)u_x - \zeta_u u_x^2 - \psi_x u_t - \psi_u u_x u_t \\ \chi^{(xx)} &= \chi_{xx} + (\chi_{ux} - \zeta_{xx})u_x - \zeta_{ux}u_x^2 - \psi_{xx}u_t \\ &-\psi_{ux}u_x u_t + u_x(\chi_{xu} + (\chi_{uu} - \zeta_{xu})u_x - \zeta_{uu}u_x^2 - \\ &\psi_{xu}u_t - \psi_{uu}u_x u_t) + u_{xt}(-\psi_x - \psi_u u_x) \\ &+u_{xx}((\chi_u - \zeta_x) - 2\zeta_u u_x - \psi_u u_t) \\ &-(\zeta_x + u_x\zeta_u)u_{xx} - (\psi_x + u_x\psi_u)u_{xt}. \end{aligned} \tag{7}$$

The βth extended infinitesimal related to the Riemann-Liouville fractional time derivative (see [10]) is

$$\begin{aligned} \chi^0_\beta &= \frac{\partial^\beta \chi}{\partial t^\beta} = (\chi_u - \beta D_t(\psi))\frac{\partial^\beta u}{\partial t^\beta} - u\frac{\partial^\beta \chi_u}{\partial t^\beta} \\ &- \sum_{n=1}^{\infty}\binom{\beta}{n}D_t^n(\zeta)D_t^{\beta-n}(u_x) + \sum_{n=1}^{\infty}\Big[\binom{\beta}{n}\frac{\partial^n\chi_u}{\partial t^n} \\ &-\binom{\beta}{n+1}D_t^{n+1}(\psi)\Big] + \mu, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \mu &= \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{J=2}^{m}\sum_{r=0}^{J-1}\binom{\beta}{n}\binom{n}{m}\times \\ &\binom{J}{r}\frac{1}{J!}\frac{t^{n-\beta}}{J!\Gamma(n+1-\beta)}[-u]^r\frac{\partial^m}{\partial t^m(u^{(J-r)})}\frac{\partial^{n-m+J}\chi}{\partial t^{n-m}\partial u^J}. \end{aligned} \tag{9}$$

III. Symmetry Analysis of the Fractional Fin Equation

In this section, the symmetry analysis of Eq. (3) is represented. Accordingly the generator related to this equation is

$$X = \zeta(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial y} + \psi(x,t,u)\frac{\partial}{\partial t} + \chi(x,t,u)\frac{\partial}{\partial u}. \tag{10}$$

The needs of invariance for Eq. (3) regarding to the prolonged symmetry generator leads to

$$X^{(2)} = X + \chi^(x)\frac{\partial}{\partial u_x} + \chi^0_\beta\frac{\partial}{\partial u_t^\beta} + \chi^{(xx)}\frac{\partial}{\partial u_{xx}}. \tag{11}$$

Then, we use the Lie symmetry criterion wherein the partial differential Eq. (3) is invariant under the prolonged symmetry generator (11) modulo the partial differential equation, namely,

$$X^{(2)}[J(u)u_{xx} + J_u(u_x)^2N^2g(x)u - u_t^\beta]\Big|_{PDE(3)=0} = 0. \tag{12}$$

Comparing terms containing derivatives of the dependent function u and using the results from Eq. (12) leads to the following system in ζ, η, ψ and χ:

$$\zeta_u = \psi_u = \psi_x = \chi_{uu} = 0, \tag{13}$$

$$J_u\chi + J\chi_u - 2J\zeta_x = 0, \tag{14}$$

$$J_{uu}\chi + 2J_u\chi_u - 2J_u\zeta_x = 0, \tag{15}$$

$$2J_u\chi_x - J\zeta_{xx} = 0, \tag{16}$$

$$\binom{\beta}{n}\frac{\partial^n\chi_u}{\partial t^n} - \binom{\beta}{n+1}D_t^{n+1}\psi = 0, \tag{17}$$

$$D_t^n\zeta = 0, \tag{18}$$

$$-\zeta N^2 g_x u - \chi N^2 g = 0. \tag{19}$$

To the expression μ vanishes, we assume that χ is linear in u; then, we assume that

$$\begin{aligned} \chi(u) &= c_1 u, \\ \zeta(x) &= c_2 x + c_3, \\ \psi(t) &= \frac{c_4}{\beta}t, \end{aligned} \tag{20}$$

according to the above hypotheses, equations (13), (16), (17) and (18) hold. Thus, the above system (13-19) becomes:

$$c_1 u J_u + c_1 J - 2c_2 J = 0, \tag{21}$$

$$c_1 u J_{uu} + 2c_1 J_u - 2c_2 J_u = 0, \tag{22}$$

$$-(c_2 x + c_3)N^2 g_x u - c_1 u N^2 g = 0. \tag{23}$$

By solving (21) and (22) simultaneously, we obtain

$$J(u) = u^{\frac{2c_2 - c_1}{c_1}}. \tag{24}$$

We conclude that equations (21) and (22) hold if and only if $c_1 = c_2 = 0$; or $J(u) = u^{\frac{2c_2-c_1}{c_1}}$.

IV. Classification

In this section, we present the full classification of the solution to this problem (3). Starting with two cases resulting from (21) and (22):

(I) $c_1 = c_2 = 0$ and $J(u)$ is any function in u,

(II) $c_1 \neq 0,\ c_2 \neq 0$ and $J(u) = u^{\frac{2c_2-c_1}{c_1}}$.

For complete categorization, we analyze one by one.

4.1 Case I

If $c_1 = c_2 = 0$ and $J(u)$ is any function in u, then Eq. (23) becomes $c_3 g_x = 0$. This leads to the following cases:

(I) $c_3 = 0$ and $g(x)$ is any function in x.

(II) $c_3 \neq 0$ and $g(x) = A$, where A is constant.

4.1.1. Subcase (I) $c_3 = 0$ and $g(x)$ is any function in x.
In this case, the infinitesimal generators ζ, ψ and χ have the following form

$$\zeta = 0,\ \psi = \frac{c_4}{\beta}t,\ \chi = 0. \tag{25}$$

In accordance with above infinitesimals, the symmetry generator is given by

$$X_2 = \frac{t}{\beta}\frac{\partial}{\partial t}. \tag{26}$$

4.1.2. Subcase (II) $c_3 \neq 0$ and $g(x) = A$
The forms of the symmetry generators for ζ, ψ and χ corresponding to this case are

$$\zeta = c_3,\ \psi = \frac{c_4}{\beta}t,\ \chi = 0. \tag{27}$$

Then, the associated generators to the above infinitesimals are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x} + \frac{t}{\beta}\frac{\partial}{\partial t}. \tag{28}$$

4.2 Case II

If $c_1 \neq 0$, $c_2 \neq 0$, and $J(u) = u^{\frac{2c_2-c_1}{c_1}}$, then Eq. (23) becomes $-(c_2 x + c_3)g_x - c_1 g = 0$, which means that $g(x) = (c_2 x + c_3)^{-\frac{c_1}{c_2}}$.

The expressions for the infinitesimal symmetry generators for ζ, ψ and χ corresponding to this case are

$$\zeta = c_2 x + c_3,\ \psi = \frac{c_4}{\beta} t,\ \chi = c_1 u. \quad (29)$$

The above infinitesimals yield the following generators

$$X_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad X_2 = x\frac{\partial}{\partial x} + \frac{t}{\beta}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \quad (30)$$

V. Symmetry Generators

In this section, we consider the symmetry generators obtained above for special values of $g(x)$ and $J(u)$.

1- $J(u)$ is any function in u.

a- If $g(x)$ is any function in x, then we only have one symmetry generator,

$$X_2 = \frac{t}{\beta}\frac{\partial}{\partial t}. \quad (31)$$

b- Let $g(x) = A$, where A is constant, then the symmetry generators corresponding to this case are

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial x} + \frac{t}{\beta}\frac{\partial}{\partial t}. \quad (32)$$

2- $J(u) = u^{\frac{2c_2 - c_1}{c_1}}$ and $g(x) = (c_2 x + c_3)^{\frac{-c_1}{c_2}}$.
This leads to the following generators

$$X_1 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad X_2 = x\frac{\partial}{\partial x} + \frac{t}{\beta}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \quad (33)$$

VI. Some Reductions

In this section, we use similarity variables obtained through the symmetry generators to present some reductions of Eq. (3).

A. Case 1: $J(u) = u$ and $g(x) = \frac{1}{x}$

In this case, we introduce a reduction to Eq. (3) by using the symmetry generators obtained in the previous section. We consider the symmetry generator $X_2 = x\frac{\partial}{\partial x} + \frac{t}{\beta}\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}$. This generator has the following characteristic equation

$$\frac{dx}{x} = \frac{\beta dt}{t} = \frac{du}{u}. \quad (34)$$

Solving the above equation, the similarity variables: $u = t^\beta w(z)$ and $z = x\,t^{-\beta}$ are obtained. We use these similarity variables to transform equation (3) into a nonlinear ordinary differential equation of fractional order. The below theorem shows this transformation.

Theorem(1) The fin Eq. (3) reduces under similarity variables $z = x\,t^{-\beta}$ and $u = t^\beta w(z)$ to the following nonlinear ODE of order β with $0 < \beta \le 1$, namely:

$$ww'' + (w')^2 - N^2 z^{-1} w = \left(1 - \beta z\frac{d}{dz}\right)\left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z), \quad (35)$$

where $(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w)(z)$ is the Erdélyi-Kober fractional integral operator defined in general in the following form:

$$\left(E^{\psi,\beta}_{\delta} w\right)(z) := \begin{cases} \dfrac{1}{\Gamma(\beta)}\displaystyle\int_1^\infty (\theta-1)^{\beta-1}\theta^{-(\psi+\beta)} w(z\theta^{\frac{1}{\delta}})\,d\theta, & \beta > 0, \\ w(z), & \beta = 0. \end{cases} \quad (36)$$

Proof:
The Riemann-Liouville fractional derivative for the similarity transformations $z = x\,t^{-\beta}$ and $u = t^\beta w(z)$ for $0 < \beta \le 1$ is

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial}{\partial t}\left[\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta} s^\beta w(x\,s^{-\beta})\,ds\right]. \quad (37)$$

We set $s = \frac{t}{\theta}$; then, $\theta = \frac{t}{s}$, and hence the above equation becomes

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial}{\partial t}\left[\frac{t}{\Gamma(1-\beta)}\int_1^\infty (\theta-1)^{-\beta}\theta^{-2} w(z\theta^\beta)\,d\theta\right], \quad (38)$$

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial}{\partial t}\left(tE^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z). \quad (39)$$

Consider $g(z) \in C^1(0,\infty)$ with $z = x\,t^{-\beta}$, then

$$t\frac{\partial g(z)}{\partial t} = tx(-\beta t^{-\beta-1})g'(z) = -\beta z g'(z). \quad (40)$$

In accordance with (40), we infer that

$$\begin{aligned} &\frac{\partial}{\partial t}\left(tE^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z) \\ &= \left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z) + t\frac{d}{dz}\left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z)\frac{dz}{dt} \\ &= \left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z) + t\frac{d}{dz}\left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z)(-x\beta t^{-\beta-1}) \\ &= \left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z) - \beta z\frac{d}{dz}\left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z) \\ &= \left(1 - \beta z\frac{d}{dz}\right)\left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z). \end{aligned} \quad (41)$$

Therefore, (38) becomes

$$\frac{\partial^\beta u}{\partial t^\beta} = \left(1 - \beta z\frac{d}{dz}\right)\left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z). \quad (42)$$

Thus, the time-fractional fin Eq. (3) reduces to the following nonlinear ordinary differential equation of fractional order:

$$ww'' + (w')^2 - N^2 z^{-1} w = \left(1 - \beta z\frac{d}{dz}\right)\left(E^{1+\beta,1-\beta}_{\frac{1}{\beta}} w\right)(z). \quad (43)$$

B. Case 2: $J(u)$ is an arbitrary function, and $g(x)$ is constant

In this case, we introduce a reduction to Eq. (3) by using the symmetry generators obtained in the previous section. We consider the symmetry generator $X_2 = \frac{\partial}{\partial x} + \frac{t}{\beta}\frac{\partial}{\partial t}$. The characteristic equation corresponding to this generator is

$$\frac{dx}{1} = \frac{\beta dt}{t} = \frac{du}{0}. \quad (44)$$

Solving the above equation, we obtain the following similarity variables: $z = e^x t^{-\beta}$ and $u = w(z)$. Using these similarity variables, the equation (3) is transformed into a nonlinear ordinary differential equation of fractional order, and the following theorem shows this transformation.

Theorem(2) The similarity variables $z = e^x t^{-\beta}$ and $u = w(z)$ reduce the time-fractional fin Eq. (3) to a nonlinear ordinary differential equation of fractional order of the form

$$\begin{aligned} &z^2 J(w) w'' + zJ(w)w' + z^2 J_w (w')^2 - N^2 A w \\ &= \left(1 - \beta - \beta z\frac{d}{dz}\right)\left(E^{1,1-\beta}_{\frac{1}{\beta}} w\right)(z), \end{aligned} \quad (45)$$

where $(E^{1,1-\beta}_{\frac{1}{\beta}} w)(z)$ is the Erdélyi-Kober fractional integral operator.

Proof:
The proof of this Theorem is similar to that of the above Theorem.

VII. Exact solution for particular case of the time Fractional Fin Equation

Our aim in this section is to find an exact solution to problem (3) for particular cases of $g(x)$ and $J(u)$. To this end, we use the sub-equation method that will be described in the following steps (see, e.g. [27], [26], [28]:
Consider the PDE in two variables x and t, with the form

$$F(u, u_t, u_x, u_t^{\beta}, u_x^{\beta}, ...) = 0, \quad 0 < \beta \leq 1 \tag{46}$$

where u_x^{β} and u_t^{β} are the modified Riemann Liouville derivatives of u with respect to x and t, respectively.

Step 1. We assume the appropriate transformation in order to reduce Eq. (17) into nonlinear ordinary differential equation, namely

$$u = h(w), \quad w = w(x, t), \tag{47}$$

where $h(w)$ is a differentiable function of w.

Step 2. Then the solution of Eq. (18) is assumed to be in the form

$$w(\eta) = I(\varphi), \quad \varphi = \varphi(\eta), \quad \eta = \eta(x, t). \tag{48}$$

where $I(\varphi)$ is a differentiable function of φ, and φ satisfy the following fractional Riccati equation as the equation that we will use in our problem

$$D_{\eta}^{\beta}\varphi = \kappa + \varphi^2, \quad 0 < \beta \leq 1. \tag{49}$$

here D_{η}^{β} is the modified Riemann-Liouville derivative and κ is a constant. The Eq. (20) has five explicit solution obtained by method of Exp-function (see [30], [29]). These solution are of the following forms

$$\varphi(\eta) = \begin{cases} -\sqrt{-\kappa}\tanh(\sqrt{-\kappa}\eta, \beta), & \kappa < 0, \\ -\sqrt{-\kappa}\coth(\sqrt{-\kappa}\eta, \beta), & \kappa < 0, \\ \sqrt{\kappa}\tan(\sqrt{\kappa}\eta, \beta), & \kappa > 0, \\ -\sqrt{\kappa}\cot(\sqrt{\kappa}\eta, \beta), & \kappa > 0, \\ -\frac{\Gamma(1+\kappa)}{\eta^{\beta}+\omega}, \quad \omega = constant, & \kappa = 0, \end{cases} \tag{50}$$

with the generalized trigonometric and hyperbolic functions

$$\begin{aligned} &\tanh_{\beta}(\eta) = \frac{\sinh_{\beta}(\eta)}{\cosh_{\beta}(\eta)}, \quad \coth_{\beta}(\eta) = \frac{\cosh_{\beta}(\eta)}{\sinh_{\beta}(\eta)}, \\ &\tan_{\beta}(\eta) = \frac{\sin_{\beta}(\eta)}{\cos_{\beta}(\eta)}, \quad \cot_{\beta}(\eta) = \frac{\cos_{\beta}(\eta)}{\sin_{\beta}(\eta)}, \\ &\sinh_{\beta}(\eta) = \frac{E_{\beta}(\eta^{\beta}) - E_{\beta}(-\eta^{\beta})}{2}, \\ &\cosh_{\beta}(\eta) = \frac{E_{\beta}(\eta^{\beta}) + E_{\beta}(-\eta^{\beta})}{2}, \\ &\sin_{\beta}(\eta) = \frac{E_{\beta}(i\eta^{\beta}) - E_{\beta}(-i\eta^{\beta})}{2}, \\ &\cosh_{\beta}(\eta) = \frac{E_{\beta}(i\eta^{\beta}) + E_{\beta}(-i\eta^{\beta})}{2}, \end{aligned} \tag{51}$$

where $E_{\beta}(\eta)$ is the Mittag-Leffler function defined by

$$E_{\beta}(\eta) = \sum_{i=0}^{\infty} \frac{\eta^i}{\Gamma(1+i\beta)}. \tag{52}$$

Step 3. We suppose in this case the $I(\varphi)$ takes the form

$$I(\varphi) = \sum_{i=0}^{n} b_i \varphi_i \quad and \quad \eta = x + l\,t + \eta_0. \tag{53}$$

where $b_i (i = 0, 1, 2, ..., n)$ and l are constants, η_0 is an arbitrary constants and n is a positive integer obtained by balancing nonlinear terms and highest order derivatives in the reduced equation.

Step 4. Substituting Eq. (22) along with Eq. (20) in the reduced fractional ordinary differential equation and making the coefficients of φ^i be zero, we get a set of algebraic equations in unknowns $b_i (i = 0, 1, 2, ..., n)$ and l. Then using Maple to find these unknowns.

Step 5. Then substitute these constants and solutions of Riccati equation into the Eq. (19), we obtain the exact and explicit solutions of the nonlinear partial fractional differential equation.

A. Applications to the time Fractional Fin equation

We apply the method of sub-equation described above to solve the problem (3) when $J(u) = \frac{1}{u}$ and $g(x) = m$, m is a constant
the Eq. (3) in this case becomes

$$uu_{xx} - u_x^2 - N^2 mu^3 = u^2 D_t^{\beta} u. \tag{54}$$

To solve Eq. (23), using sub-equation method, we follow the steps mentioned above. First, we consider the following transformations:

$$u(x, t) = u(\eta), \quad \eta = x + lt. \tag{55}$$

here l is a constant.Then we substitute (24) into (23), we obtain the following fractional ordinary differential equation

$$uu_{\eta\eta} - u_{\eta}^2 - N^2 mu^3 = l^{\beta} u^2 D_{\eta}^{\beta} u. \tag{56}$$

We assume that the solution of Eq. (25) takes the following form

$$u(\eta) = \sum_{i=0}^{n} bi(\varphi)^i, \tag{57}$$

Then we find the positive integer n by balancing the highest order derivative terms with nonlinear terms in Eq. (23), we infer

$$u(\eta) = b_0 + b_1(\varphi) + b_2(\varphi)^2. \tag{58}$$

Substituting Eq. (27) along with Eq. (20) in the reduced fractional ordinary differential equation and making the coefficients of $(\varphi)^i$ be zero, we get a set of algebraic equations in unknowns $b_i (i = 0, 1, 2, ..., n)$ and l as follows

$$\begin{aligned} &2b_0b_2\kappa^2 - b_1^2b_0\kappa^2 - N^2mb_0^3 - l^{\beta}b_0^2b_1\kappa = 0 \\ &2b_0b_1\kappa + 2b_1b_2\kappa^2 - 3N^2mb_0^2b_1 - 3N^2mb_0^2b_2 \\ &-2l^{\beta}b_0^2b_2\kappa - 2l^{\beta}b_0b_1^2\kappa - 2l^{\beta}b_0b_1b_2\kappa = 0 \\ &8b_0b_2\kappa - 2b_2^2\kappa^2 - 3N^2mb_0b_1^2 - 6N^2mb_0b_1b_2 \\ &-3N^2mb_0b_2 - l^{\beta}b_0^2b_1 - 4l^{\beta}b_0b_1b_2\kappa \\ &-4l^{\beta}b_0b_2^2\kappa - 2l^{\beta}b_1b_2\kappa - l^{\beta}b_1b_2\kappa = 0 \\ &2b_0b_1 + 2b_1b_2\kappa - N^2mb_1 - 3N^2mb_1^2b_2 - 3N^2mb_1b_2 \\ &-N^2mb_2^3 - 2l^{\beta}b_0^2b_2 - 2l^{\beta}b_0b_1^2 - 2l^{\beta}b_0b_1b_2 \\ &-2l^{\beta}b_1^2b_2\kappa - 4l^{\beta}b_1b_2^2\kappa - 2l^{\beta}b_2^2\kappa + l^{\beta}b_1^3 = 0 \\ &6b_0b_2 + b_1^2 - 4l^{\beta}b_0b_1b_2 - 4l^{\beta}b_0b_2^2 - l^{\beta}b_1^3 \\ &-2l^{\beta}b_1^2b_2 - l^{\beta}b_1b_2 = 04b_1b_2 - 2l^{\beta}b_1^2b_2 \\ &-4l^{\beta}b_1b_2^2 - 2l^{\beta}b_2^2 = 0 \\ &2b_2^2 = 0 \end{aligned} \tag{59}$$

Solving this set of algebraic equations leads to

$$b_2 = 0, \quad , b_0 = 0, \quad b_1 = N^2m, \quad l = \left(\frac{1}{N^2m}\right)^{\frac{1}{\beta}}, \quad \kappa = \kappa. \tag{60}$$

In accordance with (29), we obtain the following explicit solutions of Eq. (3)

$$\begin{aligned} u_1 &= -N^2m\sqrt{-\kappa}\tanh(\sqrt{-\kappa}\eta, \beta), \\ u_2 &= -N^2m\sqrt{-\kappa}\coth(\sqrt{-\kappa}\eta, \beta), \end{aligned} \tag{61}$$

where $\kappa < 0, \quad \eta = x + l\,t;$

$$u_3 = N^2 m\sqrt{\kappa}\tan(\sqrt{\kappa}\eta, \beta),$$
$$u_4 = N^2 m\sqrt{\kappa}\cot(\sqrt{\kappa}\eta, \beta), \tag{62}$$

where $\kappa > 0, \quad \eta = x + l\,t;$

$$u_5 = -N^2 m\frac{\Gamma(1+\kappa)}{\eta^\beta + \omega}, \quad \omega = constant, \ \kappa = 0, \quad \eta = x + l\,t. \tag{63}$$

VIII. Conclusion

According to the results obtained in this paper, the fin equation of fractional order in time can be transformed into second-order ordinary differential equations in terms of the Erdélyi-Kober fractional integral operator by using Lie symmetry analysis. The solution of some kinds of these ODE's cannot be solved easily. However, we may use other approaches including the symmetry method to solve them and it will be more simpler than to solve original partial differential equation. On the other hand, we used sub-equation method to find an explicit and exact solution for particular case of fin equation of fractional order.

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