Triple Positive Solutions of a Fractional Differential Equation Model with the Nonlinear Term Involving the Derivative

Yang Yang, Baiyan Xu

Abstract—**Few papers have discussed the solution of fractional equation in which the derivative was involved in the nonlinear term. Because in this case, there will be a lot of difficulties, but in this paper, we study the existence of solutions for such a class of problems. This result will provide a platform for further study of fractional differential equation.**

Index Terms—**Four-point nonlocal boundary value problems, Caputo's fractional derivative, Fractional integral, Fixed point theorems.**

I. Introduction

THE definitions of derivatives and integrals of non-integer order have been proposed since 1965, but it was not until the late nineteenth century that the definition of fractional derivative was first introduced by Liouville and Riemann. Fractional derivatives play a very important role in describing the memory and genetic properties of various materials and processes, for more information, please refer to the literature Diethelm [1], Kilbas et al [2], Miller and Ross [3], Podlubny [4] and Tarasor [5] and the Refs [6-18]. Just for this reason, many aspects of fractional problem existence theory need to be explored.

If you want to see the latest research, check the literature ([21-31,34]). Zhou and Chu [20] discussed the following fractional differential equation with multi-point boundary condition

$$^{c}D^{q}_{0+}v(t) + f(t, v(t), (Kv)(t), (Hv)(t)) = 0, \quad t \in (0, 1),$$

$$a_1 v(0) - b_1 v'(0) = d_1 v(\xi_1), \quad a_2 v(1) + b_2 v'(1) = d_2 v(\xi_2),$$

where $1 < q \leq 2$ is a real number.

In [19], using some fixed point theorems, the authors found the unique solution for the nonlinear fractional integro-differential equations

$$^{c}D^{q}x(t) = f(t, x(t), (\phi x)(t), (\psi x)(t)), \quad 0 < t < 1,$$

$$1 < q \leq 2,$$

$$x'(0)+ax(\eta_1) = 0, \ \ bx'(1)+x(\eta_2) = 0, \quad 0 < \eta_1 \leq \eta_2 < 1.$$

When the BVPs of nonlinear differential equations which the first order derivative is involved in the nonlinear term explicitly, we often use the Avery-Peterson fixed point theorem to find the existence of positive solution for the problem.

This work is sponsored by Youth Research Fund Project of Changchun Guanghua University(QNJJZD2019005).

Yang Yang is with the Basic Research Section, Changchun Guanghua University, Changchun, Jilin 130033, China

Corresponding author: Baiyan Xu is with the Basic Research Section, Changchun Guanghua University, Changchun, Jilin 130033, China (E-mail: yangyangccgh@163.com).

Bai [32] found triple positive solutions for the boundary value problem

$$v^{(4)}(t) = h(t)f(t, v(t), v''(t)), \quad 0 < t < 1,$$

$$v(0) = v(1) = 0,$$

$$c_1 v''(\xi) - c_2 v'''(\xi) = 0, \quad c_3 v''(1) + c_4 v'''(1) = 0.$$

Inspired by the literature above, we are concerned with the following fractional model with the nonlinear term involving the derivative

$$D^{\alpha}_{0+}v(t) + f(t, v(t), v'(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$v(0) - \beta v'(\xi) = 0, \quad v(1) - \gamma v'(\eta) = 0, \tag{2}$$

where α is a real number with $1 < \alpha \leq 2, \ \ 0 < \xi < \eta < 1$.

However, very few papers, as far as we know, which have combined the fractional differential equation with the nonlinear differential equations which the first order derivative is involved in the nonlinear term explicitly. The main difficulty is that we can't derive the concavity or convexity of function $u(t)$ by the sign of its fractional order derivative. In this paper, we overcome the difficulty by obtaining some new inequalities and defining a special cone.

$$\Delta = 1 + \beta - \gamma.$$

We give the assumption.
(H_1) $(\gamma - 1)(1 - \eta) - (\alpha - 1)\beta\gamma > 0, \ 1 - \eta - (\alpha - 1)\gamma > 0, \ \Delta > 0, \ \gamma > 1.$
(H_2) $f \in C\left([0, 1] \times [0, +\infty) \times (-\infty, +\infty)\right) \to [0, +\infty).$

II. The preliminary lemmas

We now list some basic theory for the fractional derivative.

Definition 2.1 [33] For function y, α order Riemann-Liouville integral is

$$I^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1}y(s)ds, \quad \alpha > 0.$$

Definition 2.2 [33] For function y, α order Caputo's derivative is

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n - \alpha)}\int_0^t \frac{y^{(n)}(s)ds}{(t - s)^{\alpha+1-n}}, \quad n = [\alpha] + 1,$$

here $[\alpha]$ stands the integer part of real number α.

Lemma 2.1 If $\alpha > 0$, then

$$u(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}, \quad c_i \in R,$$

$$i = 1, 2, \cdots, n, n = [\alpha] + 1$$

is a solution of the fractional equation

$$D^{\alpha}_{0+}u(t) = 0.$$

Lemma 2.2 *[33]* If $\alpha > 0$, there are

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + c_1 + c_2t + c_3t^2 + \cdots + c_nt^{n-1},$$

for some $c_i \in R$, $i = 1, 2, \cdots, n, n = [\alpha] + 1$.

Definition 2.3 We have a real Banach space E. $P \subset E$ is a nonempty convex closed set. We called P is a cone if
(1) $\alpha u \in P$, for all $u \in P$, $\alpha \geq 0$,
(2) $u, -u \in P$ implies $u = 0$.

Definition 2.4 If $\alpha : K \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in K$ and $0 \leq t \leq 1$. We called α a nonnegative continuous concave functional. If $\gamma : K \to [0, \infty)$ is continuous and

$$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y),$$

for all $x, y \in K$ and $0 \leq t \leq 1$. We called γ a nonnegative continuous convex functional.

Let γ and θ are two convex functionals on K, and these two functions are nonnegative continuous. Let α is a concave functional on K, and this function is nonnegative continuous. Let ψ is a functional on K, and this function is nonnegative continuous. We give the convex sets as follows, basing on the following positive real numbers a, b, c, and d,

$$\begin{aligned}
&K(\gamma,\ d) = \{x \in K \mid \gamma(x) < d\},\\
&K(\gamma,\ \alpha,\ b,\ d) = \{x \in K \mid b \leq \alpha(x),\ \gamma(x) \leq d\},\\
&K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)\\
&= \{x \in K \mid b \leq \alpha(x),\ \theta(x) \leq c, \gamma(x) \leq d\},\\
&R(\gamma,\ \psi,\ a,\ d) = \{x \in K \mid\ a \leq \psi(x),\ \gamma(x) \leq d\}.
\end{aligned}$$

Theorem 2.3 ([32]) We define a Banach space E. K is a cone of E.Let γ and θ are two convex functionals on K, and these two functions are nonnegative continuous. Let α is a concave functional on K, and this function is nonnegative continuous. Let ψ is a functional on K, and this function is nonnegative continuous. The formula holds $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, and for M and d,

$$\alpha(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x) \tag{3}$$

for all $x \in \overline{K(\gamma,\ d)}$. Suppose

$$T : \overline{K(\gamma,\ d)} \to \overline{K(\gamma,\ d)}$$

is completely continuous and there exist positive numbers $a,\ b$, and c with $a < b$ such that

(S_1) $\{x \in K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)\ |\alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$;
(S_2) $\alpha(Tx) > b$ *for* $x \in K(\gamma,\ \alpha,\ b,\ d)$ *with* $\theta(Tx) > c$;
(S_3) $0 \notin R(\gamma,\ \psi,\ a,\ d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma,\ \psi,\ a,\ d)$ *with* $\psi(x) = a$.

Then T has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{K(\gamma,\ d)}$, such that

$$\gamma(x_i) \leq d \qquad for \quad i = 1, 2, 3,$$

$$b < \alpha(x_1),$$

$$a < \psi(x_2), \qquad with \quad \alpha(x_2) < b,$$

and

$$\psi(x_3) < a.$$

Lemma 2.4 If $y \in C[0, 1]$, $1 < \alpha \leq 2$, then

$$u(t) = \int_0^1 G(t, s)y(s)ds,$$

is a unique solution of

$$\begin{cases} D^{\alpha}_{0+}u(t) + y(t) = 0, \quad 0 < t < 1,\\ u(0) - \beta u'(\xi) = 0, \quad u(1) - \gamma u'(\eta) = 0, \end{cases}$$

here

$$G(t,s) = \begin{cases}
-\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(\beta+t)(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} + \dfrac{(\beta^2 - \Delta\beta + \beta t)(\xi - s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} - \dfrac{(\beta\gamma + \gamma t)(\eta - s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & s \leq \xi,\ s \leq t,\\
\dfrac{(\beta+t)(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} + \dfrac{(\beta^2 - \Delta\beta + \beta t)(\xi - s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)} - \dfrac{(\beta\gamma + \gamma t)(\eta - s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & s \leq \xi,\ t \leq s,\\
-\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(\beta+t)(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{(\beta\gamma + \gamma t)(\eta - s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & \xi \leq s \leq \eta,\ s \leq t,\\
\dfrac{(\beta+t)(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)} - \dfrac{(\beta\gamma + \gamma t)(\eta - s)^{\alpha-2}}{\Delta\Gamma(\alpha-1)}, & \xi \leq s \leq \eta,\ t \leq s,\\
-\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \dfrac{(\beta+t)(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & \eta \leq s,\ s \leq t,\\
\dfrac{(\beta+t)(1-s)^{\alpha-1}}{\Delta\Gamma(\alpha)}, & \eta \leq s,\ t \leq s.
\end{cases} \tag{4}$$

Proof:

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + c_1 + c_2t,$$

for some $c_1,\ c_2 \in R$ can be got from $D^{\alpha}_{0+}u(t) + y(t) = 0$ by using Lemma 2.2 and Definition 2.1.

$$c_1 = -\frac{\beta}{\Gamma(\alpha-1)}\int_0^{\xi} (\xi - s)^{\alpha-2}y(s)ds + c_2\beta,$$

$$\begin{aligned}
&-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds + c_1 + c_2\\
&= -\frac{\gamma}{\Gamma(\alpha-1)}\int_0^{\eta} (\eta - s)^{\alpha-2}y(s)ds + c_2\gamma,
\end{aligned}$$

can be got from condition (2).

Here it is

$$\begin{aligned}
c_1 = &\frac{\beta}{\Delta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds\\
&+ \frac{\beta^2}{\Delta\Gamma(\alpha-1)}\int_0^{\xi} (\xi - s)^{\alpha-2}y(s)ds\\
&- \frac{\beta\gamma}{\Delta\Gamma(\alpha-1)}\int_0^{\eta} (\eta - s)^{\alpha-2}y(s)ds\\
&- \frac{\beta}{\Gamma(\alpha-1)}\int_0^{\xi} (\xi - s)^{\alpha-2}y(s)ds.
\end{aligned}$$

$$\begin{aligned}c_2 = & \frac{1}{\Delta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \\ & + \frac{\beta}{\Delta\Gamma(\alpha-1)}\int_0^\xi (\xi-s)^{\alpha-2}y(s)ds \\ & - \frac{\gamma}{\Delta\Gamma(\alpha-1)}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds.\end{aligned}$$

Therefore,

$$\begin{aligned}u(t) = & -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ & + \frac{\beta+t}{\Delta\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \\ & + \frac{\beta^2-\Delta\beta+\beta t}{\Delta\Gamma(\alpha-1)}\int_0^\xi (\xi-s)^{\alpha-2}y(s)ds \\ & - \frac{\beta\gamma+\gamma t}{\Delta\Gamma(\alpha-1)}\int_0^\eta (\eta-s)^{\alpha-2}y(s)ds \\ = & \int_0^1 G(t,s)y(s)ds.\end{aligned}$$

is the unique solution of this problem. The proof is complete. ■

Let

$$M = \frac{(\beta+1)+(\alpha-1)(\beta^2-\Delta\beta+\beta)}{\Delta\Gamma(\alpha)}. \tag{5}$$

Lemma 2.5 Assume that $(H_1), (H_2)$ hold. In addition, the function $G(t,s)$ described by (4) matches the following relationship

(1) $G \in C([0,1]\times[0,1))$ and $0 \le G(t,s) \le M(1-s)^{\alpha-2}$, for $t,s \in (0,1)$;

(2) There have positive Υ satisfying

$$\min_{\frac14\le t\le\frac34} G(t,s) \ge \Upsilon M(1-s)^{\alpha-2}, \quad \frac14 \le s \le \frac34. \tag{6}$$

Proof: The definition of $G(t,s)$ implies $G \in C([0,1]\times[0,1))$.

$$G(t,s) \ge 0$$

can be got by (H_1).

$$\min_{\frac14\le t\le\frac34} G(t,s) \ge \begin{cases} \frac{[(1-\eta)-(\alpha-1)\gamma]t}{\Delta\Gamma(\alpha)}(1-s)^{\alpha-2} \\ +\frac{(\gamma-1)(1-\eta)-(\alpha-1)\beta\gamma}{\Delta\Gamma(\alpha)} \\ (1-s)^{\alpha-2}, \\ \qquad\qquad \frac14 \le s\le t\le \frac34, \\ \frac{\frac14(\beta+t)}{\Delta\Gamma(\alpha)}(1-s)^{\alpha-2}, \ \frac14\le t\le s\le\frac34, \end{cases}$$

$$\ge \begin{cases} \frac{[(1-\eta)-(\alpha-1)\gamma]\frac14}{\Delta\Gamma(\alpha)}(1-s)^{\alpha-2}, \\ +\frac{(\gamma-1)(1-\eta)-(\alpha-1)\beta\gamma}{\Delta\Gamma(\alpha)}(1-s)^{\alpha-2}, \\ \qquad\qquad \frac14\le s\le t\le\frac34, \\ \frac{\frac14(\beta+\frac14)}{\Delta\Gamma(\alpha)}(1-s)^{\alpha-2}, \qquad \frac14\le t\le s\le\frac34, \end{cases}$$

$$\begin{aligned}\ge \min\Bigg\{\Bigg\{ & \frac{[(1-\eta)-(\alpha-1)\gamma]\frac14}{\Delta\Gamma(\alpha)} \\ + & \frac{(\gamma-1)(1-\eta)-(\alpha-1)\beta\gamma}{\Delta\Gamma(\alpha)}\Bigg\}, \\ \Bigg\{ & \frac{\frac14(\beta+\frac14)}{\Delta\Gamma(\alpha)}\Bigg\}\Bigg\}(1-s)^{\alpha-2}.\end{aligned}$$

$$\begin{aligned}\max_{0\le t\le1} G(t,s) \le & \frac{(\beta+1)+(\alpha-1)(\beta^2-\Delta\beta+\beta)}{\Delta\Gamma(\alpha)} \\ & (1-s)^{\alpha-2}. \\ = & M(1-s)^{\alpha-2}, \quad 0\le s\le 1.\end{aligned}$$

Therefore,

$$\begin{aligned}\Upsilon = \min\Bigg\{\Bigg\{ & \frac{[(1-\eta)-(\alpha-1)\gamma]\frac14}{(\beta+1)+(\alpha-1)(\beta^2-\Delta\beta+\beta)} \\ + & \frac{(\gamma-1)(1-\eta)-(\alpha-1)\beta\gamma}{(\beta+1)+(\alpha-1)(\beta^2-\Delta\beta+\beta)}\Bigg\}, \\ & \Bigg\{\frac{\frac14(\beta+\frac14)}{(\beta+1)+(\alpha-1)(\beta^2-\Delta\beta+\beta)}\Bigg\}\Bigg\}.\end{aligned}$$

and

$$\min_{\frac14\le t\le\frac34} G(t,s) \ge \Upsilon M(1-s)^{\alpha-2}, \quad 0<s<1.$$

■

III. Main result

We define the norm

$$\|u\| = \max\left\{\max_{0\le t\le1}|u(t)|, \max_{0\le t\le1}|u'(t)|\right\}.$$

Thus we have $X = C^1[0,1]$ is a Banach space in the above norm case.

$K \subset X$ is a cone defined by

$$K = \{u\in X|\ u(t)\ge0,\ u(0)-\beta u'(\xi)=0,$$

$$\min_{\frac14\le t\le\frac34} u(t) \ge \Upsilon \max_{0\le t\le1} u(t)\}.$$

Lemma 3.1 Just for $u \in K$,

$$\max_{0\le t\le1}|u(t)| \le (1+\beta)\max_{0\le t\le1}|u'(t)|. \tag{7}$$

Proof:

$$u(t)-u(0) = \int_0^t u'(s)ds \le \max_{0\le t\le1}|u'(t)|, \quad t\in[0,1] \tag{8}$$

can be got by the definition of first order derivative. Moreover, taking into account that u is nonnegative, just

$$u(0) = \beta u'(\xi) \le \beta\max_{0\le t\le1}|u'(t)|.$$

Thus,

$$\max_{0\le t\le1}|u(t)| \le (1+\beta)\max_{0\le t\le1}|u'(t)|.$$

■

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functional θ, γ and the nonnegative continuous functional ψ be defined on the cone K by

$$\begin{aligned}\gamma(u) &= \max_{0\le t\le1}|u'(t)|,\ \psi(u)=\theta(u)=\max_{0\le t\le1}|u(t)|, \\ \alpha(u) &= \min_{\frac14\le t\le\frac34}|u(t)|.\end{aligned} \tag{9}$$

the relations

$$\begin{aligned}&\Upsilon\theta(u)\le\alpha(u)\le\theta(u)=\psi(u), \\ &\|u\| = \max\{\theta(u),\ \gamma(u)\} \le (1+\beta)\gamma(u).\end{aligned} \tag{10}$$

hold by lemma 2.5 and Lemma 3.1, for all $u\in K$.

So the condition (3) of Theorem 2.3 is met.

Lemma 3.2 Operator relation $T:\ K\to X$ described by

$$(Tu)(t)=\int_0^1 G(t,s)f(s,u(s),u'(s))ds. \quad (11)$$

is a completely continuous operator.

Proof:

$$(Tu)(t)=\int_0^1 G(t,s)f(s,u(s),u'(s))ds\geq 0.$$

$$(Tu)(0)-\beta(Tu)'(\xi)=0,\ \min_{\frac{1}{4}\leq t\leq\frac{3}{4}}(Tu)(t)\geq\Upsilon\max_{0\leq t\leq 1}(Tu)(t)$$

can be got by (4) and (11). Just we have $T:\ K\to K$. The operator $T:\ K\to K$ is continuous in view of the continuity of function $G(t,s)$ and $f(t,u(t),u'(t))$. If we assumed $\Omega\subset K$ is a bounded set. Then we can find a positive constant $R_1>0$ satisfies $\|u\|\leq R_1,\ u\in\Omega$. Write

$$R=\max_{0\leq t\leq 1,\ u\in\Omega}|f(t,u(t),u'(t))|+1. \quad (12)$$

Then for $u\in\Omega$, we have

$$\begin{array}{l}|Tu|\leq\int_0^1 G(t,s)|f(s,u(s),u'(s))|ds\\ \leq R\int_0^1 M(1-s)^{\alpha-2}ds=\frac{MR}{\alpha-1}.\end{array} \quad (13)$$

$$\begin{aligned}|(Tu)'(t)|\ &=\Big|-\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f(s,u(s),u'(s))ds\\ &+\frac{1}{\Delta\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,u(s),u'(s))ds\\ &+\frac{\beta}{\Delta\Gamma(\alpha-1)}\int_0^\xi(\xi-s)^{\alpha-2}f(s,u(s),u'(s))ds\\ &-\frac{\gamma}{\Delta\Gamma(\alpha-1)}\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s),u'(s))ds\Big|\\ &\leq\frac{R}{\Gamma(\alpha)}+\frac{R}{\Delta\Gamma(\alpha+1)}+\frac{\beta R\xi^{\alpha-1}}{\Delta\Gamma(\alpha)}+\frac{\gamma R\eta^{\alpha-1}}{\Delta\Gamma(\alpha)}.\end{aligned} \quad (14)$$

Hence, $T(\Omega)$ is a bounded set. For $u\in\Omega,\ t_1,t_2\in[0,1]$,

$$\begin{aligned}&|Tu(t_2)-Tu(t_1)|\\ &\leq\Big|\frac{1}{\Gamma(\alpha)}\Big(\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,u(s),u'(s))ds\\ &-\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,u(s),u'(s))ds\Big)\Big|\\ &+\frac{1}{\Delta\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,u(s),u'(s))ds\times|t_2-t_1|\\ &+\frac{\beta}{\Delta\Gamma(\alpha-1)}\int_0^\xi(\xi-s)^{\alpha-2}f(s,u(s),u'(s))ds\times|t_2-t_1|\\ &+\frac{\gamma}{\Delta\Gamma(\alpha-1)}\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s),u'(s))ds\times|t_2-t_1|\\ &\leq\frac{R|t_2^\alpha-t_1^\alpha|}{\Gamma(\alpha+1)}+\Big[\frac{R}{\Delta\Gamma(\alpha+1)}\\ &+\frac{\beta R\xi^{\alpha-1}+\gamma R\eta^{\alpha-1}}{\Delta\Gamma(\alpha)}\Big]\times|t_2-t_1|.\end{aligned} \quad (15)$$

$$\begin{aligned}&|(Tu)'(t_2)-(Tu)'(t_1)|\\ &\leq\Big|\frac{1}{\Gamma(\alpha-1)}\Big(\int_0^{t_1}(t_1-s)^{\alpha-2}f(s,u(s),u'(s))ds\\ &-\int_0^{t_2}(t_2-s)^{\alpha-2}f(s,u(s),u'(s))ds\Big)\Big|\\ &\leq\frac{R}{\Gamma(\alpha)}\times\left|t_2^{\alpha-1}-t_1^{\alpha-1}\right|.\end{aligned} \quad (16)$$

Thus

$$\|(Tu)(t_2)-(Tu)(t_1)\|\to 0 \text{ for } t_1\to t_2,\ u\in\Omega. \quad (17)$$

So $T:K\to K$ is completely continuous in view of the Arzela-Ascoli theorem. ■

Theorem 3.3 Assume that $(H_1),\ (H_2)$ hold. In addition, there exist $0<a,b,d$ satisfy $a<b<d, c=\frac{b}{\gamma}$, and if we can find f satisfies:

$$\begin{aligned}&(A_1)\ f(t,\ h,\ k)\leq\frac{\Delta\alpha\Gamma(\alpha)}{\Delta\alpha+1+\alpha(\beta\xi^{\alpha-1}+\gamma\eta^{\alpha-1})}d,\\ &for\ (t,\ h,\ k)\in[0,1]\times[0,(1+\beta)d]\times[-d,d];\\ &(A_2)\ f(t,\ h,\ k)>\frac{\alpha-1}{\Upsilon M}b,\\ &for\ (t,\ h,\ k)\in[\tfrac{1}{4},\tfrac{3}{4}]\times[b,\tfrac{b}{\Upsilon}]\times[-d,d];\\ &(A_3)\ f(t,\ h,\ k)<\frac{\alpha-1}{M}a,\\ &for\ (t,\ h,\ k)\in[0,1]\times[0,a]\times[-d,d].\end{aligned}$$

Then at least three positive solutions $u_1,\ u_2$ and u_3 can be got. In addition, the solutions satisfy

$$\begin{aligned}&\max_{0\leq t\leq 1}|u_i'(t)|\leq d\qquad for\quad i=1,2,3,\\ &b<\min_{\frac{1}{4}\leq t\leq\frac{3}{4}}|u_1(t)|,\qquad\max_{0\leq t\leq 1}|u_1(t)|\leq(1+\beta)d,\\ &a<\max_{0\leq t\leq 1}|u_2(t)|<\tfrac{b}{\Upsilon},\\ &\ with\quad\min_{\frac{1}{4}\leq t\leq\frac{3}{4}}|u_2(t)|<b,\\ &and\ \ \max_{0\leq t\leq 1}|u_3(t)|<a.\end{aligned}$$

Proof: Obviously, if $u(t)$ satisfies the relation

$$u(t)=\int_0^1 G(t,s)f(s,u(s),u'(s))ds=(Tu)(t). \quad (18)$$

then we can say the fractional problem (1)(2) has a solution $u(t)$. Thus we set out to prove that T satisfies the Avery-Peterson fixed point theorem.

For $u\in\overline{K(\gamma,d)}$, there have $\gamma(u)=\max_{0\leq t\leq 1}|u'(t)|\leq d$. Then $\max_{0\leq t\leq 1}|u(t)|\leq(1+\beta)d$ can be got by lemma 3.1. $f(t,u(t),u'(t))\leq\frac{\Delta\alpha\Gamma(\alpha)}{\Delta\alpha+1+\alpha(\beta\xi^{\alpha-1}+\gamma\eta^{\alpha-1})}d$. can be got by condition (A_1). Conversely, for $u\in K$, there have $Tu\in K$,

$$\begin{aligned}&\gamma(Tu)\\ &=\max_{0\leq t\leq 1}\Big|-\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f(s,u(s),u'(s))ds\\ &+\frac{1}{\Delta\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,u(s),u'(s))ds\\ &+\frac{\beta}{\Delta\Gamma(\alpha-1)}\int_0^\xi(\xi-s)^{\alpha-2}f(s,u(s),u'(s))ds\\ &-\frac{\gamma}{\Delta\Gamma(\alpha-1)}\int_0^\eta(\eta-s)^{\alpha-2}f(s,u(s),u'(s))ds\Big|\\ &\leq\Big(\frac{1}{\Gamma(\alpha)}+\frac{1}{\Delta\Gamma(\alpha+1)}+\frac{\beta\xi^{\alpha-1}}{\Delta\Gamma(\alpha)}+\frac{\gamma\eta^{\alpha-1}}{\Delta\Gamma(\alpha)}\Big)\\ &\frac{\Delta\alpha\Gamma(\alpha)}{\Delta\alpha+1+\alpha(\beta\xi^{\alpha-1}+\gamma\eta^{\alpha-1})}d=d.\end{aligned}$$

Just, $T:\ \overline{K(\gamma,d)}\to\overline{K(\gamma,d)}$.

In order to check condition (S_1) of Theorem 2.3, we choose $u(t)=\frac{b}{\Upsilon}\in K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$ and $\alpha(u)>b$, implies that $\{u\in K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)|\alpha(u)>b\}\neq\emptyset$. Hence, for $u\in K(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$, there is $b\leq u(t)\leq\frac{b}{\Upsilon},\quad |u'(t)|\leq d$ for $\frac{1}{4}\leq t\leq\frac{3}{4}$.

Thus, by condition (A_2) of this theorem, we have $f(t,u(t),u'(t)) > \frac{\alpha-1}{\Upsilon M}b$ for $\frac{1}{4} \le t \le \frac{3}{4}$, and combining the conditions of α and K,
we have

$$\begin{aligned}\alpha(Tu) &= \min_{\frac{1}{4}\le t\le\frac{3}{4}} \left|\int_0^1 G(t,s)f(s,u(s),u'(s))ds\right| \\ &\ge \frac{\alpha-1}{\Upsilon M}b\int_0^1 \Upsilon M(1-s)^{\alpha-2}ds = b.\end{aligned}$$

i.e., $\alpha(Tu) > b$ for all $u \in K(\gamma,\ \theta,\ \alpha,\ b,\ \frac{b}{\Upsilon},\ d\}$. This shows that condition (S_1) of Theorem 2.3 is satisfied.

Secondly,

$\alpha(Tu) \ge \Upsilon\theta(Tu) > \Upsilon\frac{b}{\Upsilon} = b$, for all $u \in K(\gamma,\ \alpha,\ b, d)$ with $\theta(Tu) > c$. can be got by Lemma 3.2.

Thus, condition (S_2) of Theorem 2.3 is satisfied.

Finally, we declare that condition (S_3) of Theorem 2.3 also holds. Clearly, as $\psi(0) = 0 < a$, there holds $0 \notin R(\gamma,\ \psi,\ a,\ d)$. Suppose that $u \in R(\gamma,\ \psi,\ a,\ d)$ with $\psi(u) = a$.
Then,

$$\begin{aligned}&\psi(Tu) = \max_{0\le t\le 1}|(Tu)(t)| \\ &= \max_{0\le t\le 1}\left|\int_0^1 G(t,s)f(s,u(s),u'(s))ds\right| \\ &\le \frac{\alpha-1}{M}a\int_0^1 M(1-s)^{\alpha-2}ds = a.\end{aligned}$$

can be got by the condition (A_3). So, the condition (S_3) of Theorem 2.3 is satisfied. Therefore, an application of Theorem 2.3, we can say the FBVP(1), (2) has at least three positive solutions $u_1,\ u_2$ and u_3 satisfy

$$\begin{aligned}&\max_{0\le t\le 1}|u_i'(t)| \le d \qquad \text{for} \quad i = 1,2,3, \\ &b < \min_{\frac{1}{4}\le t\le\frac{3}{4}}|u_1(t)|, \quad \max_{0\le t\le 1}|u_1(t)| \le (1+\beta)d, \\ &a < \max_{0\le t\le 1}|u_2(t)| < \tfrac{b}{\Upsilon}, \\ &\text{with} \quad \min_{\frac{1}{4}\le t\le\frac{3}{4}}|u_2(t)| < b, \\ &and \quad \max_{0\le t\le 1}|u_3(t)| < a.\end{aligned}$$

The proof is complete. ■

IV. Examples

Example 4.1 Consider the following FBVP

$$D_{0+}^{\frac{3}{2}}u(t) + f(t,u(t),u'(t)) = 0, \qquad 0 < t < 1, \tag{19}$$

$$u(0) - \frac{13}{20}u'(\frac{1}{20}) = 0, \quad u(1) - \frac{8}{5}u'(\frac{1}{10}) = 0, \tag{20}$$

where

$$f(t,h,k) = \begin{cases} \frac{t}{1000} + \frac{1}{800}h^4 + \frac{1}{1000}\left(\frac{k}{10000}\right)^2, \\ \text{for} \quad 0 \le h \le 10, \\ \frac{t}{1000} + \frac{25}{2} + \frac{1}{1000}\left(\frac{k}{10000}\right)^2, \\ \text{for} \quad h > 10. \end{cases}$$

We find $\alpha = \frac{3}{2},\ \beta = \frac{13}{20},\ \gamma = \frac{8}{5},\ \xi = \frac{1}{20},\ \eta = \frac{1}{10}$, $\Delta = \frac{1}{20}$, $M = \frac{434}{5\sqrt{\pi}}$, $\Upsilon = \frac{9}{434}$, $\frac{\Delta\alpha\Gamma(\alpha)}{\Delta\alpha+1+\alpha(\beta\xi^{\alpha-1}+\gamma\eta^{\alpha-1})}d \approx 323.72$, $\frac{\alpha-1}{\Upsilon M}b \approx 4.916$, $\frac{\alpha-1}{M}a \approx 0.010195$ can be got by a direct calculation.

Taking $a = 1,\ b = 10,\ d = 10000$, we get

$$\begin{aligned}&f(t,\ h,\ k) < 13 < 323.72 \\ &\quad = \frac{\Delta\alpha\Gamma(\alpha)}{\Delta\alpha+1+\alpha(\beta\xi^{\alpha-1}+\gamma\eta^{\alpha-1})}d, \\ &\quad \text{for } 0 \le t \le 1,\ 0 \le h \le 16500,\ -10000 \le k \le 10000; \\ &f(t,\ h,\ k) > 12.5 > 4.916 = \frac{\alpha-1}{\Upsilon M}b, \\ &\quad \text{for } \tfrac{1}{4} \le t \le \tfrac{3}{4},\ 10 \le h \le 482,\ -10000 \le k \le 10000; \\ &f(t,\ h,\ k) < 0.00325 < 0.010195 = \frac{\alpha-1}{M}a, \\ &\quad \text{for } 0 \le t \le 1,\ 0 \le h \le 1,\ -10000 \le k \le 10000.\end{aligned}$$

Then all conditions of Theorem 3.3 hold. Thus, with Theorem 3.3, problem (19), (20) has at least three positive solutions $u_1,\ u_2$ and u_3 such that

$$\begin{aligned}&\max_{0\le t\le 1}|u_i'(t)| \le 10000 \qquad \text{for} \quad i = 1,2,3, \\ &10 < \min_{\frac{1}{4}\le t\le\frac{3}{4}}|u_1(t)|, \quad \max_{0\le t\le 1}|u_1(t)| \le 165000, \\ &1 < \max_{0\le t\le 1}|u_2(t)| < 482, \\ &\text{with} \quad \min_{\frac{1}{4}\le t\le\frac{3}{4}}|u_2(t)| < 10, \\ &and \quad \max_{0\le t\le 1}|u_3(t)| < 1.\end{aligned}$$

References

[1] K. Diethelm, *The analysis of fractional differential equations*, Lecture Notes in Math., 2010.
[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations, in: North-Holland Mathematics Studies*, vol. 204, Elsevier Science B.V, Amsterdam, 2006.
[3] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.
[4] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[5] V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles*, Fields and Media, Springer, HEP, 2010.
[6] Ravi P. Agarwal, Mouffak Benchohra, and Samira Hamani, "A Survey on Existence Results for Boundary Value Problems of Nonlinear Fractional Differential Equations and Inclusions," Acta Appl Math, vol. 109, no. 3, pp. 973-1033, 2010.
[7] Bashir Ahmad, and Juan J. Nieto, "Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions," *Computers and Mathematics with Applications*, vol. 58, no. 9, pp. 1838-1843, 2009.
[8] Zhanbing Bai, "On positive solutions of a nonlocal fractional boundary value problem," *Nonlinear Analysis*, vol. 72, no. 2, pp. 916-924, 2010.
[9] M. Benchohra, J. Henderson, S.K. Ntouyas, and A. Ouahab, "Existence results for fractional order functional differential equations with infinite delay," *J. Math. Anal. Appl.*, vol. 338, no. 2, pp. 1340-1350, 2008.
[10] Yong-Kui Chang, and Juan J. Nieto, "Some new existence results for fractional differential inclusions with boundary conditions," *Mathematical and Computer Modelling*, vol. 49, no. 3-4, pp. 605-609, 2009.
[11] JinRong Wang, and Yong Zhou, "A class of fractional evolution equations and optimal controls," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 262-272, 2011.
[12] JinRong Wang, and Yong Zhou, "Analysis of nonlinear fractional control systems in Banach spaces," *Nonlinear Analysis*, vol. 74, no. 17, pp. 5929-5942, 2011.
[13] JinRong Wang, and Yong Zhou, "Existence and controllability results for fractional semilinear differential inclusions," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3642-3653, 2011.
[14] Shuqin Zhang, "Existence of positive solution for some class of nonlinear fractional differential equations," *J. Math. Anal. Appl.*, vol. 278, no. 1, pp. 136-148, 2003.
[15] Yong Zhou, Feng Jiao, and Jing Li, "Existence and uniqueness for fractional neutral differential equations with infinite delay," *Nonlinear Analysis*, vol. 71, no. 7-8, pp. 3249-3256, 2009.

[16] Yong Zhou, and Feng Jiao, "Nonlocal Cauchy problem for fractional evolution equations," *Nonlinear Analysis: Real World Applications*, vol. 11, no. 5, pp. 4465-4475, 2010.
[17] Hebe A. Biagioni, and Todor Gramchev, "Fractional derivative estimates in Gevrey spaces, global regularity and decay for solutions to semilinear equations in R^n," *J. Differential Equations*, vol. 194, no. 1, pp. 140-165, 2003.
[18] Arturo de Pablo, Fernando Quirós, Ana Rodríguez, and Juan Luis Vázquez, "A fractional porous medium equation," *Advances in Mathematics*, vol. 226, no. 2, pp. 1378-1409, 2011.
[19] Bashir Ahmad, and S. Sivasundaram, "On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order," *Applied Mathematics and Computation*, vol. 217, no. 2, pp. 480-487, 2010.
[20] Wen-Xue Zhou, and Yan-Dong Chu, "Existence of solutions for fractional differential equations with multi-point boundary conditions," *Commun Nonlinear Sci Numer Simulat*, vol. 17, no. 3, pp. 1142-1148, 2012.
[21] A.A. Kilbas, and J.J. Trujillo, "Differential equations of fractional order: methods results and problem-I," *Applicable Analysis*, vol. 78, no. 1-2, pp. 153-192, 2001.
[22] A.A. Kilbas, and J.J. Trujillo, "Differential Equations of Fractional Order: Methods, Results and Problems.II," *Applicable Analysis*, vol. 81, no. 2, pp. 435-493, 2002.
[23] A. Babakhani and Varsha Daftardar-Gejji, "Existence of positive solutions of nonlinear fractional differential equations," *J. Math. Anal. Appl.*, vol. 278, no. 2, pp. 434-442, 2003.
[24] Zhanbing Bai, and Haishen Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *J. Math. Anal. Appl.*, vol. 311, no. 2, pp. 495-505, 2005.
[25] Domenico Delbosco, and Luigi Rodino, " Existence and Uniqueness for a Nonlinear Fractional Differential Equation," *Journal of Mathematical Analysis and Applications*, vol. 204, no. 2, pp. 609-625, 1996.
[26] Samuil D. Eidelman, and Anatoly N. Kochubei, "Cauchy problem for fractional diffusion equations," *J. Differential Equations*, vol. 199, no. 2, pp. 211-255, 2004.
[27] Qinghua Feng, "Compact Difference Schemes for a Class of Space-time Fractional Differential Equations," *Engineering Letters*, vol. 27, no. 2, pp. 269-277, 2019.
[28] Varsha Daftardar-Gejji, and A. Babakhani," Analysis of a system of fractional differential equations," *J. Math. Anal. Appl.*, vol. 293, no. 2, pp. 511-522, 2004.
[29] Hossein Jafari, and Varsha Daftardar-Gejji, "Positive solutions of nonlinear fractional boundary value problems using Adomian decomposition method," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 700-706, 2006.
[30] Bin Zheng, "Some New Fractional Integral Inequalities in the Sense of Conformable Fractional Derivative," *Engineering Letters*, vol. 27, no. 2, pp. 287-294, 2019.
[31] Bashir Ahmad, and Guotao Wang, "A study of an impulsive four-point nonlocal boundary value problem of nonlinear fractional differential equations," *Computers and Mathematics with Applications*, vol. 62, no. 3, pp. 1341-1349, 2011.
[32] Chuanzhi Bai, "Triple positive solutions of three-point boundary value problems for fourth-order differential equations," *Computers and Mathematics with Applications*, vol. 56, no. 5, pp. 1364-1371, 2008.
[33] Xiangkui Zhao, Chengwen Chai, and Weigao Ge, "Positive solutions for fractional four-point boundary value problems," *Commun Nonlinear Sci Numer Simulat*, vol. 16, no. 9, pp. 3665-3672, 2011.
[34] Lili Wang, "Dynamic Output Feedback Guaranteed Cost Control for Linear Nominal Impulsive Systems," *Engineering Letters*, vol. 26, no. 4, pp. 405-409, 2018.