Derivation Theoretical Approach to MV-algebras

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Abstract—**As a supplement of the derivation theory, we present the concept of τ-difference derivations on MV-algebras in this paper. We investigate some properties of τ-difference derivations and their fixed point sets, and obtain that the fixed point set $Fix_\hbar(M)$ is a lattice ideal. Finally, we display some characterizes of Boolean algebras and linearly ordered Boolean algebras by the fixed point set of a simple τ-difference derivation, and prove that the set of all simple τ-difference derivations on an MV-algebra forms a Boolean algebra under suitable binary compositions.**

Index Terms—**MV-algebra, Boolean algebra, τ-difference derivation, Simple τ-difference derivation.**

I. Introduction

THE notion of derivations was first introduced in a prime ring from the analytic theory [1], and it is very helpful for studying algebraic structures in algebraic systems. Recently, the research of the derivation theory has experienced a tremendous growth, and many algebraic properties of derivations have been investigated. Inspired by derivations on rings, Jun and Xin [2] applied the notion of derivations to BCI-algebras, and introduced the notion of left-right (resp. right-left) derivations of BCI-algebras. Particularly, Xin et al. [3] put forward the notion of derivations for lattices, and characterized modular lattices and distributive lattices by isotone derivations. Continuing the work on derivations for lattices, Xin [4] established some characterizations of a chain, a distributive lattice, a modular lattice and a relatively pseudo-complemented lattice by using the fixed sets of isotone derivations. Noticing that BL-algebras, MV-algebras and Heyting algebras are particular types of residuated lattices, He et al. [5] found that it is meaningful to establish the derivation theory of residuated lattices for characterizing these particular residuated lattices. To develop the theory of derivations, Kondo applied the notion of derivations to commutative residuated lattices and studied some properties of monotone derivation [6]. Moreover, he investigated some properties of multiplicative derivations and d-filters of commutative residuated lattices in [7]. Zhu et al. considered a generalized derivation determined by a derivation in a residuated lattice, and got that a good ideal generalized derivation is determined by its fixed point set, they also showed that the relationship between good ideal generalized derivation filters and filters of the fixed point set for good ideal generalized derivations [8]. Derivations are also used to characterize generalized algebraic structure of residuated lattices, such as EQ-algebras [9], semihoops [10], G-algebras [11] and quaternion algebras [12].

The notion of derivations for MV-algebras was raised in [13], and some characterizations of a derivation on an MV-algebra were given by using isotone derivations. In analogy with Leibniz's formula for derivations in rings, Ghorbani et al. [14] presented the notions of $(\odot, \oplus)$-derivations and $(\ominus, \odot)$-derivations for MV-algebras, and studied the connection between these derivations. To give some representations of MV-algebras in terms of derivations, Wang et al. [15] investigated some properties of implicative derivations and difference derivations, and obtained their characterizations in MV-algebras. Motamed and Ehterami [16] defined the concepts of $(\wedge, \odot)$-derivation and $(\rightarrow, \vee)$-derivation for BL-algebras and discuss some related results. Alsatayhi and Moussavi [17] gave the notions of (φ, ψ)-derivations of types 1 and 2 on BL-algebras via endomorphisms. The paper [17] gives us a great deal of inspiration to introduce the notion of τ-difference derivations on MV-algebras in order to obtain fundamental properties of derivations of MV-algebras. We investigate some properties and characterizations of τ-difference derivations, and obtain that the fixed point set $Fix_\hbar(M)$ is a lattice ideal. Moreover, we characterize Boolean algebras and linearly ordered Boolean algebras by the fixed point set of a simple τ-difference derivation, and discuss the algebraic structures of the set of all simple τ-difference derivations.

II. Preliminaries

An algebra $(M, \boxplus, \neg, 0)$ is called an MV-algebra if it satisfies the following axioms: for any $a, b \in M$,

(MV1) $(M, \boxplus, 0)$ is a commutative monoid,

(MV2) $\neg\neg a = a$,

(MV3) $a \boxplus \neg 0 = \neg 0$,

(MV4) $\neg(\neg a \boxplus b) \boxplus b = \neg(\neg b \boxplus a) \boxplus a$.

For any $a, b \in M$, we set $1 = \neg 0$, $a \boxminus b = \neg(\neg a \boxplus b)$, $a \vee b = (a \boxminus b) \boxplus b$, $a \wedge b = b \boxminus (b \boxminus a)$, $a \boxtimes b = a \boxminus \neg b$ and $a \rightarrow b = \neg a \boxplus b$. It can be check that $(M, \wedge, \vee)$ is a distributive lattice. In what follows, unless mentioned otherwise, M is an MV-algebra.

Lemma 2.1: [18] The partial order $\leq$ on M is defined by $a \leq b$ iff a and b satisfy the following equivalent conditions:

(i) $a = a \wedge b$;

(ii) $a \boxminus b = 0$;

(iii) there is $c \in M$ such that $b = a \boxplus c$.

Proposition 2.2: [18] Let M be an MV-algebra. Then the following results hold: $\forall a, b, c \in M$,

(1) $a \boxminus 0 = a$, $a \boxplus \neg a = 1$;
(2) $a \boxminus b \leq c$ if and only if $a \leq b \boxplus c$;
(3) $a \boxminus b \leq a \wedge b \leq a \vee b \leq a \boxplus b$;
(4) if $a \leq b$, then $c \boxminus b \leq c \boxminus a$ and $a \boxplus c \leq b \boxplus c$;
(5) $a \boxtimes (b \vee c) = (a \boxtimes b) \vee (a \boxtimes c)$,
$a \boxplus (b \wedge z) = (a \boxplus b) \wedge (a \boxplus c)$;

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(6) $a \boxtimes (b \boxplus z) \leq b \boxplus (a \boxtimes c)$,
$a \boxtimes (b \wedge z) = (a \boxtimes b) \wedge (a \boxtimes c)$.

An MV-algebra M is a Boolean algebra if it satisfies:

$$a \boxplus a = a \text{ (or } a \boxtimes a = a)$$

for any $a \in M$, and denote by $B(M) = \{a \in M | a \boxtimes a = a\}$ be the set of all idempotent elements of M.

Proposition 2.3: [19] Then following conditions are equivalent: for any $a, b \in M$,

(i) $a \in B(M)$;
(ii) $a \boxplus a = a$;
(iii) $a \boxtimes a = a$;
(iv) $a \boxplus b = a \vee b$;
(v) $a \boxtimes b = a \wedge b$.

Proposition 2.4: [20] If $a \in B(M)$, then we have

$$a \boxtimes (u \boxplus v) = (a \boxtimes u) \boxplus (a \boxtimes v)$$

for any $u, v \in M$.

Let I be a nonempty set of M. Then I is called a lattice ideal of M if it satisfies: for any $a, b \in M$, (i) $a \leq b$ and $b \in I$ imply $a \in I$; (ii) if $a \in I$ and $b \in I$, then $a \vee b \in I$. An ideal I of M is called a lattice prime ideal if it satisfies: for any $a, b \in M$, $a \wedge b \in I$ implies $a \in I$ or $b \in I$. For any $a \in V$, $[a]$ is a lattice ideal generated by a, and it is easy to check that $[a] = \{b \in M | b \leq a\}$ [21]. A nonempty set I of M is called an MV-ideal if it satisfies (i) and (iii) $a, b \in I$ implies $a \boxplus b \in I$ for any $a, b \in M$ [18].

Definition 2.5: [19] Let $(M_1, \boxplus_1, \neg_1, 0_1)$ and $(M_2, \boxplus_2, \neg_2, 0_2)$ be two MV-algebras. A map $\tau : M_1 \to M_2$ is called an MV-homomorphism if it satisfies the following conditions: for any $a, b \in M_1$,

(i) $\tau(0_1) = 0_2$,
(ii) $\tau(a \boxplus_1 b) = \tau(a) \boxplus_2 \tau(b)$,
(iii) $\tau(\neg_1 a) = \neg_2 \tau(a)$.

Definition 2.6: [15] A map $\hbar : M \to M$ is called a difference derivation if it satisfies:

$$\hbar(u \boxminus v) = \hbar(u) \boxminus v,$$

for any $u, v \in M$.

III. τ-DIFFERENCE DERIVATIONS ON MV-ALGEBRAS

In the present section, we introduce the notion of τ-difference derivations on MV-algebras, and study some properties of these operators.

Definition 3.1: Let $\tau : M \to M$ be an MV-homomorphism. A map $\hbar : M \to M$ is called a τ-difference derivation on M if it satisfies:

$$\hbar(x \boxminus y) = \hbar(x) \boxminus \tau(y)$$

for any $x, y \in M$.

If the homomorphism τ is the identity map id_M on M in the above definition, then the id_M-difference derivation is actually the ordinary difference derivation.

Example 3.2: Let M be an MV-algebra. Define a map $\hbar : M \to M$ by

$$\hbar(x) = 0$$

for any $x \in M$, it is easy to see that $\hbar$ is a τ-difference derivation for any MV-endomorphism τ on M. Moreover, for the identity map $\tau = id_M$ on M, we define a map $\hbar : M \to M$ by

$$\hbar(x) = x$$

for any $x \in M$, then $\hbar$ is an id_M-difference derivation on M, which is called an identity id_M-difference derivation.

Example 3.3: Let $M = \{0, a, b, 1\}$, where $0 < a < 1$ and $0 < b < 1$. The operations $\boxplus$ and $\neg$ are defined as follows:

$\boxplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$\neg$	0	a	b	1
	1	b	a	0

then $(M, \boxplus, \neg, 0)$ is an MV-algebra. We define two maps $\tau : M \to M$ and $\hbar : M \to M$ as follows

$$\tau(x) = \begin{cases} 0, & x = 0, \\ b, & x = a, \\ a, & x = b, \\ 1, & x = 1, \end{cases} \qquad \hbar(x) = \begin{cases} 0, & x = 0, \\ 0, & x = a, \\ a, & x = b, \\ a, & x = 1. \end{cases}$$

By routine calculations, τ is an MV-homomorphism, and $\hbar(x)$ is a τ-difference derivation on M.

Proposition 3.4: Let $\hbar$ be a τ-difference derivation on M. Then the followings hold: for any $x, y \in M$,

(1) $\hbar(0) = 0$;
(2) $\hbar(x) \leq \tau(x)$;
(3) if $x \leq y$, then $\hbar(x) \leq \hbar(y)$,
(4) $\hbar(x) \boxminus \tau(y) \leq \tau(x) \boxminus \hbar(y)$;
(5) $\hbar(x \boxminus y) \leq \hbar(x) \boxminus \hbar(y)$;
(6) $\hbar(\neg x) \leq \neg \hbar(x)$;
(7) if $\hbar(1) = 1$, then $\hbar = \tau$, which means that $\hbar$ is an MV-endomorphism.

Proof: (1) For any $x \in M$, we get that $x \boxminus 1 = 0$, and so $\hbar(0) = \hbar(x \boxminus 1) = \hbar(x) \boxminus \tau(1) = \hbar(x) \boxminus 1 = 0$.

(2) Assume that $x, y \in M$ such that $x \leq y$, then $x \boxminus y = 0$. Thus

$$0 = \hbar(0) = \hbar(x \boxminus y) = \hbar(x) \boxminus \tau(y),$$

this means that $\hbar(x) \leq \tau(y)$, that is, $x \leq y$ implies that $\hbar(x) \leq \tau(y)$. Now since $x \leq x$, therefore $\hbar(x) \leq \tau(x)$.

(3) If $x \leq y$, then $x = y \boxminus (y \boxminus x)$, and so

$$\begin{aligned} \hbar(x) &= \hbar(y \boxminus (y \boxminus x)) \\ &= \hbar(y) \boxminus \tau(y \boxminus x) \\ &\leq \hbar(y). \end{aligned}$$

(4) Since $(\tau(x) \boxminus \hbar(y)) \boxplus \tau(y) \geq (\tau(x) \boxminus \hbar(y)) \boxplus \hbar(y) \geq \tau(x) \geq \hbar(x)$, hence $\hbar(x) \boxminus \tau(y) \leq \tau(x) \boxminus \hbar(y)$.

(5) According to (2), we get that $\hbar(y) \leq \tau(y)$ for any $y \in M$, then $\hbar(x \boxminus y) = \hbar(x) \boxminus \tau(y) \leq \hbar(x) \boxminus \hbar(y)$.

(6) From (2) and (5), it follows that

$$\begin{aligned} \hbar(\neg x) &= \hbar(1 \boxminus x) \\ &= \hbar(1) \boxminus \tau(x) \\ &\leq 1 \boxminus \tau(x) \\ &\leq 1 \boxminus \hbar(x) \\ &= \neg \hbar(x) \end{aligned}$$

for any $x \in M$.

(7) Noticing $\hbar(1) = 1$, we get that

$$\begin{aligned} \hbar(\neg x) &= \hbar(1 \boxminus x) \\ &= \hbar(1) \boxminus \tau(x) \\ &= 1 \boxminus \tau(x) \\ &= \neg \tau(x) \end{aligned}$$

for any $x \in M$. Moreover,

$$\hbar(x) = \hbar(\neg(\neg x)) = \neg(\neg \tau(x)) = \tau(x),$$

thus $\hbar = \tau$. ■

In the following, we derive a condition for a τ-difference derivation to be an MV-endomorphism.

Proposition 3.5: Let τ be an MV-homomorphism on M, and $\hbar : M \to M$ be a τ-difference derivation. Then $\hbar = \tau$ if and only if $\hbar(x) \boxminus \tau(y) = \tau(x) \boxminus \hbar(y)$ for any $x, y \in M$.

Proof: Suppose that $\hbar(x) \boxminus \tau(y) = \tau(x) \boxminus \hbar(y)$ for any $x, y \in M$, then

$$\begin{aligned}\hbar(x) &= \hbar(x \boxminus 0)\\ &= \hbar(x) \boxminus \tau(0)\\ &= \tau(x) \boxminus \hbar(0)\\ &= \tau(x) \boxminus 0\\ &= \tau(x).\end{aligned}$$

The converse is obviously. ■

Theorem 3.6: Let $\hbar : M \to M$ be a map. Then we have: for any $x, y \in M$,

(1) $\hbar$ is a τ-difference derivation on M;
(2) $\hbar(x) = \hbar(1) \boxtimes \tau(x)$;
(3) $\hbar(x \boxtimes y) = \hbar(x) \boxtimes \tau(y)$.

Proof: (1) $\Rightarrow$ (2) Suppose that $\hbar$ is a τ-difference derivation, then

$$\begin{aligned}\hbar(x) &= \hbar(1 \boxminus \neg x)\\ &= \hbar(x) \boxminus \tau(\neg x)\\ &= \hbar(x) \boxminus \neg\tau(x)\\ &= \hbar(1) \boxtimes \tau(x),\end{aligned}$$

for any $x \in M$.

(2) $\Rightarrow$ (3) Assume that (2) is valid. For any $x, y \in M$, we have

$$\begin{aligned}\hbar(x \boxtimes y) &= \hbar(1) \boxtimes \tau(x \boxtimes y)\\ &= \hbar(1) \boxtimes (\tau(x) \boxtimes \tau(y))\\ &= (\hbar(1) \boxtimes \tau(x)) \boxtimes \tau(y)\\ &= \hbar(x) \boxtimes \tau(y).\end{aligned}$$

(3) $\Rightarrow$ (1) Suppose that $\hbar(x \boxtimes y) = \hbar(x) \boxtimes \tau(y)$ for any $x, y \in M$. It follows that

$$\begin{aligned}\hbar(x \boxminus y) &= \hbar(x \boxtimes \neg y)\\ &= \hbar(x) \boxtimes \tau(\neg y)\\ &= \hbar(x) \boxtimes \neg\tau(y)\\ &= \hbar(x) \boxminus \tau(y),\end{aligned}$$

thus $\hbar$ is a τ-difference derivation. ■

Proposition 3.7: Let $\hbar$ be a τ-difference derivation on M. Then for any $x, y \in M$,

(1) $\hbar(x \vee y) = \hbar(x) \vee \hbar(y)$;
(2) $\hbar(x \wedge y) = \hbar(x) \wedge \hbar(y)$.

Proof: (1) Since $\hbar$ is a τ-difference derivation, it follows from Theorem 3.6 that

$$\begin{aligned}\hbar(x \vee y) &= \hbar(1) \boxtimes \tau(x \vee y)\\ &= \hbar(1) \boxtimes (\tau(x) \vee \tau(y))\\ &= (\hbar(1) \boxtimes \tau(x)) \vee ((\hbar(1) \boxtimes \tau(y))\\ &= \hbar(x) \vee \hbar(y).\end{aligned}$$

(2) Similar to the proof of (1). ■

Proposition 3.8: Let $\hbar : M \to M$ be a τ-difference derivation and $\hbar(1) \in B(M)$. Then for any $x, y \in M$,

(1) $\hbar(x \boxtimes y) = \hbar(x) \boxtimes \hbar(y)$;
(2) $\hbar(x \boxplus y) = \hbar(x) \boxplus \hbar(y)$;
(3) $\hbar(B(M)) \subseteq B(M)$.

Proof: (1) According to Theorem 3.6, we get that

$$\begin{aligned}\hbar(x \boxtimes y) &= \hbar(1) \boxtimes \tau(x \boxtimes y)\\ &= \hbar(1) \boxtimes (\tau(x) \boxtimes \tau(y))\\ &= (\hbar(1) \boxtimes \tau(x)) \boxtimes ((\hbar(1) \boxtimes \tau(y))\\ &= \hbar(x) \boxtimes \hbar(y)\end{aligned}$$

for any $x, y \in M$.

(2) Since $\hbar(1) \in B(M)$, then

$$\begin{aligned}\hbar(x \boxplus y) &= \hbar(1) \boxtimes \tau(x \boxplus y)\\ &= \hbar(1) \boxtimes (\tau(x) \boxplus \tau(y))\\ &= (\hbar(1) \boxtimes \tau(x)) \boxplus ((\hbar(1) \boxtimes \tau(y))\\ &= \hbar(x) \boxplus \hbar(y).\end{aligned}$$

(3) For any $x \in B(M)$, we have $x = x \boxtimes x$. It follows from (1) that

$$\hbar(x) = \hbar(x \boxtimes x) = \hbar(x) \boxtimes \hbar(x),$$

and so $\hbar(x) \in B(M)$. Therefore, $\hbar(B(M)) \subseteq B(M)$ is valid. ■

Theorem 3.9: Let M be an MV-algebra. Then the following statements are equivalent:

(1) M is a Boolean algebra;
(2) for any τ-difference derivation $\hbar$ on M, if $\hbar(1) \in B(M)$, then $\hbar(x \wedge y) = \hbar(x) \boxtimes \hbar(y)$.

Proof: (1) $\Rightarrow$ (2) Suppose that M is a Boolean algebra, then $x \wedge y = x \boxtimes y$ for any $x, y \in M$. Let $\hbar$ be a τ-difference derivation on M. If $\hbar(1) \in B(M)$, then

$$\hbar(x \wedge y) = \hbar(x \boxtimes y) = \hbar(x) \boxtimes \hbar(y)$$

by Proposition 3.8.

(2) $\Rightarrow$ (1) Assume that (2) is valid. From Example 3.2 we get that the identity map $\hbar : M \to M$ is an id_M-difference derivation on M, and $\hbar(1) = 1 \in B(M)$. It follows that

$$\hbar(x) = \hbar(x \wedge x) = \hbar(x) \boxtimes \hbar(y)$$

for any $x \in M$, that is, $x = x \wedge x = x \boxtimes x$, therefore M is a Boolean algebra. ■

From Proposition 3.7 and Proposition 3.8, we can get the following result.

Theorem 3.10: Let $\hbar : M \to M$ be a τ-difference derivation and $\hbar(1) \in B(M)$. Then $(\hbar(M), \boxplus, *, \hbar(0))$ is an MV-algebra, where $(\hbar(x))^* := \hbar(\neg x)$ for any $x \in M$.

Proposition 3.11: Let $\hbar$ be a τ-difference derivation on M, and $Fix_\hbar(M) := \{x \in M | \hbar(x) = \tau(x)\}$ be the set of all fixed elements of $\hbar$, which is call a fixed point set. Then

(1) $Fix_\hbar(M)$ is a lattice ideal of M;
(2) if M is linearly ordered MV-aglebra, then $Fix_\hbar(M)$ is a lattice prime ideal of M;
(3) $\hbar(x \boxtimes y) = \hbar(x) \boxtimes \hbar(y)$, for any $x \in M$ and $y \in Fix_\hbar(M)$;
(4) $\hbar(x \boxplus y) \leq \hbar(x) \boxplus \hbar(y)$, for any $x, y \in Fix_\hbar(M)$;
(5) if $\hbar(1) \in B(M)$, then $Fix_\hbar(M)$ is an MV-ideal of M.

Proof: (1) Let $x \leq y$ and $y \in Fix_{\hbar}(M)$. Then $\hbar(y) = \tau(y)$, and

$$\begin{aligned}\hbar(x) &= \hbar(x \wedge y)\\ &= \hbar(y \boxminus (y \boxminus x))\\ &= \hbar(y) \boxminus \tau(y \boxminus x)\\ &= \tau(y) \boxminus (\tau(y) \boxminus \tau(x))\\ &= \tau(y) \wedge \tau(x)\\ &= \tau(y \wedge x)\\ &= \tau(x),\end{aligned}$$

thus $x \in Fix_{\hbar}(M)$. Now for any $x, y \in M$, if $x \in Fix_{\hbar}(M)$ and $y \in Fix_{\hbar}(M)$, then $\hbar(x) = \tau(x)$ and $\hbar(y) = \tau(y)$. According to Proposition 3.7, we get that

$$\begin{aligned}\hbar(x \vee y) &= \hbar(x) \vee \hbar(y)\\ &= \tau(x) \vee \tau(y)\\ &= \tau(x \vee y),\end{aligned}$$

this meas that $x \vee y \in Fix_{\hbar}(M)$. Hence, $Fix_{\hbar}(M)$ is a lattice ideal of M.

(2) It is known that every ideal is prime in a linearly ordered lattice, thus $Fix_{\hbar}(M)$ is a prime lattice ideal of M by hypotheses and (1).

(3) For any $x \in M$ and $y \in Fix_{\hbar}(M)$, we get that $\hbar(y) = \tau(y)$. Then $\hbar(x \boxtimes y) = \hbar(x) \boxtimes \tau(y) = \hbar(x) \boxtimes \hbar(y)$.

(4) For any $x, y \in Fix_{\hbar}(M)$, we have $\hbar(x) = \tau(x)$ and $\hbar(y) = \tau(y) = \hbar(1) \boxtimes \tau(y)$. Then

$$\begin{aligned}\hbar(x \boxplus y) &= \hbar(1) \boxtimes \tau(x \boxplus y)\\ &= \hbar(1) \boxtimes (\tau(x) \boxplus \tau(y))\\ &\leq \tau(x) \boxplus (\hbar(1) \boxtimes \tau(y))\\ &= \hbar(x) \boxplus \hbar(y).\end{aligned}$$

(5) If $x, y \in Fix_{\hbar}(M)$, then $\hbar(x) = \tau(x)$ and $\hbar(y) = \tau(y) = \tau(y)$. Since $\hbar(1) \in B(M)$, then

$$\begin{aligned}\hbar(x \boxplus y) &= \hbar(x) \boxplus \hbar(y)\\ &= \tau(x) \boxplus \tau(y)\\ &= \tau(x \boxplus y),\end{aligned}$$

and so $x \boxplus y \in Fix_{\hbar}(M)$ by Proposition 3.8. Combination with (1), we get that $Fix_{\hbar}(M)$ is an MV-ideal of M. ■

IV. Simple τ-difference derivations

In the section, we give a special type of τ-difference derivations, which is called simple τ-difference derivations. Some properties of simple τ-difference derivations and their fixed point sets are discussed.

Proposition 4.1: Let $\tau : M \to M$ be an MV-homomorphism. For a fixed element $a \in M$, if a map $\hbar_a : M \to M$ is defined by

$$\hbar_a(x) = \tau(x) \boxtimes a$$

for any $x \in M$, then $\hbar_a$ is a τ-difference derivation on M. And $\hbar_a$ is called a simple τ-difference derivation.

Proof: For any $x, y \in M$,

$$\begin{aligned}\hbar_a(x \boxtimes y) &= \tau(x \boxtimes y) \boxtimes a\\ &= (\tau(x) \boxtimes \tau(y)) \boxtimes a\\ &= (\tau(x) \boxtimes a) \boxtimes \tau(y)\\ &= \hbar_a(x) \boxtimes \tau(y).\end{aligned}$$

According to Theorem 3.6, we get that $\hbar_a$ is a τ-difference derivation on M. ■

Proposition 4.2: Let $(M, \boxplus, \neg, 0)$ be an MV-algebra. If $Fix_{\hbar_{\tau(a)}}(M) = [a]$ for any $a \in M$, then $(\tau(M), \boxplus, \neg, 0)$ is a Boolean algebra.

Proof: It is easy to check that $(\tau(M), \boxplus, \neg, 0)$ is an MV-algebra. For any $b \in \tau(M)$, there exists $a \in M$ such that $b = \tau(a)$. Notice that $Fix_{\hbar_{\tau(a)}}(M) = [a]$ and $a \in [a]$, we get $a \in Fix_{\hbar_{\tau(a)}}(M)$, and therefore

$$\hbar_{\tau(a)}(a) = \tau(a) \boxtimes \tau(a) = \tau(a),$$

that is, $b \boxtimes b = b$. Hence, $\tau(M)$ is a Boolean algebra. ■

Theorem 4.3: Let $\tau : M \to M$ be an MV-homomorphism. If τ is an MV-monomorphism on M, then the followings are equivalent:

(1) For any $a \in M$, $Fix_{\hbar_{\tau(a)}}(M) = [a]$;
(2) $\tau(M)$ is a Boolean algebra.

Proof: (1) $\Rightarrow$ (2) It follows immediately from Proposition 4.2.

(2) $\Rightarrow$ (1) Suppose that $\tau(M)$ is a Boolean algebra, then

$$\tau(x) \boxtimes \tau(x) = \tau(x)$$

for any $x \in M$. Since $\hbar_{\tau(a)}(a) = \tau(a) \boxtimes \tau(a) = \tau(a)$ for any $a \in M$, therefore $a \in Fix_{\hbar_{\tau(a)}}(M)$. From Proposition 3.11, we see that $Fix_{\hbar_{\tau(a)}}(M)$ is a lattice ideal of M. For any $x \in [a]$, we have $x \leq a$, and so $x \in Fix_{\hbar_{\tau(a)}}(M)$, which implies that $[a] \subseteq Fix_{\hbar_{\tau(a)}}(M)$. To prove the converse inclusion, suppose that $x \in Fix_{\hbar_{\tau(a)}}(M)$, then

$$\begin{aligned}\tau(x) &= \hbar_{\tau(a)}(x)\\ &= \tau(x) \boxtimes \tau(a)\\ &= \tau(x) \wedge \tau(a)\\ &= \tau(x \wedge a).\end{aligned}$$

Since τ is an MV-monomorphism on M, then $x = x \wedge a$, and so $x \leq a$. It follows that $x \in [a]$, hence $Fix_{\hbar_{\tau(a)}}(M) \subseteq [a]$. In view of the above natural result, we get that $Fix_{\hbar_{\tau(a)}}(M) = [a]$. ■

It is known that if an MV-homomorphism is an identity map, then it is an MV-monomorphism. Hence we can get the following result.

Corollary 4.4: Let $\tau : M \to M$ be an MV-homomorphism. If τ is an identity map, then the following conditions are equivalent:

(1) For any $a \in M$, $Fix_{\hbar_a}(M) = [a]$;
(2) M is a Boolean algebra.

Proposition 4.5: Let M be an MV-algebra, and $\tau : M \to M$ be an MV-homomorphism. If τ is an MV-monomorphism on M, then the following assertions are equivalent:

(1) For any $a \in M$, $Fix_{\hbar_{\tau(a)}}(M) = [a]$ and $Fix_{\hbar_{\tau(a)}}(A)$ is a lattice prime ideal of M;
(2) M is a linearly ordered Boolean algebra.

Proof: (1) $\Rightarrow$ (2) Using Theorem 4.3, we get that M is a Boolean algebra. For any $a, b \in M$,

$$\begin{aligned}\hbar_{\tau(a\wedge b)}(a \wedge b) &= \tau(a \wedge b) \boxtimes \tau(a \wedge b)\\ &= \tau(a \wedge b),\end{aligned}$$

which implies that $a \wedge b \in Fix_{\hbar_{\tau(a\wedge b)}}(M)$. Noticing that $Fix_{\hbar_{\tau(a\wedge b)}}(M)$ is a lattice prime ideal of M, we get that

$a \in Fix_{\hbar_{\tau(a\wedge b)}}(M)$ or $b \in Fix_{\hbar_{\tau(a\wedge b)}}(M)$. If $a \in Fix_{\hbar_{\tau(a\wedge b)}}(M)$, then

$$\begin{aligned}\tau(a) &= \hbar_{\tau(a\wedge b)}(a)\\ &= \tau(a) \boxtimes \tau(a \wedge b)\\ &= \tau(a \boxtimes (a \wedge b)).\end{aligned}$$

Since τ is an MV-monomorphism on M, therefore $a = a \boxtimes (a \wedge b) \leq a \wedge b$, and so $a = a \wedge b$, thus $a \leq b$. If $b \in Fix_{\hbar_{\tau(a\wedge b)}}(M)$, we can get $b \leq a$ in a similar way. Consequently, M is a linearly ordered Boolean algebra.

(2) $\Rightarrow$ (1) Assume that M is a linearly ordered Boolean algebra, it follows from Proposition 3.11 (2) that $Fix_{\hbar_{\tau(a)}}(M)$ is a lattice prime ideal of M. Moreover, it is easy to show that $\tau(M)$ is a Boolean algebra. According to Theorem 4.3, we get that $Fix_{\hbar_{\tau(a)}}(M) = [a]$ for any $a \in M$. ■

In what follows, we focus on the algebraic structure of the set $H(M) = \{\hbar_{\tau(a)} | a \in M\}$.

Proposition 4.6: Let M be an MV-algebra. Then $(H(M), \boxplus, \star, \hbar_{\tau(0)})$ is a Boolean algebra, where

$$(\hbar_{\tau(a)} \boxplus \hbar_{\tau(b)})(x) := \hbar_{\tau(a)}(x) \vee \hbar_{\tau(b)}(x)$$

and

$$\hbar^{\star}_{\tau(a)} := \hbar_{\tau(\neg a)},$$

for any $\hbar_{\tau(a)}, \hbar_{\tau(b)} \in H(M)$ and $x \in M$.

Proof: For any $\hbar_{\tau(a)}, \hbar_{\tau(b)} \in H(M)$,

$$\begin{aligned}(\hbar_{\tau(a)} \boxplus \hbar_{\tau(b)})(x) &= \hbar_{\tau(a)}(x) \vee \hbar_{\tau(b)}(x)\\ &= (\tau(x) \boxtimes \tau(a)) \vee (\tau(x) \boxtimes \tau(b))\\ &= \tau(x) \boxtimes (\tau(a) \vee \tau(b))\\ &= \tau(x) \boxtimes \tau(a \vee b)\\ &= \hbar_{\tau(a\vee b)}(x).\end{aligned}$$

According to the definition of MV-algebras, it is easy to show that $(H(M), \boxplus, \star, \hbar_{\tau(0)})$ is an MV-algebra. Moreover,

$$\hbar_{\tau(a)} \boxplus \hbar_{\tau(a)} = \hbar_{\tau(a\vee a)} = \hbar_{\tau(a)},$$

thus $H(M)$ is a Boolean algebra. ■

V. Conclusions

In this paper, we presented the notions of τ-difference derivations and simple τ-difference derivations on MV-algebras. In our framework, we gave some properties of τ-difference derivations and their fixed point sets, and displayd some characterizes of Boolean algebras and linearly ordered Boolean algebras by the fixed point set of simple τ-difference derivations. The research embodies the relationship between MV-algebras and Boolean algebras. The thoughts and methods in this paper can be completely applied to special cases of residuated lattices, and the corresponding τ-difference derivations can be introduced and their characterizations can be obtained on these logical algebras.

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