B-Spline Curve Interpolation Model by using Intuitionistic Fuzzy Approach

Mohammad Izat Emir Zulkifly, Abd. Fatah Wahab and Rozaimi Zakaria

Abstract—**In this paper, B-spline curve interpolation model by using intuitionistic fuzzy set approach is introduced. Firstly, intuitionistic fuzzy control point relation is defined based on the intuitionistic fuzzy concept. Later, the intuitionistic fuzzy control point relation is blended with B-spline basis function. Through interpolation method, intuitionistic fuzzy B-spline curve model is visualized. Finally, some numerical examples and an algorithm to generate the desired curve is shown.**

Index Terms—**B-spline curve, interpolation method, intuitionistic fuzzy control point relation, intuitionistic fuzzy B-spline curve.**

I. Introduction

Curve is a necessary and inevitable in order to represent data point [1]. However, the nature of data point obtained is difficult to understand, process and represent as it is affected by noise and uncertainty. Thus, fuzzy set theory together with geometric modeling are used to solve this problem. Tuohy and Patrikalakis [2] proposed a method for reconstruction of surfaces from spatially distributed geophysical data with uncertainties. They used bi-quadratic uniform integral enveloping or interval B-spline surface to represent uncertainty data. Later, their work is extended to representation of volumetric data with interval B-spline volumetric functions [3]–[4]. Patrikalalis et al. developed enveloping or interval explicit B-spline geometries that represent the underwater geophysical data and the uncertainty of the data resulting from sensor measurement in [5] based on [2]. Tuohy and Patrikalakis [6] also proposed a method for the representation of functions with uncertainty describing a measured geophysical property by using interval B-spline.

The techniques mentioned above have been extended by Anile et al. in [7] and have been applied in data modeling and data reduction problem [8]–[10]. In [11], Anile et al. generalized the modeling techniques proposed by Patrikalakis et al. [5] to include the use of fuzzy numbers. Their approach starts by reducing an original large data set to a smaller set whose elements are fuzzy number with suitable membership functions. Later, they introduced fuzzy B-spline to describe the fuzzy data and provide efficient algorithms for the computation of $\alpha -$levels of the approximating splines. Moreover, Anile & Spinella [12] described uncertainty data through the concepts of fuzzy arithmetic and applying these methods to uncertain sparse data that are caused by several sources such as measurement errors, data reduction, modeling errors and defined the fuzzy B-splines method. They also presented rigorous algorithms for constructing fuzzy B-splines fitting uncertain sparse data and for their interrogation.

Wahab et. al [13] use the theory of fuzzy set from Zadeh [14] and its properties to solve the uncertainty problems through the concept of fuzzy numbers. They applied fuzzy number as uncertainty data, defined fuzzy control points and introduced fuzzy Bézier and fuzzy B-spline curve. By using the definitions introduced, they construct fuzzy spline curve and provide examples of fuzzy Bézier and fuzzy B-spline curve in the context of Computer-Aided Geometric Design (CAGD). Through their techniques, all the uncertainty data is in the form of fuzzy number, which means that each of crisp control point is consist of left and right fuzzy control points where each control point has certain degree of belonging to original control points (crisp control points). This means that its membership functions are left and right continuous in closed bounded interval at each of α value.

Several years later, Wahab et al. [15], introduced fuzzy control point and fuzzy surface model for CAGD by using the same concept in [13]. They study the properties concerning approximation of fuzzy control points by means of fuzzy Bézier, fuzzy B-spline and fuzzy Non-uniform Rational B-spline (NURBS). They provide a set of points in 2D and 3D space to produce a certain type of curve and surface in CAGD or geometric modeling and they stated that it often occurs that points approximation are not sets of real numbers but ranges of qualitative. These sets of point are given incomplete knowledge that is characterized whether or not that the set of points really belong to their domain of definition. They consider these points as fuzzy control points which their degrees of truth take the value from 0 to 1. Several studies were extended from the above researches for uncertainty data in geometric modeling such as interpolation of Bézier curves [16]–[18], interpolation of B-spline curve [19]–[20] and fuzzy B-spline surface modeling [21]. Others research involving uncertain data and spline have been successfully carried out in [22]–[29].

Intuitionistic fuzzy set (IFS) is the generalization of fuzzy set theory [14]. IFS was first introduced by Atanassov [30] and it is very compatible to deal with uncertainty. The idea of IFS concept is an alternative approach to define a fuzzy set in case where the available information is insufficient for the definition of an imprecise concept by

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means of a conventional fuzzy set. Fuzzy set theory only considers the membership function while IFS considers membership function, non-membership and non-determinacy function so that the sum of those values is equal to one [31]–[33]. Presently, IFSs have been studied and successfully used in different fields of science and mathematics. Among the works on these sets are as in [34]–[44]. IFS is generally defined by three functions (membership, non-membership and uncertainty) with the constraint that summation must be equal to one [45].

Research of IFS with geometric modeling have been done by Zulkifly & Wahab. In [46], they introduced an idea of IFS in spline curve and surface which focused on Bézier spline where the curve and surface are blended with intuitionistic fuzzy control point. Wahab et. al [47] discussed intuitionistic fuzzy Bézier model and generated intuitionistic fuzzy Bézier curve using interpolation method. They visualized intuitionistic fuzzy Bézier curve that consists of membership, non-membership and uncertainty curve by blending the Bernstein polynomial with intuitionistic fuzzy control point that have been defined. Later, Zulkifly & Wahab defined intuitionistic fuzzy control point relation (IFCPR) through intuitionistic fuzzy concept with some properties. They illustrate intuitionistic fuzzy bicubic Bézier surface through the approximation method by using data point with intuitionistic features [48]. By using IFCPR, they also generate cubic Bézier curve through interpolation method and intuitionistic fuzzy B-spline curve (IFB-SC) using approximation method [49]–[50].

The aim of this paper is to generate and illustralized IFB-SC through interpolation method by using IFCPR. This paper is organized as follows. Section 1 discussed some introduction and previous works related to this research. In section 2, intuitionistic fuzzy point relation (IFPR), its properties and IFCPR is shown. Section 3 introduces IFB-SC through interpolation method by using IFCPR. Section 4 shows some numerical example and visualization of intuitionistic fuzzy B-spline curve interpolation together with its algorithm. Finally, section 5 will conclude this research.

II. Intuitionistic Fuzzy Point Relation

IFPR is developed and introduced based on the concept of IFS. Let *V*, *W* be a collection of universal space of points in the Euclidean space and $V,W \in \mathbb{R}^2$, then IFPR is defined as follows:

Definition 1. Let X,Y be a collection of universal space of points with non-empty set and $V,W,I \subseteq \mathbb{R}\times\mathbb{R}\times\mathbb{R}$, then IFPR is defined as

$$T^* = \left\{\left\langle\left(v_i,w_j\right),\mu_T\left(v_i,w_j\right),\nu_T\left(v_i,w_j\right),\pi_T\left(v_i,w_j\right)\right\rangle \mid \left(\mu_T\left(v_i,w_j\right),\nu_T\left(v_i,w_j\right),\pi_T\left(v_i,w_j\right)\right)\in I\right\} \quad (1)$$

where $\left(v_i,w_j\right)$ is an ordered pair of points or coordinates and $\left(v_i,w_j\right)\in V\times W$. $\mu_T\left(v_i,w_j\right)$, $\nu_T\left(v_i,w_j\right)$ and $\pi_T\left(v_i,w_j\right)$ are the grades of membership, non-membership and uncertainty of the ordered pair of points respectively in $[0,1]\in I$. Furthermore the condition $0\le \mu_T\left(v_i,w_j\right)+\nu_T\left(v_i,w_j\right)\le 1$ is follows and the degree of uncertainty is denoted by

$$\pi_T\left(v_i,w_j\right)=1-\left(\mu_T\left(v_i,w_j\right)+\nu_T\left(v_i,w_j\right)\right) \quad (2)$$

IFPR is based on fuzzy point in the Euclidean space and intuitionistic fuzzy point (IFP) is in IFS. Hence, IFPR is in intuitionistic fuzzy relation (IFR) and denoted by $T^*\in R^*$ and $P^*\times Q^*\in A^*\times B^*$.

Definition 2. Let P^* be an IFP and A^* is intuitionistic fuzzy number (IFN) in *V*. Hence, P^* is said to be in A^* and denoted by $P^*\in A^*$ if and only if $\mu_P\left(v_i\right)\le\mu_A\left(v_i\right)$ and $\nu_P\left(v_i\right)\ge\nu_A\left(v_i\right)$ for all $v_i\in V$. Every A^* can be expressed as the union of all IFP that belong to A^* which if $\mu_A\left(v_i\right)$ and $\nu_P\left(v_i\right)$ is non-zero for $v_i\in V$, then $\mu_A\left(v_i\right)=\sup\left\{y:\mu_P\left(v_i\right)\right.$ is IFP (membership) and $\left.0<y\le\mu_A\left(v_i\right)\right\}$ and $\nu_A\left(v_i\right)=\inf\left\{y:\nu_P\left(v_i\right)\right.$ is IFP (non-membership) and $\left.0<\nu_A\left(v_i\right)\le y\right\}$ respectively. Therefore, each and every IFP P^* in A^* can be written as $P^*=\{P_i^*\mid i=1,2,\ldots,n,i\in\mathbb{N}\}$ and $A^*=P_1^*\cup P_2^*\cup\ldots\cup P_n^*$.

Theorem 1. If $A^*=\bigcup_{i\in I}A_i^*$ where $I=\left\{1,2,\ldots,n\right\}$ and *I* is any index, then $P^*\in A^*$ if and only if $P^*\in A_i^*$ for some $i\in I$.

Proof: Let the support for P^* denoted by v_0, then

$$\begin{aligned}\mu_A\left(v_0\right)&=\sup_{i\in I}\mu_{A_i}\left(v_0\right),\\ \nu_A\left(v_0\right)&=\inf_{i\in I}\nu_{A_i}\left(v_0\right)\end{aligned} \quad (3)$$

i) There exists some $i_0\in I$ such as $\mu_{A_{i_0}}\left(v_0\right)=\mu_A\left(v_0\right)$ and $\nu_{A_{i_0}}\left(v_0\right)=\nu_A\left(v_0\right)$.

ii) $\mu_{A_i}\left(v_0\right)\le\mu_A\left(v_0\right)$ and $\nu_{A_i}\left(v_0\right)\ge\nu_A\left(v_0\right)$ for all $i\in I$.

For (i) $P^*\in A_{i_0}^*$. For (ii) $P^*\in A^*$ implies that $\mu_P\left(v_0\right)\le\mu_A\left(v_0\right)$, $\nu_P\left(v_0\right)\ge\nu_A\left(v_0\right)$ and considering that $\mu_A\left(v_0\right)=\sup_{i\in I}\mu_{A_i}\left(v_0\right)$, $\nu_A\left(v_0\right)=\inf_{i\in I}\nu_{A_i}\left(v_0\right)$, it follows that $\mu_P\left(v_0\right)\le\mu_{A_{i_0}}\left(v_0\right)$, $\nu_P\left(v_0\right)\ge\nu_{A_{i_0}}\left(v_0\right)$ for some i_0. Thus $P^*\in A_{i_0}^*$.

Definition 3. Let P^* and Q^* be an IFP and A^* and B^* is IFN in *V* and *W* respectively. Hence, IFPR T^* on P^* and Q^*, $P^*\times Q^*$ is said to be in R^*, and denoted by $P^*\times Q^*\in A^*\times B^*$ if and only if $\mu_T\left(v_i,w_j\right)\le\mu_R\left(v_i,w_j\right)$ and

$\nu_T(v_i, w_i) \ge \nu_R(v_i, w_i)$ for all $(v_i, w_j) \in V \times W$. Obviously, every R^* can be expressed as the union of all IFPR that belong to R^* which if $\mu_T(v_i, w_j)$ and $\nu_T(v_i, w_j)$ is non-zero for $(v_i, w_j) \in V \times W$, then $\mu_R(v_i, w_j) = \sup$ $\{\mu_{P\times Q}(v_i, w_j) : \mu_{P\times Q}(v_i, w_j)$ is IFPR (membership) and $0 < \mu_{P\times Q}(v_i, w_j) \le \mu_R(v_i, w_j)\}$ and $\nu_R(v_i, w_j) = \inf$ $\{\nu_{P\times Q}(v_i, w_j) : \nu_{P\times Q}(v_i, w_j)$ is IFPR (non-membership) and $0 < \nu_R(v_i, w_j) \le \nu_{P\times Q}(v_i, w_j)\}$ respectively. Therefore, each and every T^* in R^* can be written as $T^* = \{ T_i^* \mid i = 0,1,\ldots,n, i \in \mathbb{N}\}$ and $R^* = T_1^* \cup T_2^* \cup \ldots \cup T_n^*$.

Theorem 2. If $R^* = \bigcup_{i\in I} R_i^*$ where $I = \{1,2,\ldots,n\}$ and I is any index, then $T^* \in R^*$ if and only if $T^* \in R_i^*$ for some $i \in I$.

Proof: Let the support for T^* denoted by (v_0, w_0), then

$$\begin{aligned} \mu_R(v_0, w_0) &= \sup_{i\in I} \mu_{R_i}(v_0, w_0), \\ \nu_R(v_0, w_0) &= \inf_{i\in I} \mu_{R_i}(v_0, w_0) \end{aligned} \tag{4}$$

i) There exists some $i_0 \in I$ such as $\mu_{R_{i_0}}(v_0, w_0) = \mu_R(v_0, w_0)$ and $\nu_{R_{i_0}}(v_0, w_0) = \nu_R(v_0, w_0)$.
ii) $\mu_{R_i}(v_0, w_0) \le \mu_R(v_0, w_0)$ and $\nu_{R_i}(v_0, w_0) \ge \nu_R(v_0, w_0)$ for all $i \in I$.
For (i) $T^* \in R_{i_0}^*$. For (ii) $T^* \in R^*$ implies that $\mu_T(v_0, w_0) \le \mu_R(v_0, w_0), \nu_T(v_0, w_0) \ge \nu_R(v_0, w_0)$ and considering that $\mu_R(v_0, w_0) = \sup_{i\in I} \mu_{R_i}(v_0, w_0)$, $\nu_R(v_0, w_0) = \inf_{i\in I} \nu_{R_i}(v_0, w_0)$, it follows that $\mu_T(v_0, w_0) \le \mu_{R_{i_0}}(v_0, w_0)$, $\nu_T(v_0, w_0) \ge \nu_{R_{i_0}}(v_0, w_0)$ for some i_0. Thus $T^* \in R_{i_0}^*$.

The collection of all points or set of points that are used to determine the shape of a spline curve is called control point. The control point plays an important role in the process of generating, controlling, and producing smooth curve. IFCPR is defined as follows:

Definition 4. Let T^* be an IFPR, then IFCPR is defined as set of points $n+1$ that indicates the positions and coordinates of a location and is used to describe the curve and is denoted by

$$C_i^* = \{C_1^*, C_2^*, \ldots, C_{n+1}^*\} \tag{5}$$

where the control polygon vertices or the control point is numbered from 1 to $n+1$.

III. Intuitionistic Fuzzy B-Spline Curve Interpolation Model

The IFB-SC is obtained by blending IFCPR with B-spline basis function and defined as follows:

Definition 5. Let $C_i^* = \{C_1^*, C_2^*, \ldots, C_{n+1}^*\}$ where $i = 1,2,\ldots,$ $n+1$ be the IFCPR and IFB-SC denoted by $S^*(t)$ with the position vector along the curve as a function of the parameter *t*, hence by blended it with the blending function, IFB-sC is written as

$$S^*(t) = \sum_{i=1}^{n+1} C_i^* N_i^k(t) \tag{6}$$

with $t_{\min} \le t \le t_{\max}$ and $2 \le k \le n+1$ where C_i^* are the position vectors of $n+1$ control polygon vertices, and N_i^k are the normalized B-spline basis functions. The $N_i^k(t)$ is defined as

$$N_i^1(t) = \begin{cases} 1 & if \quad t_i \le t < t_{i+1} \\ 0 & otherwise \end{cases} \tag{7}$$

and

$$N_i^k(t) = \frac{(t - t_i)}{t_{i+k-1} - t_i} N_i^{k-1}(t) + \frac{(t_{i+k} - t)}{t_{i+k} - t_{i+1}} N_{i+1}^{k-1}(t) \tag{8}$$

IFB-SC in (6) is parametric function consists of membership curve, non-membership curve and uncertainty curve and denoted as follows

$$S^\mu(t) = \sum_{i=1}^{n+1} C_i^\mu N_i^k(t) \tag{9}$$

$$S^\nu(t) = \sum_{i=1}^{n+1} C_i^\nu N_i^k(t) \tag{10}$$

$$S^\pi(t) = \sum_{i=1}^{n+1} C_i^\pi N_i^k(t) \tag{11}$$

If the data points lies in IFB-SC then the data point should follow (6). Equation (6) is rewritten for all data point denoted by *j* as:

$$\begin{aligned} D_1^*(t_1) &= N_1^k(t_1)T_1^* + N_2^k(t_1)T_2^* + \ldots + N_{n+1}^k(t_1)T_{n+1}^* \\ D_2^*(t_2) &= N_1^k(t_2)T_{1\,1}^* + N_2^k(t_2)T_2^* + \ldots + N_{n+1}^k(t_2)T_{n+1}^* \\ &\vdots \\ D_j^*(t_j) &= N_1^k(t_j)T_1^* + N_2^k(t_j)T_2^* + \ldots + N_{n+1}^k(t_j)T_{n+1}^* \end{aligned} \tag{12}$$

where $2 \le k \le n+1 \le j$. Equation (12) is written in matrix form as

$$[D^*] = [N][T^*] \tag{13}$$

where

$$\left[D^*\right]^T = \left[D_1^*(t_1) \quad D_2^*(t_2) \quad \cdots \quad D_j^*(t_j)\right]$$

$$[N] = \begin{bmatrix} N_1^k(t_1) & \cdots & \cdots & N_{n+1}^k(t_1) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ N_1^k(t_j) & \cdots & \cdots & N_{n+1}^k(t_j) \end{bmatrix} \tag{14}$$

$$\left[T^*\right]^T = \left[T_1^* \quad T_2^* \quad \cdots \quad T_{n+1}^*\right]$$

Parametric value t_j for every output is the measurement of data point along IFB-SC. For data point *j*, the parametric value on data point to ℓ is as follows;

$$t_1 = 0$$

$$\frac{t_\ell}{t_{\text{maks}}} = \frac{\sum_{r=2}^{\ell} \left|D_r^* - D_{r-1}^*\right|}{\sum_{r=2}^{j} \left|D_r^* - D_{r-1}^*\right|} \quad \ell \geq 2 \tag{15}$$

The maximum parameter is denoted by $t_{\max}$, normally taken as maximum value for knot vector. If $2 \leq k \leq n+1 = j$, then *N* is squared matrix and control polygon is obtained directly through inverse matrix such as

$$\left[T^*\right] = [N]^{-1}\left[D^*\right] \quad \leq k \leq n+1 = j \tag{16}$$

Therefore, IFB-SC interpolation can be obtained through (16).

IV. Numerical Example and Algorithm

To illustrate IFB-SC interpolation, let's consider IFB-SC with five intuitionistic fuzzy control point relation as in Table I.

TABLE I
INTUITIONISTIC FUZZY CONTROL POINT RELATION

Intuitionistic fuzzy control point relation	Membership	Non-membership	Uncertainty
$C_1^* = (2,2)$	0.6	0.3	0.1
$C_2^* = (7,8)$	0.4	0.4	0.2
$C_3^* = (11,13)$	0.7	0.2	0.1
$C_4^* = (17,18)$	0.5	0.1	0.4
$C_5^* = (25,23)$	0.2	0.3	0.5

Intuitionistic fuzzy control point relation with its respective degree

By using (6), the desired interpolation curve is visualized separately from Fig. 1 until Fig.3 with their respective data points (black dots) and intuitionistic control points (red dots). The line connecting the control points is called intuitionistic control polygon consists of membership, non-membership, and uncertainty control polygon. Fig. 1 until Fig.3 are also called membership, non-membership, and uncertainty B-spline curve interpolation. The intuitionistic control point and control polygon controlled the curve and make sure that the curve interpolate the data points.

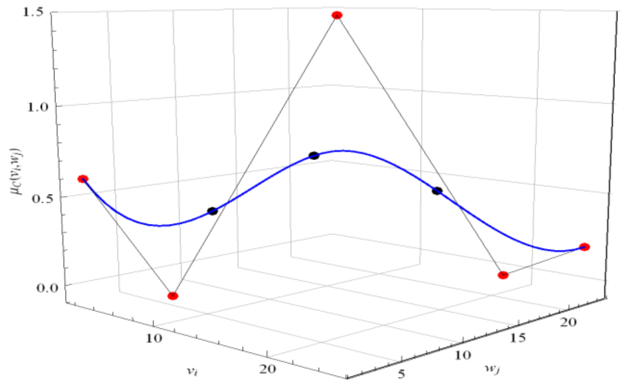

Fig. 1. IFB-SC interpolation (membership) with its respective data points, control points and membership control polygon.

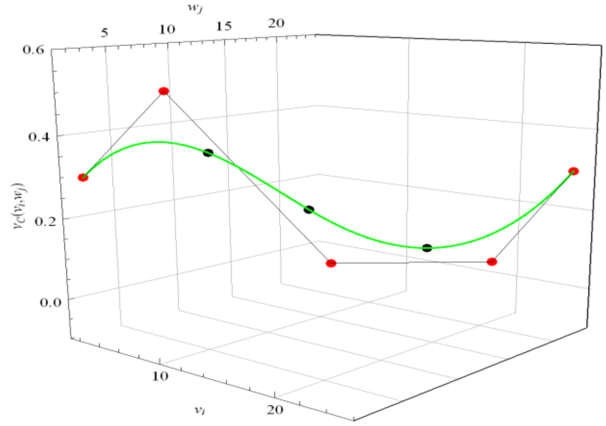

Fig. 2. IFB-SC interpolation (non-membership) with its respective data points, control points and non-membership control polygon.

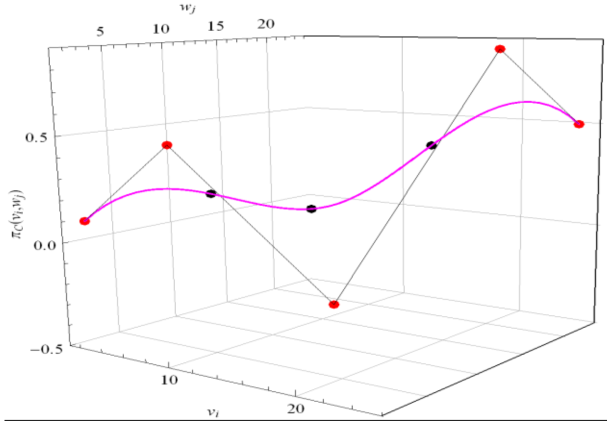

Fig. 3. IFB-SC interpolation (uncertainty) with its respective data points, control points and uncertainty control polygon.

Fig. 4 until Fig. 6 visualized IFB-SC interpolation consists of membership, non-membership and uncertainty curve with its data points and connecting data points respectively.

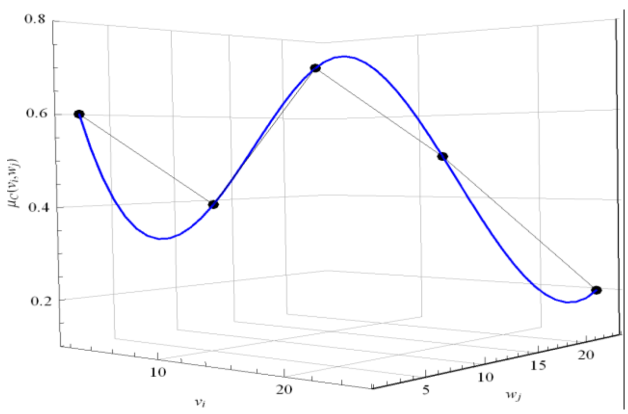

Fig. 4. IFB-SC interpolation (membership) with its respective data points.

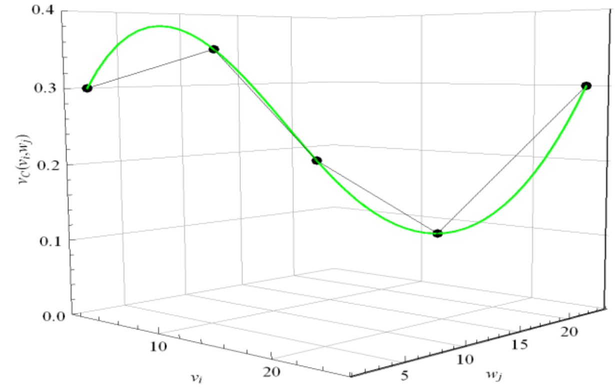

Fig. 5. IFB-SC interpolation (non-membership) with its respective data points.

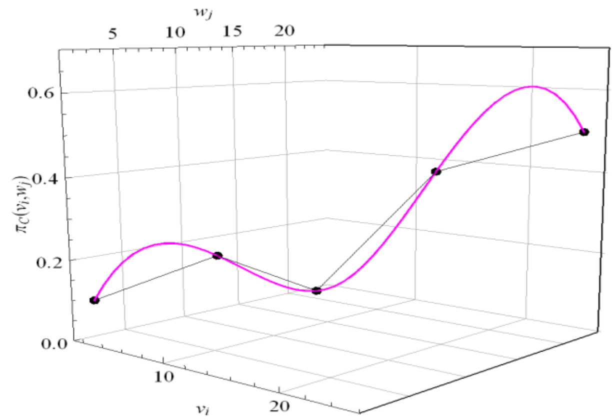

Fig. 6. IFB-SC interpolation (uncertainty) with its respective data points.

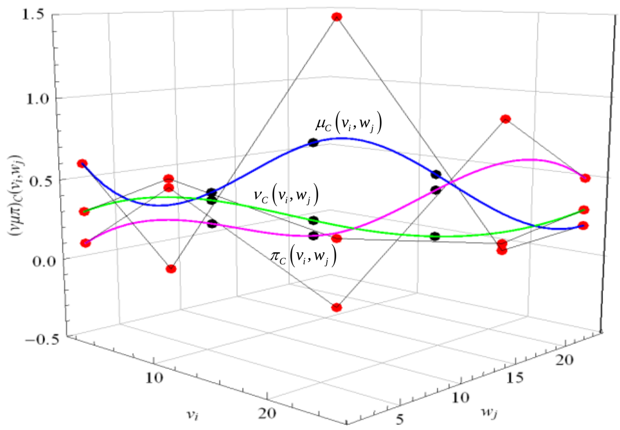

Fig. 7. IFB-SC interpolation with its respective data points, control points and intuitionistic control polygon.

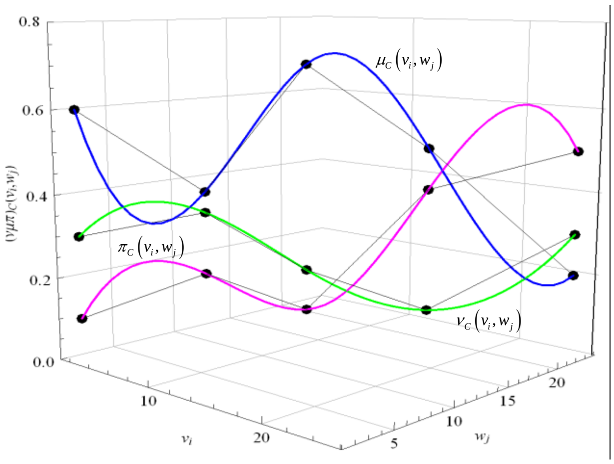

Fig. 8. IFB-SC interpolation with its respective data points and connecting data points.

Fig. 7 until Fig. 9 visualized IFB-SC interpolation in different perspective. Fig. 7 visualized IFB-SC interpolation with its respective data points, intuitionistic control points and control polygons. Fig. 8 visualized IFB-SC interpolation with its respective data points and connecting data points. Lastly, Fig. 9 is IFB-SC interpolation with its data points.

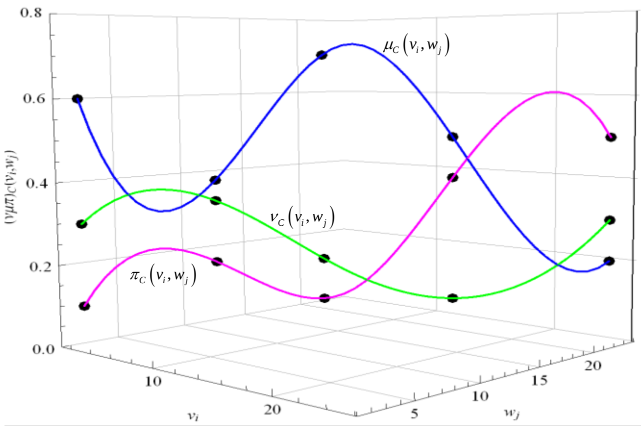

Fig. 9. IFB-SC interpolation with its respective data points.

Next, the algorithm to obtain IFB-SC interpolation is summarized as follows:

Algorithm

Step 1: Intuitionistic fuzzy data point relation and its respective knot vector are determined with $D^{z^*} = \left\{ D_i^{z^*} \right\}_{i=1}^{n+1}$ and $k = \left\{ k_i \right\}_{i=1}^{n+1}$.

Step 2. Find the parametric value along intuitionistic fuzzy B-spline curve correspond with each intuitionistic fuzzy control point relation through (15).

Step 3. Determine the chord lengths between each point where

$$\left|D_2^* - D_1^*\right|, \left|D_3^* - D_2^*\right|, \ldots, \left|D_r^* - D_{r-1}^*\right|$$

(b) Normalized parameter is calculated where

$$\sum_{r=2}^{r} \left(D_r^* - D_{r-1}^* \right)$$

and

$$t_1, \frac{t_2}{t_{\text{maks}}}, \ldots, \frac{t_\ell}{t_{\text{maks}}}$$

Step 4. Find B-spline basis function based on knot vector in Step 1 by preparing $[N]$ matrix through (13) and (14).

Step 5. Next, intuitionistic fuzzy control point relation is obtained through (16).

Step 6. Finally, the intuitionistic fuzzy control point relation is blended with the B-spline basis function as in (6)–(11) and yield IFB-SC interpolation.

V. Conclusion

This paper has introduced IFB-SC interpolation model by defining IFCPR. In modeling data involving intuitionistic features, IFB-SC interpolation model is an ideal approach because it is characterized by membership, non-membership, and uncertainty functions. With these functions, all data will be processed and analyzed. IFB-SC interpolation model can be applied in economy, real time tracking, stock market, data mining, databases, wireless sensor networks, management decision-making field, stochastic processes, routing, and remote sensing. The characteristic of intuitionistic fuzzy data combines with visualization using B-spline curve interpolation plays an important role for analyzing and describing the nature of some problems or situations with its reasoning. The method and the resulting model will be able to contribute to the field of fuzzy modeling techniques. This model also can be extended to surface and can be used to solve intuitionistic fuzzy data problems.

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