Existence and Non-existence of Traveling Wave for a Holling-Tanner Model

Xiangkui Zhao, Zhizhi He

***Abstract*—In 2017, Ai, Du and Peng[7] studied the traveling wave of a singularity Holling-Tanner predator-prey model. By employing an auxiliary system to overcome the singularity, they proved the existence of traveling waves when the parameter of functional response belongs to a limited range. This article aims to extend the above results and simplify the process. The details are stated below: (i) Improving the parameter to ∞; (ii) Simplifying the process by constructing a non-zero lower solution to overcome the singularity.**

***Index Terms*—Predator-prey model, traveling wave solution, upper and lower solution.**

I. Introduction

THE relationship between prey and predator has long been a topic of interest to many researchers, see [1-9] and their references. Murray and Renshaw[1-2] presented the Holling-Tanner predator-prey model:

$$\begin{cases} u_t = u_{xx} + ru(1-u) - \frac{rku}{a+bu}v, \\ v_t = dv_{xx} + sv(1-\frac{v}{u}). \end{cases} \tag{1}$$

Chen, Guo and Yao studied traveling waves of (1) when $a = 1, b = 0$, $0 < k < 1$ in reference [3]. Zhao studied traveling waves of (1) when $a = 1, b = 0$, $k = 1$ in reference [4].

Ai, Du and Peng considered the traveling waves of the following generalized Holling-Tanner predator-prey model:

$$\begin{cases} u_t = u_{xx} + u(1-u) - \frac{\alpha u^m}{1+\beta u^m}v, \\ v_t = dv_{xx} + sv(1-\frac{v}{u}) \end{cases} \tag{2}$$

in reference [7].

If $m = 1$, they proved the existence of traveling waves only when $\alpha \leq \frac{\sqrt{2}}{\sqrt{2}-1}$.

Since the predator equation has a singularity at zero prey population, they constructed an auxiliary function $sv(1 - \frac{v}{\sigma_\varepsilon(u)})$ where

$$\sigma_\varepsilon(u) = \begin{cases} u, & u \geq \varepsilon, \\ u + \varepsilon e^{\frac{1}{u-\varepsilon}}, & u \leq \varepsilon. \end{cases} \tag{3}$$

to overcome the singularity, but this complicate the research process.

In this paper, we will extend the research results of the model (2) when $m = 1$, which is the following model

$$\begin{cases} u_t = u_{xx} + u(1-u) - \frac{\alpha v}{1+\beta u}, \\ v_t = dv_{xx} + sv(1-\frac{v}{u}). \end{cases} \tag{4}$$

We will construct a non-zero lower solution to overcome the singularity. The method can also be applied to the generalized Holling-Tanner predator-prey model (2). Where u and v are the population sizes of prey and predator respectively, the parameters α, d and s are positive, and β is nonnegative. When $\beta = 0$, we refer the reader to reference [3-4]. For the specific details about functions u, v and these constants, we refer the reader to reference [7]. (4) has two constant steady states $(1, 0)$ and (u^*, v^*) with

$$u^* = v^* = \frac{2}{\sqrt{(\beta-1-\alpha)^2 + 4\beta} + 1 + \alpha - \beta}.$$

A traveling wave of (4) is the following form:

$$u(x,t) = \phi_1(x+ct) = \phi_1(z),$$
$$v(x,t) = \phi_2(x+ct) = \phi_2(z),\ z \in \mathbb{R},$$

where the constant $c > 0$ is the wave speed; $z = x + ct$ is called the moving coordinate. The wave profile (ϕ_1, ϕ_2) satisfies the following system of equations:

$$\begin{cases} \phi_1''(z) - c\phi_1'(z) + \phi_1(z)(1 - \phi_1(z) - \frac{\alpha\phi_2(z)}{1+\beta\phi_1(z)}) = 0, \\ d\phi_2''(z) - c\phi_2'(z) + s\phi_2(z)(1 - \frac{\phi_2(z)}{\phi_1(z)}) = 0, \end{cases} \tag{5}$$

where $z \in \mathbb{R}$. We will study the traveling waves connecting $(1, 0)$ and (u^*, v^*). The tail behavior of wave profile (ϕ_1, ϕ_2) at ∞ is discussed by comparing ϕ_1^-, ϕ_2^- and ϕ_1^+, ϕ_2^+, where

$$\phi_i^- = \liminf_{z\to+\infty} \phi_i(z), \quad \phi_i^+ = \limsup_{z\to+\infty} \phi_i(z),\ i = 1, 2.$$

We will organize the rest of this paper as follows. In section 2, the upper and lower solutions of the model are introduced. In section 3, the existence of traveling waves is considered. In section 4, the model (4) is discussed.

II. Upper and lower solutions

THROUGHOUT the paper, assume that α, β satisfy one of the conditions
(i) $\alpha < 1$, $\beta \in \mathbb{R}$; (ii) $\alpha \geq 1, \beta > 2\alpha + 2\sqrt{\alpha^2 - \alpha} - 1$.
And denote

$$\lambda_1 = \frac{c + \sqrt{c^2+4}}{2},\ \lambda_2 = \frac{c - \sqrt{c^2 - 4ds}}{2d},$$
$$\lambda_3 = \frac{c + \sqrt{c^2 - 4ds}}{2d},\ c^* = 2\sqrt{ds}.$$

Definition 2.1: The functions $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ are called a pair of upper and lower solutions of (4), if $\overline{\phi}_i'$, $\underline{\phi}_i'$, $\overline{\phi}_i''$, $\underline{\phi}_i''$, $i = 1, 2$ are bounded and the inequalities

$$\begin{cases} \overline{\phi}_1''(z) - c\overline{\phi}_1'(z) + \overline{\phi}_1(z)\big[1 - \overline{\phi}_1(z) - \frac{\alpha\underline{\phi}_2(z)}{1+\beta\overline{\phi}_1(z)}\big] \leq 0, \\ \underline{\phi}_1''(z) - c\underline{\phi}_1'(z) + \underline{\phi}_1(z)\big[1 - \underline{\phi}_1(z) - \frac{\alpha\overline{\phi}_2(z)}{1+\beta\underline{\phi}_1(z)}\big] \geq 0, \\ d\overline{\phi}_2''(z) - c\overline{\phi}_2'(z) + s\overline{\phi}_2(z)(1 - \frac{\overline{\phi}_2(z)}{\overline{\phi}_1(z)}) \leq 0, \\ d\underline{\phi}_2''(z) - c\underline{\phi}_2'(z) + s\underline{\phi}_2(z)(1 - \frac{\underline{\phi}_2(z)}{\underline{\phi}_1(z)}) \geq 0 \end{cases} \tag{6}$$

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Xiangkui Zhao and Zhizhi He are both with the Department of Mathematics, School of Mathematics and physics, University of Science and Technology Beijing, Beijing 100083, P. R. China e-mail: xiangkuizh@ustb.edu.cn.

hold for $z \in \mathbb{R}\backslash D$ with some finite set $D = \{z_1, z_2, \cdots, z_m\}$.

A. The case $c > c^*$.

Then, we consider the case $c > c^*$. Similar to reference [3], for given constants $\mu, q > 1$, we introduce functions

$$f_1(z) = e^{\lambda_2 z} - qe^{\mu\lambda_2 z},\ f_2(z) = e^{\lambda_2 z},\ z \in \mathbb{R},$$

then $z_0 = -\ln q/[(\mu-1)\lambda_2]$ is the unique zero point of f_1, $z_M = -\ln(q\mu)/[(\mu-1)\lambda_2] < z_0$ is the unique maximum point of f_1 in $(-\infty, z_0)$, f_1 is positive in $(-\infty, z_0)$, f_2 is positive and strictly increasing in $(-\infty, z_0)$.

Since $z_M < z_0 < 0$, $f_2(0) = 1$, we can choose a small enough positive number $\delta << 1$, a negative number $z_2 \in (z_M, z_0)$ satisfying

$$f_1(z_2) = \delta, \quad f_1'(z_2) < 0. \tag{7}$$

We choose the constants μ, η, p, q satisfying the following assumptions.
(A1) $\mu \in (1, \min\{\lambda_3/\lambda_2, 2\})$, $\eta > 0$ small enough such that $\lambda_2 > \eta\lambda_1$ and $(\eta\lambda_1)^2 - c(\eta\lambda_1) - \kappa < 0$;
(A2) $p > \frac{\alpha}{-[(\eta\lambda_1)^2 - c(\eta\lambda_1) - \kappa]}$,
$q > \max\{1, \frac{s}{-\kappa[d(\mu\lambda_2)^2 - c(\mu\lambda_2)) + s]}\}$.
Introduce the functions $\overline{\phi}_1(z), \underline{\phi}_1(z), \overline{\phi}_2(z), \underline{\phi}_2(z)$ as following:

$$\overline{\phi}_1(z) = 1, \qquad z \in \mathbb{R}, \tag{8}$$

$$\underline{\phi}_1(z) = \begin{cases} \kappa, & z > z_1, \\ 1 - pe^{\eta\lambda_1 z}, & z \le z_1, \end{cases} \tag{9}$$

$$\overline{\phi}_2(z) = \begin{cases} 1, & z > 0, \\ e^{\lambda_2 z}, & z \le 0, \end{cases} \tag{10}$$

$$\underline{\phi}_2(z) = \begin{cases} \delta, & z > z_2, \\ e^{\lambda_2 z} - qe^{\mu\lambda_2 z}, & z \le z_2, \end{cases} \tag{11}$$

where $\kappa = \frac{\beta - 1 + \sqrt{(\beta-1)^2 + 4\beta(1-\alpha)}}{2\beta}$, $z_1 < 0$ is defined by $pe^{\eta\lambda_1 z_1} = 1 - \kappa$, $\delta \le \kappa$. It is obvious that $\kappa < 1$, when $\alpha, \beta > 0$.

Lemma 2.1: Assume that $c > c^*$, then the functions $(\overline{\phi}_1(z), \overline{\phi}_2(z))$, $(\underline{\phi}_1(z), \underline{\phi}_2(z))$ defined by (8)-(11) are a pair of upper and lower solutions of (1.5).
The proof is similar to *Lemma 3.1* in reference [4]. We omit it here.

Remark 2.1: Assume that $c > c^*$, κ satisfy $(1-\kappa)(1+\beta\kappa^m) = \alpha\kappa^{m-1}$, then we can choose proper constants μ, η, p, q such that the functions $(\overline{\phi}_1(z), \overline{\phi}_2(z))$, $(\underline{\phi}_1(z), \underline{\phi}_2(z))$ defined by (8)-(11) are a pair of upper and lower solutions of the generalized Holling-Tanner predator-prey model (2).

B. The case $c = c^*$.

Next we consider the case $c = c^*$. For given constants $h = \lambda_2 e^2/2, q > h\sqrt{2/\lambda_2}$, introduce functions

$$g_1(z) = [-hz - q(-z)^{1/2}]e^{\lambda_2 z},\ g_2(z) = -hze^{\lambda_2 z},$$

$$g_3(z) = -hz - q(-z)^{1/2},\ z \le 0.$$

It is from reference [3] that $z_0 = -(q/h)^2 < -2/\lambda_2$ is the unique zero of g_1 in $(-\infty, 0)$, $g_1 > 0$ has a unique maximum point $\tilde{z}$ in $(-\infty, z_0)$, g_2 strictly increases on $(\infty, -2/\lambda_2]$, $g_2(-2/\lambda_2) = 1$, g_3 is positive and strictly decreases in $(-\infty, z_0)$.

Since $\tilde{z} < z_0 < -2/\lambda_2$, $g_2(-2/\lambda_2) = 1$, we can choose a small enough positive number $\delta << 1$, a negative number $z_2 \in (\tilde{z}, z_0)$ satisfying

$$g_1(z_2) = \delta,\ g_1'(z_2) < 0,\ \texttt{and}\ g_2(z_3) \le 1 - \delta. \tag{12}$$

Now we consider the existence of the upper and lower solutions of (5), when $c = c^*$.

For $p > e$, there exists $z_1 \le -2/\lambda_2$ with $pe^{\eta\lambda_1 z_1} = 1$, $\lambda_2 > 2\eta\lambda_1$, since $pe^{\eta\lambda_1 z}$ is increasing in z and

$$pe^{-2\eta\lambda_1/\lambda_2} > pe^{-1} > 1.$$

Next, we choose the constants η, p, q satisfying the following assumptions.
(B1) $0 < \eta << 1$ satisfies

$$(\eta\lambda_1)^2 - c(\eta\lambda_1) - \kappa < 0, \lambda_2 > 2\eta\lambda_1;$$

(B2) $p > \max\left\{e, \frac{\alpha h}{-(\eta\lambda_1 e)[(\eta\lambda_1)^2 - c(\eta\lambda_1) - \kappa]}\right\}$;
(B3) $q > \max\left\{h\sqrt{2/\lambda_2}, \frac{4sh^2}{d\delta}\left(\frac{7}{2e\lambda_2}\right)^{7/2}\right\}$.

We introduce the functions $\overline{\phi}_1(z), \underline{\phi}_1(z), \overline{\phi}_2(z), \underline{\phi}_2(z)$ as following:

$$\overline{\phi}_1(z) = 1, \qquad z \in \mathbb{R} \tag{13}$$

$$\underline{\phi}_1(z) = \begin{cases} \kappa, & z \ge z_1, \\ 1 - pe^{\eta\lambda_1 z}, & z \le z_1, \end{cases} \tag{14}$$

$$\overline{\phi}_2(z) = \begin{cases} 1, & z \ge -2/\lambda_2, \\ -hze^{\lambda_2 z}, & z \le -2/\lambda_2, \end{cases} \tag{15}$$

$$\underline{\phi}_2(z) = \begin{cases} \delta, & z \ge z_2, \\ (-hz - q(-z)^{1/2})e^{\lambda_2 z}, & z \le z_2. \end{cases} \tag{16}$$

Where $pe^{\eta\lambda_1 z_1} = 1 - \kappa$, $\delta \le \kappa$. Then the following lemma holds.

Lemma 2.2: Assume that $c = c^*$, then the functions $(\overline{\phi}_1(z), \overline{\phi}_2(z))$, $(\underline{\phi}_1(z), \underline{\phi}_2(z))$ defined by (13)-(16) are a pair of upper and lower solutions of (4).
The proof is similar to Lemma 3.2 in reference [4]. We omit it here.

Remark 2.2: Assume that $c = c^*$, κ satisfy $(1-\kappa)(1+\beta\kappa^m) = \alpha\kappa^{m-1}$, then we can choose proper constants η, p, q such that the functions $(\overline{\phi}_1(z), \overline{\phi}_2(z))$, $(\underline{\phi}_1(z), \underline{\phi}_2(z))$ defined by (13)-(16) are a pair of upper and lower solutions of the generalized Holling-Tanner predator-prey model (2).

III. The existence of traveling wave

SIMILAR to reference [3], we will consider the existence of traveling waves for (4) by Schauder's fixed point theorem.

First, we introduce the sets

$$X = \{\Phi = (\phi_1, \phi_2) \mid \Phi \text{ is a continuous function from } \mathbb{R} \text{ to } \mathbb{R}^2\}$$

and

$$X_\kappa = \{\Phi \in X \mid \kappa \le \phi_1 \le 1 \text{ and } 0 \le \phi_2 \le 1 \text{ for all } z \in \mathbb{R}\}.$$

Next, we define the functions

$$\begin{cases} F_1(x,y) := \omega x + x(1 - x - \frac{\alpha y}{1+\beta x}), \\ F_2(x,y) := \omega y + sy(1 - \frac{y}{x}) \end{cases} \tag{17}$$

for constant ω. If $\omega > \max\{(1 + \frac{\alpha}{1+\beta\kappa}), s(2-\kappa)/\kappa\}$, we know that F_1 is nondecreasing in x and nonincreasing in y for $\kappa \le x \le 1$ and $0 \le y \le 1$. Also, F_2 is nondecreasing with respect to x and y for $\kappa \le x \le 1$ and $0 \le y \le 1$.

Let

$$d_1 = 1,\ d_2 = d,\ \lambda_{i1}(c) = \frac{c - \sqrt{c^2 + 4\omega d_i}}{2d_i},$$

$$\lambda_{i2}(c) = \frac{c + \sqrt{c^2 + 4\omega d_i}}{2d_i},\ i = 1, 2.$$

For $\Phi = (\phi_1, \phi_2) \in X_\kappa$, define operator $P = (P_1, P_2)$ as following:

$$P_i(\phi_1,\phi_2)(z) = \frac{1}{d_i(\lambda_{i2} - \lambda_{i1})}\bigg(\int_{-\infty}^{z} e^{\lambda_{i1}(z-s)} + \int_{z}^{+\infty} e^{\lambda_{i2}(z-s)}\bigg) F_i(\phi_1,\phi_2)(s)ds$$

for $i = 1, 2, z \in \mathbb{R}$. Obviously

$$d_i(P_i(\phi_1,\phi_2))''(z) - c(P_i(\phi_1,\phi_2))'(z) - \beta P_i(\phi_1,\phi_2)(z) + F_i(\phi_1,\phi_2)(z) = 0,$$

for $i = 1, 2, z \in \mathbb{R}$.

Lemma 3.1: If (5) has a pair of upper and lower solutions $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ in X_κ satisfying
(C1) $\overline{\phi}_i(z) \ge \underline{\phi}_i(z),\ z \in \mathbb{R},\ i = 1, 2;$
(C2) $\overline{\phi}'_i(z-) \ge \overline{\phi}'_i(z+),\ \underline{\phi}'_i(z-) \le \underline{\phi}'_i(z+)\ z \in D,\ i = 1, 2$
where

$$\overline{\phi}'_i(z\pm) := \lim_{\xi\to z\pm} \overline{\phi}'_i(\xi),\quad \underline{\phi}'_i(z\pm) := \lim_{\xi\to z\pm} \underline{\phi}'_i(\xi).$$

Then it has a solution (ϕ_1, ϕ_2) such that $\overline{\phi}_i(z) \ge \phi_i(z) \ge \underline{\phi}_i(z)$ for all $z \in \mathbb{R}$, $i = 1, 2$.
The proof is similar to Lemma 2.3 in reference [3]. We omit it here.

Theorem 3.2: Assume that $c \ge c^*$, then there exists a positive solution (ϕ_1, ϕ_2) of (5) such that

$$\lim_{z\to-\infty}(\phi_1,\phi_2)(z) = (1, 0),$$

$$\underline{\phi}_i(z) \le \phi_i(z) \le \overline{\phi}_i(z),\ i = 1, 2,\ z \in \mathbb{R}.$$

Proof: Now we consider the case $c > c^*$. By Lemma 2.1, we know that (8)-(11) are a pair of upper and lower solutions of (5).

Now we will prove the condition(C1) and (C2) hold for the case $c > c^*$.
When $z \ge z_1$,we have

$$\overline{\phi}_1(z) - \underline{\phi}_1(z) = 1 - \kappa > 0.$$

When $z < z_1$, we have

$$\overline{\phi}_1(z) - \underline{\phi}_1(z) = pe^{\eta\lambda_1 z} > 0.$$

Similarly, it can be proven that $\overline{\phi}_2(z) \ge \underline{\phi}_2(z)$. Thus, condition (C1) holds.

For condition (C2), we have

$$\begin{cases} \underline{\phi}'_1(z_1+) = 0 > \underline{\phi}'_1(z_1-), \\ \overline{\phi}'_2(0+) = 0 < \overline{\phi}'_2(0-), \\ \underline{\phi}'_2(z_2+) = 0 > \underline{\phi}'_2(z_2-). \end{cases} \tag{18}$$

Hence there exists a positive solution (ϕ_1, ϕ_2) of (5) such that

$$\underline{\phi}_i(z) \le \phi_i(z) \le \overline{\phi}_i(z),\ i = 1, 2,\ z \in \mathbb{R}$$

by Lemma 3.1. It is obvious that $\lim_{z\to-\infty}(\phi_1,\phi_2)(z) = (1, 0)$. The case $c = c^*$ can be proven similarly. The proof is complete. ■

A. The properties of traveling waves

Now we consider the tail behavior at ∞ of the traveling waves for (4), which is the solution of (5).

Proposition 3.3: Assume that $c \ge c^*$, then the traveling waves (ϕ_1, ϕ_2) of (4) satisfies $\kappa \le \phi_1^- \le \phi_2^- \le \phi_2^+ \le \phi_1^+ \le 1$.

Proof: By Theorem 3.2, we know that

$$1 \ge \overline{\phi}_1(z) \ge \phi_1(z) \ge \underline{\phi}_1(z) \ge \kappa.$$

for all $z \in \mathbb{R}$.

Then we will prove that $\phi_1^- \le \phi_2^-$. For the contradiction, we suppose that $\phi_1^- > \phi_2^-$. If ϕ_2 is eventually monotone, we have $\phi_2(\infty)$ exists, since ϕ_2 is bounded on $\mathbb{R}$. Hence $\phi_2(\infty) = \phi_2^- = \phi_2^+$. Since

$$\int_0^\infty \phi'_2(s)ds = \phi_2(\infty) - \phi_2(0)$$

is finite, either $\liminf_{s\to+\infty} \phi'_2(s) = 0$ when $\phi'_2(s) \ge 0$ for $s >> 1$ or $\limsup_{s\to+\infty} \phi'_2(s) = 0$ when $\phi'_2(s) \le 0$ for $s >> 1$. Then we can find a sequence $\{z_n\}$ with $z_n \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} \phi_2(z_n) = \phi_2^- < \phi_1^-,\quad \lim_{n\to+\infty} \phi'_2(z_n) = 0.$$

Hence

$$\liminf_{n\to+\infty}\left\{1 - \frac{\phi_2(z_n)}{\phi_1(z_n)}\right\} \ge \{1 - \frac{\phi_2^-}{\phi_1^-}\bigg\} > 0.$$

Integrating the second equation of the system (5) from 0 to z_n, we have

$$d(\phi'_2(z_n) - \phi'_2(0)) - c[\phi_2(z_n) - \phi_2(0)] = -s\int_0^{z_n} \phi_2(s)\left[1 - \frac{\phi_2(s)}{\phi_1(s)}\right]ds. \tag{19}$$

When $n \to +\infty$, it is a contradiction because since the left side of (19) is bounded and the right side of (19) tends to $-\infty$.

If ϕ_2 is oscillatory at ∞, then we can choose a sequence $\{z_n\}$ of minimal points of ϕ_2 with $z_n \to \infty$ as $n \to +\infty$ such that $\lim_{n\to+\infty} \phi_2(z_n) = \phi_2^-$. Note that

$$d\phi''_2(z_n) - c\phi'_2(z_n) \ge 0$$

for all n. Also, we have

$$\liminf_{n\to+\infty}\left\{1 - \frac{\phi_2(z_n)}{\phi_1(z_n)}\right\} \ge \left\{1 - \frac{\phi_2^-}{\phi_1^-}\right\} > 0.$$

This implies that

$$\liminf_{n\to+\infty}\left\{d\phi_2''(z_n)-c\phi_2'(z_n)+s\phi_2(z_n)\left[1-\frac{\phi_2(z_n)}{\phi_1(z_n)}\right]\right\}>0$$

is a contradiction. To sum up, we know that $\phi_1^- \le \phi_2^-$ holds.

The case $\phi_2^+ \le \phi_1^+$ can be treated similarly. Consequently,

$$\kappa \le \phi_1^- \le \phi_2^- \le \phi_2^+ \le \phi_1^+ \le 1$$

holds. The proof is complete. ■

Proposition 3.4: Assume that $c \ge c^*$, then the traveling wave (ϕ_1, ϕ_2) of (4) satisfies

$$1-\phi_1^+-\frac{\alpha\phi_2^-}{1+\beta\phi_1^+}\ge 0,\quad 1-\phi_1^--\frac{\alpha\phi_2^+}{1+\beta\phi_1^-}\le 0.$$

Proof: Now we will prove that

$$1-\phi_1^+-\frac{\alpha\phi_2^-}{1+\beta\phi_1^+}\ge 0.$$

For contradiction, we suppose that $1-\phi_1^+-\frac{\alpha\phi_2^-}{1+\beta\phi_1^+}<0$. If ϕ_1 is eventually monotone, similar to Proposition 3.3, we know that $\phi_1(\infty)$ exists. We can also find a sequence $\{z_n\}$ with $z_n \to +\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty}\phi_1(z_n)=\phi_1^+,\quad \lim_{n\to+\infty}\phi_1'(z_n)=0.$$

Hence

$$\limsup_{n\to+\infty}\left[1-\phi_1(z_n)-\frac{\alpha\phi_2(z_n)}{1+\beta\phi_1(z_n)}\right]$$
$$\le\left[1-\phi_1^+-\frac{\alpha\phi_2^-}{1+\beta\phi_1^+}\right]<0.$$

Integrating the first equation of the system (5) from 0 to z_n, we have

$$(\phi_2'(z_n)-\phi_2'(0))-c[\phi_2(z_n)-\phi_2(0)]$$
$$=-r\int_0^{z_n}\phi_1(s)\left[1-\phi_1(s)-\frac{\alpha\phi_2(s)}{1+\beta\phi_1(s)}\right]ds. \tag{20}$$

Let $n \to +\infty$, we get a contradiction, since the left side of (20) is bounded and the right side of (20) tends to $+\infty$.

If ϕ_1 is oscillatory at ∞, we can choose a sequence $\{z_n\}$ of maximal points of ϕ_1 with $z_n \to \infty$ as $n \to +\infty$ such that $\lim\limits_{n\to+\infty}\phi_1(z_n)=\phi_1^+$. Note that

$$\phi_1''(z_n)-c\phi_1'(z_n)\le 0$$

for all n. Also, we have

$$\limsup_{n\to+\infty}\left[1-\phi_1(z_n)-\frac{\alpha\phi_2(z_n)}{1+\beta\phi_1(z_n)}\right]$$
$$\le\left[1-\phi_1^+-\frac{\alpha\phi_2^-}{1+\beta\phi_1^+}\right]<0.$$

This implies that

$$\limsup_{n\to+\infty}\left\{\phi_1''(z_n)-c\phi_1'(z_n)\right.$$
$$\left.+r\phi_1(z_n)\left[1-\phi_1(z_n)-\frac{\alpha\phi_2(z_n)}{1+\beta\phi_1(z_n)}\right]\right\}<0$$

is a contradiction.

The above analysis of the two cases of ϕ_1 lead us to the conclusion that

$$1-\phi_1^+-\frac{\alpha\phi_2^-}{1+\beta\phi_1^+}\ge 0.$$

The case $1-\phi_1^--\frac{\alpha\phi_2^+}{1+\beta\phi_1^-}\le 0$ can be proven similarly. The proof is complete. ■

Theorem 3.5: Assume that $c \ge c^*$, then the traveling wave (ϕ_1, ϕ_2) of (4) such that

$$\lim_{z\to+\infty}(\phi_1,\phi_2)(z)=(u^*,v^*), \tag{21}$$

Proof: From Proposition 3.3 and 3.4, we can see

$$\phi_1^+ +\frac{\alpha\phi_1^-}{1+\beta\phi_1^+}\le\phi_1^+ +\frac{\alpha\phi_2^-}{1+\beta\phi_1^+}$$
$$\le\phi_1^- +\frac{\alpha\phi_2^+}{1+\beta\phi_1^-}\le\phi_1^- +\frac{\alpha\phi_1^+}{1+\beta\phi_1^-}. \tag{22}$$

Case $\alpha<1$. From (22), we have

$$\phi_1^+-\phi_1^-\le\alpha\left[\frac{\phi_1^+}{1+\beta\phi_1^-}-\frac{\phi_1^-}{1+\beta\phi_1^+}\right]\le\alpha(\phi_1^+-\phi_1^-).$$

So $\phi_1^+=\phi_1^-$. From Proposition 3.3, we can see $\phi_1^+=\phi_2^+=\phi_2^-=\phi_1^-$. So

$$\lim_{z\to+\infty}(\phi_1,\phi_2)(z)=(u^*,v^*). \tag{23}$$

Case $\alpha\ge 1$, $\beta>2\alpha+2\sqrt{\alpha^2-\alpha}-1$. From (22), we have

$$\phi_1^+-\phi_1^-\le\alpha\left[\frac{\phi_1^+}{1+\beta\phi_1^-}-\frac{\phi_1^-}{1+\beta\phi_1^+}\right]$$
$$\le\alpha\frac{(\phi_1^+-\phi_1^-)[1+\beta(\phi_1^++\phi_1^-)]}{(1+\beta\phi_1^+)(1+\beta\phi_1^-)}.$$

Also, from $\kappa \le \phi_1^- \le \phi_2^- \le \phi_2^+ \le \phi_1^+ \le 1$, after calculation,

$$\frac{1+\beta(\phi_1^++\phi_1^-)}{(1+\beta\phi_1^+)(1+\beta\phi_1^-)}\le\frac{4\beta}{(\beta+1)^2}$$

is proven. So

$$\phi_1^+-\phi_1^-\le\frac{4\alpha\beta}{(\beta+1)^2}(\phi_1^+-\phi_1^-).$$

Since $\beta>2\alpha+2\sqrt{\alpha^2-\alpha}-1$, that is $\alpha<\frac{(\beta+1)^2}{4\beta}$, we have $\frac{4\alpha\beta}{(\beta+1)^2}<1$. So $\phi_1^+=\phi_1^-$. Similar to the case $\alpha<1$, we can see

$$\lim_{z\to+\infty}(\phi_1,\phi_2)(z)=(u^*,v^*). \tag{24}$$

■

Remark 3.1: There is no traveling wave of (4) for the case $c<c^*$, which means $c=c^*$ is the minimal speed.
The proof is similar to Theorem 2.6 in reference [3]. We will not repeat it.

IV. Conclusion

In this paper, we consider the traveling waves of a Holling-Tanner predator-prey model. Our work can simplify the research of reference [7] by employing a non-zero lower solution. We also enlarged the range of α to $(0,+\infty)$, which supplement the reference [7]. The research method can also be applied to the generalized Holling-Tanner predator-prey model (2). The restrict of our work is that the upper and lower solutions must be modified as constant m.

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