Weyl Fractional Integral of Multi-Index Mittag-Leffler Function and I-Function

C. Gammeng, U. K. Saha, S. Maity

Abstract- In this paper we develop two new interesting theorems that establishes relationships connecting the Weyl fractional integral to the product of multi-index Mittag-Leffler function with *I*-function and *H*-function. Subsequently, certain very intriguing special cases are obtained from the main theorems.

Index Terms- *I*-function, *H*-function, Multi-index Mittag-Leffler function, Weyl fractional integral.

I. INTRODUCTION

Fractional calculus is the area that attends to the integration and differentiation of arbitrary order that could be rational, irrational or complex. The subject has attained substantial reputation in the past few decades in varied fields of Physics, Chemistry, Biological Science and Engineering. Contradictory to the classical calculus, there are numerous interpretations of fractional order integrals and derivatives. Of late, many authors have published innumerable papers [3, 4, 8-10, 13-16, 18, 23, 28, 29, 31], where they have thoroughly investigated the properties, applications and various extensions of the fractional calculus operators associating distinct arrays of special functions.

The Weyl fractional integral [18] is defined as

$$W_{s,\infty}^{c} f(s) = \frac{1}{\Gamma(c)} \int_{s}^{\infty} (r-s)^{c-1} f(r)dr, Re(c) > 0. \quad (1)$$

The Mittag-Leffler (M-L) function E_μ (Mittag-Leffler, 1902-1905) [19] is given as

$$E_\mu(y) = \sum_{r=0}^{\infty} \frac{y^r}{\Gamma(\mu r+1)}, \mu > 0. \quad (2)$$

Wiman 1905 [32, 33] and Agarwal 1953 [1] also defined the Mittag-Leffler function $E_{\mu,\nu}$ by the power series:

$$E_{\mu,\nu}(y) = \sum_{r=0}^{\infty} \frac{y^r}{\Gamma(\mu r+\nu)}, \mu > 0, \nu > 0. \quad (3)$$

The multi-index Mittag-Leffler function was introduced by Kiryakova [13] and is defined as:

Let $\delta_1, \cdots, \delta_m > 0$ and $\xi_1, \cdots, \xi_m$ be arbitrary real numbers and $m > 1$ be an integer, then the multi-index M-L function is defined by power series:

$$E_{\left(\frac{1}{\delta_i}\right),(\xi_i)}(y) = \sum_{r=0}^{\infty} \frac{y^r}{\prod_{j=1}^{m} \Gamma\left(\xi_j + \frac{r}{\delta_j}\right)}. \quad (4)$$

For $m = 1, (4)$ becomes the classical M-L function $E_{\left(\frac{1}{\delta}\right),(\xi)}(y)$ and on taking $\frac{1}{\delta} = \mu > 0$ and $\xi = \nu > 0$, it reduces to the M-L function $E_{\mu,\nu}(y)$ of Wiman 1905 and Agarwal 1953 given in (3).

V. P. Saxena [25] established the *I*-function and defined it as

$$I(s) = I_{X_\beta,Y_\beta:R}^{U,V}[s]$$

$$= I_{X_\beta,Y_\beta:R}^{U,V}\left[s \left| \begin{matrix} (a_k,e_k)_{1,V}; (a_{k\beta},e_{k\beta})_{V+1,X_\beta} \\ (b_k,f_k)_{1,U}; (b_{k\beta},f_{k\beta})_{U+1,Y_\beta} \end{matrix} \right.\right]$$

$$= \frac{1}{2\pi\beta} \oint_L \varphi(t) s^t dt \quad (5)$$

where,

$$\varphi(t) = \frac{\prod_{k=1}^{U} \Gamma(b_k - f_k t) \prod_{k=1}^{V} \Gamma(1-a_k+e_k t)}{\sum_{\beta=1}^{R} \left\{ \prod_{k=U+1}^{Y_\beta} \Gamma(1-b_{k\beta}+f_{k\beta} t) \prod_{k=V+1}^{X_\beta} \Gamma(a_{k\beta}-e_{k\beta} t) \right\}} \quad (6)$$

$X_\beta, Y_\beta (\beta = 1,2,\cdots,R), U, V$ are integers satisfying $0 \le V \le X_\beta, 0 \le U \le Y_\beta$; $e_k, f_k, e_{k\beta}, f_{k\beta}$ are real and positives; $a_k, b_k, a_{k\beta}, b_{k\beta}$ are complex numbers. L is a suitable contour of the Mellin-Barnes types from $\gamma - ie$ to $\gamma + ie$ (γ is real) in the complex *t*-plane. One may refer [25] for details regarding existence and various other parametric restrictions of *I*-function. Recently, many researchers have dedicated to study the *I*-function further. Shukla et al. [27] established integral relations involving *I*-function with the product of generalized hyper-geometric function. Jain et al. [7] studied multiple integral involving *I*-function and Bessel-Maitland function. Habenom et al. [6] developed new results by

Manuscript received April 1, 2020; revised September 29, 2020.
This work was supported under the project SG-TEQIP-III/BAS/UKS of the National Institute of Technology, Arunachal Pradesh, India.
C. Gammeng, Research Scholar, Department of Basic & Applied Science, National Institute of Technology, Arunachal Pradesh, India. Phone no. 8974365634; Email: cgammeng@gmail.com
U. K. Saha, Assistant Professor, Department of Basic & Applied Science, National Institute of Technology, Arunachal Pradesh, India. Email: utpal@nitap.ac.in
S. Maity, Assistant Professor, Department of Basic & Applied Science, National Institute of Technology, Arunachal Pradesh, India. Email: susanta@nitap.ac.in

applying finite integral on *I*-function. Saha et al. [21] evaluated certain integrals involving product of *I*-function with exponential function, hypergeometric function and *H*-function.

For $R = 1$, (5) corresponds to the renowned *H*-function introduced by C. Fox in 1961 [5],

$$H_{X,Y}^{U,V}\left[s\left|\begin{matrix}(a_k,e_k)_{1,V};(a_k,e_k)_{V+1,X}\\(b_k,f_k)_{1,U};(b_k,f_k)_{U+1,Y}\end{matrix}\right.\right] = H_{X,Y}^{U,V}\left[s\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right] \tag{7}$$

More recently, due to its close affiliation to the Mellin transforms, the *H*-function has been recognized to play a pivotal role in fractional calculus and in their applications, some of which involving non-Gaussian stochastic processes and phenomena of non-standard diffusion and relaxation, see [11, 12, 17, 24, 26, 30].

According to Braksma [2], when $D > 0$ so that $|arg\, y| < \frac{1}{2}D\pi$, then the integral in the right side of (7) is absolutely convergent, where

$$D = \sum_{k=1}^{V} e_k - \sum_{k=V+1}^{X} e_k + \sum_{k=1}^{U} f_k - \sum_{k=U+1}^{Y} f_k \tag{8}$$

Here, we stress upon the main definition of Weyl fractional integral and present some new identities that includes Weyl fractional integrals to the product of multi-index Mittag-Leffler function, *I*-function and also to *H*-function.

II. MATHEMATICAL PRE-REQUISITES

The following pre-requisites will help us constitute the main result. The *H*-function satisfies the following integral [22]

$$\int_z^\infty u^{\gamma-1}(u-z)^{\chi-1}H_{X,Y}^{U,V}\left[su^\theta(u-z)^\phi\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]du = z^{\gamma+z-1}H_{X+2,Y+1}^{U+1,V+1}\left[sz^{\theta\phi}\left|\begin{matrix}(1-\chi,\theta),(a_k,e_k)_{1,X},(1-\gamma,\theta)\\(1-\gamma-\chi,\theta+\phi),(b_k,f_k)_{1,Y}\end{matrix}\right.\right] \tag{9}$$

where,

(i) $|\arg s| < \frac{1}{2}D\pi$, D is defined in (8),

(ii) θ, ϕ are positive real numbers and γ, χ are complex numbers,

(iii) $min\left[Re\left(\frac{1-\gamma-\chi}{\theta+\phi}\right), min_{1\le k\le U}\left[Re\left(\frac{b_k}{f_k}\right)\right]\right]$

$> max\left[-Re\left(\frac{\chi}{\phi}\right), max_{1\le k\le V}\left[Re\left(\frac{a_k-1}{e_k}\right)\right]\right].$

Lemma 1.1 From Rainville [20] we have an important result

$$\sum_{w=0}^{\infty}\sum_{i=0}^{\infty} D(i,w) = \sum_{w=0}^{\infty}\sum_{i=0}^{w} D(i,w-i) \tag{10}$$

III. MAIN RESULTS

In this section, we established two theorems connecting the Weyl fractional integral to the product of multi-index Mittag-Lefler function with *I*-function and with *H*-function. Our results are encased in the following:

Theorem 3.1 Let U, V, $X_\beta, Y_\beta\,(\beta = 1,\cdots,R)$ be integers satisfying $0 \le V \le X_\beta, 0 \le U \le Y_\beta$; $e_k, f_k, e_{k\beta}, f_{k\beta}$ be positive and real and $a_k, b_k, a_{k\beta}, b_{k\beta}$ be complex numbers. L is the path of integration separating the increasing and decreasing sequence of the poles of integration. Then for all values of c,

$$W_{s,\infty}^c\left\{y^{\gamma-1}(s-y)^{\chi-1}e^{-yu}E_{\left(\frac{1}{\delta_i}\right),(\xi_i)}\left(hy^\zeta\right)\times I_{X_\beta,Y_\beta:R}^{U,V}\left[y^\lambda\right]\right\}$$

$$= e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w} f(i)\frac{u^{w-i}}{(w-i)!}(-1)^{\chi+w-i-1}$$

$$\frac{1}{2\pi\beta}\oint\varphi(\lambda)\times W_{s,\infty}^c f\left(s^{\lambda+\gamma+\zeta i-1}\right)d\lambda\ , \tag{11}$$

$$\text{where,}\quad f(i) = \frac{h^i}{\Pi_{j=1}^m\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}. \tag{12}$$

Proof: Denoting the L.H.S by Δ

$$\Delta = W_{s,\infty}^c\left\{y^{\gamma-1}(s-y)^{\chi-1}e^{-yu}E_{\left(\frac{1}{\delta_i}\right)(\xi_i)}\left(hy^\zeta\right)\right.$$

$$\left.\times I_{X_\beta,Y_\beta:R}^{U,V}\left[y^\lambda\right]\right\}$$

$$= W_{s,\infty}^c\left\{e^{-su}y^{\gamma-1}(s-y)^{\chi-1}e^{(s-y)u}E_{\left(\frac{1}{\delta_i}\right),(\xi_i)}\left(hy^\zeta\right)\right.$$

$$\left.\times I_{X_\beta,Y_\beta:R}^{U,V}\left[y^\lambda\right]\right\}.$$

Now, replacing $e^{(s-y)u}$ with $\sum_{w=0}^{\infty}\frac{(s-y)^w u^w}{w!}$ and making use of (4) and (5), we get

$$\Delta = W_{s,\infty}^c\left\{e^{-su}\quad y^{\gamma-1}(s-y)^{\chi-1}\sum_{w=0}^{\infty}\frac{(s-y)^w u^w}{w!}\right.$$

$$\left.\sum_{i=0}^{\infty}\frac{h^i y^{\zeta i}}{\Pi_{\mathrm{j=1}}^{\mathrm{m}}\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}\times\frac{1}{2\pi\beta}\oint\varphi(\lambda)y^\lambda d\lambda\right\}$$

$$= W_{s,\infty}^c\left\{e^{-su}y^{\gamma-1}(s-y)^{\chi-1}\sum_{w=0}^{\infty}\sum_{i=0}^{\infty}\frac{h^i y^{\zeta i}}{\Pi_{\mathrm{j=1}}^{\mathrm{m}}\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}\right.$$

$$\left.\frac{(s-y)^w u^w}{w!}\times\frac{1}{2\pi\beta}\oint\varphi(\lambda)y^\lambda d\lambda\right\}.$$

Applying (10), the above expression becomes

$$\Delta = W_{s,\infty}^{c}\left\{e^{-su}y^{\gamma-1}(s-y)^{\chi-1}\sum_{w=0}^{\infty}\sum_{i=0}^{w}\frac{h^{i}y^{\zeta i}}{\Pi_{j=1}^{m}\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}\right.$$

$$\left.\frac{(s-y)^{w-i}u^{w-i}}{(w-i)!}\times\frac{1}{2\pi\beta}\oint\varphi(\lambda)y^{\lambda}d\lambda\right\}.$$

Now, making use of (1), the expression changes to

$$\Delta = e^{-su}\frac{1}{\Gamma(c)}\int_{s}^{\infty}(r-s)^{c-1}r^{\gamma-1}(s-r)^{\chi-1}$$

$$\sum_{w=0}^{\infty}\sum_{i=0}^{w}\frac{h^{i}r^{\zeta i}}{\Pi_{j=1}^{m}\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}\frac{(s-r)^{w-i}u^{w-i}}{(w-i)!}$$

$$\times\frac{1}{2\pi\beta}\oint\varphi(\lambda)r^{\lambda}d\lambda dr$$

$$= e^{-su}\frac{1}{\Gamma(c)}\int_{s}^{\infty}(r-s)^{c-1}r^{\gamma+\zeta i-1}(s-r)^{\chi+w-i-1}$$

$$\sum_{w=0}^{\infty}\sum_{i=0}^{w}f(i)\frac{u^{w-i}}{(w-i)!}\times\frac{1}{2\pi\beta}\oint\varphi(\lambda)r^{\lambda}d\lambda dr,$$

where $f(i)$ is given in (12).

Interchanging the order of integration and summation

$$\Delta = e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w}f(i)\frac{u^{w-i}}{(w-i)!}(-1)^{\chi+w-i-1}\frac{1}{2\pi\beta}\oint\varphi(\lambda)$$

$$\times\left\{\frac{1}{\Gamma(c)}\int_{s}^{\infty}(r-s)^{\chi+c+w-i-2}r^{\lambda+\gamma+\zeta i-1}dr\right\}d\lambda$$

$$= e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w}f(i)\frac{u^{w-i}}{(w-i)!}(-1)^{\chi+w-i-1}\frac{1}{2\pi\beta}\oint\varphi(\lambda)$$

$$\times W_{s,\infty}^{c}f\left(s^{\lambda+\gamma+\zeta i-1}\right)d\lambda.$$

Theorem 3.2 Let $\sum_{k=1}^{V}e_k-\sum_{k=V+1}^{X}e_k+\sum_{k=1}^{U}f_k-\sum_{k=U+1}^{Y}f_k>0$ and $0\le V\le X, 0\le U\le Y$, such that U,V,X and Y are non-negative integers along with equation (9) and its set of conditions (i)-(iii). Then, for every value of c,

$$W_{s,\infty}^{c}\left\{y^{\gamma-1}(s-y)^{\chi-1}e^{-yu}E_{\left(\frac{1}{\delta_i}\right),(\xi_i)}\left(hy^{\zeta}\right)\right.$$

$$\left.\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$=\frac{(-1)^{\chi-1}}{\Gamma(c)}s^{\gamma+\chi+c-2}e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w}f(i)\frac{u^{w-i}}{(w-i)!}(-1)^{w-i}$$

$$s^{(\zeta-1)i+w}\times H_{X+2,Y+1}^{U+1,V+1}$$

$$\left[gs^{\theta+\phi}\left|\begin{matrix}(2-c-\chi+i-w,\phi),(a_k,b_k)_{1,X},(1-\gamma-\zeta i,\phi)\\(2-c-\gamma-\chi-(\zeta-1)i-w,\theta+\phi),(b_k,f_k)_{1,Y}\end{matrix}\right.\right] \tag{13}$$

where, $$f(i)=\frac{h^{i}}{\Pi_{j=1}^{m}\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}. \tag{14}$$

Proof: Denoting the L.H.S of equation (13) by Ω

$$\Omega = W_{s,\infty}^{c}\left\{y^{\gamma-1}(s-y)^{\chi-1}e^{-yu}E_{\left(\frac{1}{\delta_i}\right),(\xi_i)}\left(hy^{\zeta}\right)\right.$$

$$\left.\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$= W_{s,\infty}^{c}\left\{e^{-su}y^{\gamma-1}(s-y)^{\chi-1}e^{(s-y)u}E_{\left(\frac{1}{\delta_i}\right),(\xi_i)}\left(hy^{\zeta}\right)\right.$$

$$\left.\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}.$$

Now, replacing $e^{(s-y)u}$ with $\sum_{w=0}^{\infty}\frac{(s-y)^{w}u^{w}}{w!}$ and using (4) and (7), we get

$$\Omega = W_{s,\infty}^{c}\left\{e^{-su}y^{\gamma-1}(s-y)^{\chi-1}\sum_{w=0}^{\infty}\frac{(s-y)^{w}u^{w}}{w!}\right.$$

$$\left.\sum_{i=0}^{\infty}\frac{h^{i}y^{\zeta i}}{\Pi_{j=1}^{m}\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$= W_{s,\infty}^{c}\left\{e^{-su}y^{\gamma-1}(s-y)^{\chi-1}\sum_{w=0}^{\infty}\sum_{i=0}^{\infty}\frac{h^{i}y^{\zeta i}}{\Pi_{j=1}^{m}\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}\right.$$

$$\left.\frac{(s-y)^{w}u^{w}}{w!}\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right].\right.$$

Using (10), the above expression reduces to

$$\Omega = W_{s,\infty}^{c}\left\{e^{-su}y^{\gamma-1}(s-y)^{\chi-1}\sum_{w=0}^{\infty}\sum_{i=0}^{w}\frac{h^{i}y^{\zeta i}}{\Pi_{j=1}^{m}\Gamma\left(\xi_j+\frac{i}{\delta_j}\right)}\right.$$

$$\left.\frac{(s-y)^{w-i}u^{w-i}}{(w-i)!}\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$= e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w}f(i)\frac{u^{w-i}}{(w-i)!}W_{s,\infty}^{c}\{y^{\gamma+\zeta i-1}(s-y)^{w+\chi-i-1}$$

$$\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\},$$

where, $f(i)$is given in (14).

Applying (1), the above expression takes the form

$$\Omega = e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w} f(i)\frac{u^{w-i}}{(w-i)!}\left\{\frac{1}{\Gamma(c)}\int_{s}^{\infty}(r-s)^{c-1}\,r^{\gamma+\zeta i-1}\right.$$

$$\left.(s-r)^{\chi+w-i-1}\times H_{X,Y}^{U,V}\left[gr^{\theta}(r-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]dr\right\}$$

$$= e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w} f(i)\frac{u^{w-i}}{(w-i)!}\frac{(-1)^{\chi+w-i-1}}{\Gamma(c)}\left\{\int_{s}^{\infty}(r-s)^{\chi+c+w-i-2}\right.$$

$$\left.r^{\gamma+\zeta i-1}\times H_{X,Y}^{U,V}\left[gr^{\theta}(r-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]dr\right\}.$$

Now, making use of (9) on the integral part of R.H.S, we get

$$\Omega = e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w} f(i)\frac{u^{w-i}}{(w-i)!}\frac{(-1)^{\chi+w-i-1}}{\Gamma(c)}$$

$$s^{\gamma+\chi+c+(\zeta-1)i+w-2}\times H_{X+2,Y+1}^{U+1,V+1}$$

$$\left[gs^{\theta+\phi}\left|\begin{matrix}(2-c-\chi+i-w,\phi),(a_k,e_k)_{1,X},(1-\gamma-\zeta i,\phi)\\(2-c-\gamma-\chi-(\zeta-1)i-w,\theta+\phi),(b_k,f_k)_{1,Y}\end{matrix}\right.\right]$$

$$=\frac{(-1)^{\chi-1}}{\Gamma(c)}s^{\gamma+\chi+c-2}e^{-su}\sum_{w=0}^{\infty}\sum_{i=0}^{w} f(i)\frac{u^{w-i}}{(w-i)!}$$

$$(-1)^{w-i}s^{(\zeta-1)i+w}\times H_{X+2,Y+1}^{U+1,V+1}$$

$$\left[gs^{\theta+\phi}\left|\begin{matrix}(2-c-\chi+i-w,\phi),(a_k,b_k)_{1,X},(1-\gamma-\zeta i,\phi)\\(2-c-\gamma-\chi-(\zeta-1)i-w,\theta+\phi),(b_k,f_k)_{1,Y}\end{matrix}\right.\right].$$

IV. SPECIAL CASES

In this section, we shall extend various succeeding corollaries.

Corollary 4.1 Taking $u = 0$, $h = 0$ in (13), the exponential function and the multi-index Mittag-Leffler function reduces to unity. Also, substituting γ by $\lambda+1$ and χ by 1, the result in (13) reduces to:

$$W_{s,\infty}^{c}\left\{y^{\lambda}\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{\Gamma(c)}s^{\lambda+c}H_{X+2,Y+1}^{U+1,V+1}\left[gs^{\theta+\phi}\left|\begin{matrix}(1-c,\phi),(a_k,e_k)_{1,X},(-\lambda,\theta)\\(-c-\lambda,\theta+\phi),(b_k,f_k)_{1,Y}\end{matrix}\right.\right] \tag{15}$$

Corollary 4.2 Now on replacing θ by $-\theta$ and ϕ by $-\phi$, the result (15) further reduces to:

$$W_{s,\infty}^{c}\left\{y^{\lambda}\times H_{X,Y}^{U,V}\left[gy^{-\theta}(y-s)^{-\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{\Gamma(c)}s^{\lambda+c}\times H_{X+1,Y+2}^{U+1,V+1}$$

$$\left[gs^{-\theta-\phi}\left|\begin{matrix}(1+c+\lambda,\theta+\phi)(a_k,e_k)_{1,X}\\(c,\phi),(b_k,f_k)_{1,Y},(1+\lambda,\theta)\end{matrix}\right.\right] \tag{16}$$

Corollary 4.3 Further, substituting $-\theta$ for θ, (15) produces the following results accordingly

for $\theta > \phi$,

$$W_{s,\infty}^{c}\left\{y^{\lambda}\times H_{X,Y}^{U,V}\left[gy^{-\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{\Gamma(c)}s^{\lambda+c}\times H_{X+2,Y+1}^{U,V+2}$$

$$\left[gs^{-\theta+\phi}\left|\begin{matrix}(1-c,\phi),(1+c+\lambda,\theta-\phi),(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y},(1+\lambda,\theta)\end{matrix}\right.\right] \tag{17}$$

for $\theta = \phi$,

$$W_{s,\infty}^{c}\left\{y^{\lambda}\times H_{X,Y}^{U,V}\left[gy^{-\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{\Gamma(c)\Gamma(1+c+\lambda)}s^{\lambda+c}\times H_{X+1,Y+1}^{U,V+1}\left[g\left|\begin{matrix}(1-c,\phi),(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y},(1+\lambda,\theta)\end{matrix}\right.\right] \tag{18}$$

for $\theta < \phi$,

$$W_{s,\infty}^{c}\left\{y^{\lambda}\times H_{X,Y}^{U,V}\left[gy^{-\theta}(y-s)^{\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{\Gamma(c)}s^{\lambda+c}\times H_{X+1,Y+2}^{U+1,V+1}$$

$$\left[gs^{-\theta+\phi}\left|\begin{matrix}(1-c,\phi),(a_k,e_k)_{1,X}\\(-c-\lambda,\phi-\theta).(b_k,f_k)_{1,Y},(1+\lambda,\theta)\end{matrix}\right.\right] \tag{19}$$

Corollary 4.4 Lastly, substituting $-\phi$ for ϕ, (15) produces the following results accordingly

for $\theta > \phi$,

$$W_{s,\infty}^{c}\left\{y^{\lambda}\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{-\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{\Gamma(c)}s^{\lambda+c}\times H_{X+1,Y+2}^{U+2,V}$$

$$\left[gs^{\theta-\phi}\left|\begin{matrix}(a_k,e_k)_{1,X},(-\lambda,\theta)\\(c,\phi),(-c-\lambda,\theta-\phi),(b_k,f_k)_{1,Y}\end{matrix}\right.\right] \tag{20}$$

for $\theta = \phi$,

$$W_{s,\infty}^{c}\left\{y^{\lambda}\times H_{X,Y}^{U,V}\left[gy^{\theta}(y-s)^{-\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$=\frac{1}{\Gamma(c)\Gamma(1+c+\lambda)}s^{\lambda+c}\times H_{X+1,Y+1}^{U+1,V}\left[g\left|\begin{matrix}(a_k,e_k)_{1,X},(-\lambda,\theta)\\(c,\phi),(b_k,f_k)_{1,Y},\end{matrix}\right.\right] \tag{21}$$

for $\theta < \phi$,

$$W^{c}_{s,\infty}\left\{y^{\lambda} \times H^{U,V}_{X,Y}\left[gy^{\theta}(y-s)^{-\phi}\left|\begin{matrix}(a_k,e_k)_{1,X}\\(b_k,f_k)_{1,Y}\end{matrix}\right.\right]\right\}$$

$$= \frac{1}{\Gamma(c)} s^{\lambda+c} \times H^{U+1,V+1}_{X+2,Y+1}$$

$$\left[gs^{\theta-\phi}\left|\begin{matrix}(1+c+\lambda,\phi-\theta),(a_k,e_k)_{1,X},(-\lambda,\theta)\\(c,\phi).(b_k,f_k)_{1,Y}\end{matrix}\right.\right] \qquad (22)$$

REFERENCES

[1] R. P. Agarwal, *"A propos d'une Note M. Pierre Humbert"*, C. R. Acad. Sci. Paris, vol. 236, pp. 2031-2032, 1953.

[2] B. L. J. Braksma, "*Asymptotic expansions and analytic continuations for a class of Barnes-integral*", Compositio Mathematica, vol. 15, pp. 339-341, 1963.

[3] J. Choi, P. Agarwal, *"A note on fractional integral operator associated with multi index Mittag-Lefflerr function"*, Filomat. Vol. 30(7), pp. 1931-1939, 2016.

[4] A. Faraj, T. Salim, S. Sadek & J. Ismail, *"Generalized Mittag-Leffler function associated with Weyl fractional calculus operators"*, Journal of Mathematics, vol. 2013, http://dx.doi.org/10.1155/2013/821762

[5] C. Fox, "*The G- and H-function as symmetrical Fourier kernels*", Transaction of The American Mathematical Society, vol. 98, pp. 395-429, 1961.

[6] H. Habenom, H. Tadesse & G. V. Reddy, *"Unified finite integral associated with generalized I-function"*, International Journal of Mathematics and its application, vol. 6(2-B), pp. 253-259, 2018. http://ijmaa.in/ .

[7] P. Jain, A. Gupta, V. P. Saxena, *"Multiple integral involving I-function and Bessel-Maitland functions"*, International Journal of Mathematics Trends and Technology, vol. 39 (3), pp. 232-237, 2016. http://www.ijmttjournal.org

[8] M. Kamarujjama, O. Khan, *"Computation of new class of integrals involving generalized galue type struve function"*, Journal of Computational and Applied Mathematics, vol. 351, pp. 228-236, 2019. https://doi.org/10.1016/j.cam.2018.11.014

[9] M. Kamarujjama, N. U. Khan, O. Khan, *"The generalized p-k-Mittag Leffler function and solution of fractional kinetic equations"*, The Journal of Analysis, 2019. http://dx.doi.org/10.1007/s41478-018-0160-z

[10] A. A. Kilbas, *"Fractional calculus of the generalized Wright function"*, Fractional Calculus and Applied Analysis, vol. 8 (2), pp. 113-126, 2005.

[11] A. A. Kilbas, M. Saigo, *"On the H-function"*, Journal of Applied Mathematics and Stochastic Analysis, vol. 12, pp. 191-204, 1999.

[12] A. A. Kilbas, M. Saigo, *"H-transforms: Theory and Applications"*, CRC Press, Boca Raton, FL, 2004.

[13] V. Kiryakova, "*Multiindex (multiindex) Mittag-Leffler functions and relations to generalized fractional calculus*", Journal of Computational and Applied Mathematics, vol. 118, pp. 241-259, 2000.

[14] V. Kiryakova, *"The special functions of fractional calculus as generalized fractional calculus operators of some basic functions"*, Computer & Mathematics with Applications, vol. 59 (3), pp. 128-141, 2010.

[15] V. Kiryakova, Y. Luchko, *"Riemann-Liouville and Caputo type multiple Erdelyi-Kober operators"*, Central European Journal of Physics, vol. 11 (10), pp. 1314-1336, 2013.

[16] C. Lavault. "Fractional Calculus and generalized Mittag-Leffler type functions", 2017. https://hal.archives-ouvertes.fr/hal-01482060v2.

[17] A. M. Mathai, R. K. Saxena, *"The H-function with Applications in Statistics and Other Disciplines"*, Wiley Eastern Ltd., New Delhi, 1978.

[18] K. S. Miller, "*The Weyl fractional calculus, Fractional calculus and its applications, Lecture Notes in Math*", New York: Springer-Verlag, vol. 457, pp 80-89, 1975.

[19] G. M. Mittag-Leffler, *"Sur la representation analytique dune branche uniforme dune fontion monogene"*, Acta. Mathematica, vol. 29, pp. 101-181, 1905.

[20] E. D. Rainville, "*Special functions*", Bronx, New Delhi: Chelsea Publication Company, 1971.

[21] U. K. Saha, L. K. Arora, B. K. Dutta, *"Integrals involving I-function"*, General Mathematics Notes, vol. 6(1), pp. 1-14, 2011.

[22] M. Saigo, R. K. Saxena & J. Ram, "*On the fractional calculus operator associated with the H-function*", GanitaSandesh, vol. 6(1), pp36-47, 1992.

[23] R. K. Saxena, S. L. Kalla, R. Saxena, *"On multivariate analogue of generalized Mittag-Leffler function"*, Integral Transform and Special Functions, vol. 22 (7), pp. 533-548, 2011.

[24] R. K. Saxena, T. F. Nonnenmacher, *"Applications of the H-function in Markovian and non-Markovian chain models"*, Fractional Calculus and Applied Analysis, vol. 7, pp. 135-148, 2004.

[25] V. P. Saxena, *"The I-Function"*, Anamaya Publishers, New Delhi, 2008.

[26] W. R. Schneider, *"Stable distributions: Fox function representation and generalization, in: S. Albeverio, G. Casati, D. Merlini (Eds.), Stochastic Processes in Classical and Quantum Systems"*, Lecture Notes in Physics, Springer, Berlin, vol. 262, pp. 497-511, 1986.

[27] M. K. Shukla, S. Pandey, R. Shrivastava, *"On some integral Relations involving I-functions with application"*, Research Journal of Mathematical and Statistical Sciences, vol. 1 (1), pp. 14-18, 2013. http://www.isca.in

[28] H. M. Srivastava, *"Some families of Mittag-Leffler type functions and associated operators of fractional calculus"*, TWMS Journal of Pure and Applied Mathematics, vol. 7, pp. 123-145, 2016.

[29] H. M. Srivastava, P. Agarwal, *"Certain fractional integral operators and the generalized incomplete hypergeometric functions"*, Applications and Applied Mathematics, vol. 8 (2), pp. 333-345, 2013.

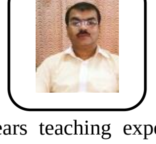
[30] H. M. Srivastava, K. C. Gupta, S. P. Goyal, *"The H-functions of One and Two Variables with Applications"*, South Asian Publishers, New Delhi, 1982.

[31] H. M. Srivastava, Z. Tomovski, *"Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel"*, Applied Mathematics and Computation, vol. 211, pp. 198-210, 2009.

[32] A. Wiman, *"Über den Fundamental satz in der Theorie der Funktionen $E_\alpha(x)$"*, Acta. Math, vol. 29, pp. 191-201, 1905.

[33] A. Wiman, *"Über die Nullstellun der Funktionen $E_\alpha(x)$"*, Acta. Math, vol. 29, pp. 217-234, 1905.

Ms. Cristina Gammeng is pursuing her Ph.D degree in the Department of Basic and Applied Science at National Institute of Technology, Arunachal Pradesh, India. She received her M.Sc degree in Mathematics from Rajiv Gandhi University, Arunachal Pradesh in 2013. She is presently serving as a Lecturer in Govt. Polytechnic College, Arunachal Pradesh. She has published research papers in various Scopus indexed journals.



Dr. U. K. Saha is currently working as an assistant professor in the Department of Basic & Applied Science at National Institute of Technology, Arunachal Pradesh, India. Before joining the institution he has worked as assistant professor in a Govt. College of Assam, India. He has more than 24 years teaching experience and 8 years of research experience. He has published more than 30 papers in different national and international journals of reputed; He is a life member of various professional bodies.

Dr. S. Maity is an Assistant Professor of Mathematics and Head of Department of Basic and Applied Science at National Institute of Technology, Arunachal Pradesh. He received his M.sc and Ph. D degrees in Applied Mathematics from the University of Calcutta in 2004 and 2012 respectively. His research work includes the modeling of thin film, heat and mass transfer, nanofluids, thermocapillary flows. He has published several research articles in many world leading journals.