Resistance Distance and Kirchhoff Index of the Diamond Hierarchical Graph and the Generalized Corona Graph

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***Abstract*—Given simple graphs $G, H_1, \cdots, H_n$, where $n =| V(G) |$, the generalized corona, denoted by $G\tilde{\circ} \wedge_{i=1}^{n} H_i$, is the graph obtained by taking one copy of graphs $G, H_1, \cdots, H_n$ and joining the ith vertex of G to every vertex of H_i. The diamond hierarchical graph S_G is formed by adding two new vertices v_e, w_e for each edge $e = uv$ and then deleting edge e and adding in edges uv_e, uw_e and v_ev, w_ev. In this paper, closed-form formulas for resistance distance and Kirchhoff index of $G\tilde{\circ} \wedge_{i=1}^{n} H_i$ whenever G and H_i are arbitrary graphs are obtained. And the resistance distance and Kirchhoff index of S_G whenever G is an arbitrary graph are obtained.**

***Index Terms*—Kirchhoff index; Resistance distance; Diamond hierarchical graph; Generalized corona graph**

I. Introduction

All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let d_i be the degree of vertex i in G and $D_G = diag(d_1, d_2, \cdots, d_{|V(G)|})$ be the diagonal matrix with all vertex degrees of G as its diagonal entries. For a graph G, let A_G and B_G denote the adjacency matrix and vertex-edge incidence matrix of G, respectively. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of G, where D_G is the diagonal matrix of vertex degrees of G. We use $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$ to denote the eigenvalues of L_G. The Kronecker product of matrices $A = (a_{ij})$ and B, denoted by $A \otimes B$, is defined to be the partition matrix $(a_{ij}B)$.

The resistance distance is a tool motivated by ideas from electrical network theory and applications in chemistry that has proven valuable in the study of graphs. The resistance distance between vertices u and v of G was defined by Klein and Randić [8] to be the effective resistance between nodes u and v as computed with Ohm's law when all the edges of G are considered to be unit resistors. The Kirchhoff index $Kf(G)$ was defined in [8] as $Kf(G) = \sum_{u<v} r_{uv}(G)$, where $r_{uv}(G)$ denotes the resistance distance between u and v in G. Resistance distance are, in fact, intrinsic to the graph, with some nice purely mathematical interpretations and other interpretations. In complex networks, represented by graphs, the effective resistance characterizes the difficulty of transport in a network. As a robustness indicator, the effective resistance allows to compare graphs and is applied in improving the robustness of complex networks, especially against cascading failures in electrical networks. The Kirchhoff index was introduced in chemistry as a better alternative to other parameters used for discriminating different molecules with similar shapes and structures. See [2]. For more information on resistance distance and Kirchhoff index of graphs, the readers are referred to Refs.([8]- [10], [12]- [21]) and the references therein.

The resistance distance and Kirchhoff index of some composite operations between two graphs are studied, such as product, lexicographic product [15], corona [10], subdivision-vertex join and subdivision-edge join [4] and so on. Given a connected graph G with n vertices and m edges, the diamond hierarchical graph S_G [7] is formed by adding two new vertices v_e, w_e for each edge $e = uv$ in G and then deleting edge e and adding in edges uv_e, uw_e and v_ev, w_ev. It is routine to check that the order of S_G is $n + 2m$ and the size of S_G is $4m$. Figure 1 shows an example of the diamond hierarchical graph when G is K_4 by deleting one edge.

In [6], the generalized corona are introduced, and their A-spectrum(resp., L-spectrum) are investigated. Let G and H_i be vertex-disjoint graphs. The generalized corona of G and H_i for $i = 1, 2, ..., n$, denoted by $G\tilde{\circ} \wedge_{i=1}^{n} H_i$, is the graph obtained by taking one copy of graphs $G, H_1, \cdots, H_n$ and joining the ith vertex of G to every vertex of H_i. Figure 2 shows an example of the generalized corona graph $G\tilde{\circ} \wedge_{i=1}^{3} H_i$ when $G = K_3, H_1 = K_2, H_2 = P_2, H_3 = K_3$. Bu et al. [4] investigated resistance distance in subdivision-vertex join and subdivision-edge join of graphs. Liu et al. [9] gave the resistance distance and Kirchhoff index of R-vertex join and R-edge join of two graphs. Liu [10] obtained the resistance distance and Kirchhoff index of corona and edge corona of two graphs. Motivated by these, in this paper we consider the generalized corona to the case of n different graphs and we obtain the resistances distance and Kirchhoff index in terms of the corresponding parameters of the factors. And the resistance distance and Kirchhoff index of the diamond hierarchical graphs S_G whenever G is an arbitrary graph.

II. Preliminaries

The $\{1\}$-inverse of M is a matrix X such that $MXM = M$. If M is singular, then it has infinite $\{1\}$-inverse [1]. For a square matrix M, the group inverse of M, denoted by $M^{\#}$, is the unique matrix X such that $MXM = M$, $XMX = X$ and $MX = XM$. It is known that $M^{\#}$ exists if and only if

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$rank(M) = rank(M^2)$ ([1], [5]). If M is real symmetric, then $M^{\#}$ exists and $M^{\#}$ is a symmetric $\{1\}$-inverse of M. Actually, $M^{\#}$ is equal to the Moore-Penrose inverse of M since M is symmetric [1].

It is known that resistance distance in a connected graph G can be obtained from any $\{1\}$-inverse of G ([2], [3]). We use $M^{(1)}$ to denote any $\{1\}$-inverse of a matrix M, and let $(M)_{uv}$ denote the (u, v)-entry of M.

Lemma 2.1 ([3], [5]) Let G be a connected graph. Then

$$\begin{aligned} r_{uv}(G) &= (L_G^{(1)})_{uu} + (L_G^{(1)})_{vv} - (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} \\ &= (L_G^{\#})_{uu} + (L_G^{\#})_{vv} - 2(L_G^{\#})_{uv}. \end{aligned}$$

Let 1_n denote the column vector of dimension n with all the entries equal one. We will often use 1 to denote an all-ones column vector if the dimension can be read from the context.

Lemma 2.2 [4] For any graph, we have $L_G^{\#}1 = 0$.
For a square matrix M, let $tr(M)$ denote the trace of M.

Lemma 2.3 [11] Let G be a connected graph with n vertices. Then

$$Kf(G) = ntr(L_G^{(1)}) - 1^T L_G^{(1)} 1 = ntr(L_G^{\#}).$$

Lemma 2.4 [11] Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be the Laplacian matrix of a connected graph. If D is nonsingular, then

$$X = \begin{pmatrix} H^{\#} & -H^{\#}BD^{-1} \\ -D^{-1}B^TH^{\#} & D^{-1} + D^{-1}B^TH^{\#}BD^{-1} \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of L, where $H = A - BD^{-1}B^T$.

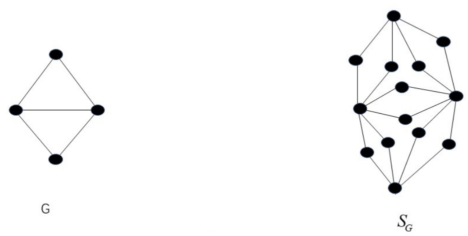

Fig. 1: G and S_G

III. The resistance distance and Kirchhoff index of the diamond hierarchical graphs

In this section, we focus on determing the resistance distance and Kirchhoff index of the diamond hierarchical graphs S_G whenever G is an arbitrary graph. Let $V(S_G) = V \cup V_1 \cup V_2$, where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of all the inherited vertices from G, $V_1 = \{v_{11}, v_{12}, \ldots, v_{1m}\}$ is the set of all

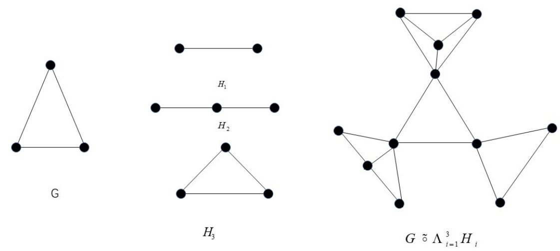

Fig. 2: the generalized corona $G\tilde{\circ} \wedge_{i=1}^{3} H_i$

the subdivision vertices, whereas $V_2 = \{v_{21}, v_{22}, \ldots, v_{2m}\}$ is the set all the rest vertices, each vertex v_{2i} corresponds to edge e_i in $E(G)$.

Theorem 3.1 Let G be a graph with n vertices and m edges and let S_G be the graph obtained from G with $V(S_G) = V \cup V_1 \cup V_2$, where V, V_1 and V_2 are defined as above. Then S_G have the resistance distance and Kirchhoff index as follows:

(i) For any $i, j \in V(G)$, we have

$$r_{ij}(S_G) \;=\; (L_G^{\#})_{ii} + (L_G^{\#})_{jj} - 2(L_G^{\#})_{ij} = r_{ij}(G).$$

(ii) For any $i, j \in V_l (l = 1, 2)$, we have
$r_{ij}(S_G)$

$$\begin{aligned} &= (\frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B)_{ii} + (\frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B)_{jj} \\ &\quad -2(\frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B)_{ij}. \end{aligned}$$

(iii) For any $i \in V(G)$, $j \in V_l (l = 1, 2)$, we have
$r_{ij}(S_G)$

$$= (L_G^{\#})_{ii} + (\frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B)_{jj} - 2(L_G^{\#}B)_{ij}.$$

(iv) The Kirchhoff index of S_G is

$$\begin{aligned} Kf(S_G) &= (n + 2m)\left(\frac{1}{n}Kf(G) + tr(D_GL_G^{\#})\right) \\ &\quad -\pi^TL_G^{\#}\pi - \frac{4m^2 - n^2 + n}{2}, \end{aligned}$$

where $\pi^T = (d_1, d_2, ..., d_n)$.

Proof Let D_G and B be the diagonal matrix and the incidence matrices of G, respectively. With a suitable labeling for vertices of S_G, the Laplacian matrix of S_G can be written as follows:

$$L(S_G) = \begin{pmatrix} 2D_G & -B & -B \\ -B^T & 2I_m & 0 \\ -B^T & 0 & 2I_m \end{pmatrix}.$$

By Lemma 2.4, we have

$$\begin{aligned} H &= 2D_G - \begin{pmatrix} -B & -B \end{pmatrix} \begin{pmatrix} \frac{1}{2}I_m & 0 \\ 0 & \frac{1}{2}I_m \end{pmatrix} \\ &\quad \begin{pmatrix} -B^T \\ -B^T \end{pmatrix} \\ &= 2D_G - (\tfrac{1}{2}BB^T + \tfrac{1}{2}BB^T) \\ &= 2D_G - BB^T \\ &= 2D_G - (D_G + A_G) = L_G. \end{aligned}$$

So $H^{\#} = L_G^{\#}$.
According to Lemma 2.4, we calculate $-H^{\#}BD^{-1}$ and $-D^{-1}B^TH^{\#}$.

$$\begin{aligned}
-H^{\#}BD^{-1} &= -L_G^{\#}\begin{pmatrix} -B & -B \end{pmatrix}\begin{pmatrix} \frac{1}{2}I_m & 0 \\ 0 & \frac{1}{2}I_m \end{pmatrix} \\
&= -L_G^{\#}\begin{pmatrix} -\frac{1}{2}B & -\frac{1}{2}B \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}L_G^{\#}B & \frac{1}{2}L_G^{\#}B \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
-D^{-1}B^TH^{\#} &= -\begin{pmatrix} \frac{1}{2}I_m & 0 \\ 0 & \frac{1}{2}I_m \end{pmatrix}\begin{pmatrix} -B^T \\ -B^T \end{pmatrix}L_G^{\#} \\
&= \begin{pmatrix} \frac{1}{2}B^T \\ \frac{1}{2}B^T \end{pmatrix}L_G^{\#} = \begin{pmatrix} \frac{1}{2}B^TL_G^{\#} \\ \frac{1}{2}B^TL_G^{\#} \end{pmatrix}.
\end{aligned}$$

We are ready to compute the $D^{-1}B^TH^{\#}BD^{-1}$.
$D^{-1}B^TH^{\#}BD^{-1}$

$$\begin{aligned}
&= -\begin{pmatrix} \frac{1}{2}B^TL_G^{\#} \\ \frac{1}{2}B^TL_G^{\#} \end{pmatrix}\begin{pmatrix} -B & -B \end{pmatrix} \\
&\quad \begin{pmatrix} \frac{1}{2}I_m & 0 \\ 0 & \frac{1}{2}I_m \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{4}B^TL_G^{\#}B & \frac{1}{4}B^TL_G^{\#}B \\ \frac{1}{4}B^TL_G^{\#}B & \frac{1}{4}B^TL_G^{\#}B \end{pmatrix}.
\end{aligned}$$

Based on Lemma 2.4, the following matrix

$$N = \begin{pmatrix} L_G^{\#} & \frac{1}{2}L_G^{\#}B & \frac{1}{2}L_G^{\#}B \\ \frac{1}{2}B^TL_G^{\#} & \frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B & \frac{1}{4}B^TL_G^{\#}B \\ \frac{1}{2}B^TL_G^{\#} & \frac{1}{4}B^TL_G^{\#}B & \frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B \end{pmatrix} \qquad (1)$$

is a symmetric $\{1\}$-inverse of L_{S_G}.

(i) For any $i, j \in V(G)$, by Lemma 2.1 and Equation (1), we have

$$r_{ij}(S_G) = (L_G^{\#})_{ii} + (L_G^{\#})_{jj} - 2(L_G^{\#})_{ij} = r_{ij}(G),$$

as stated in (i).

(ii) For any $i, j \in V_l (l = 1, 2)$, by Lemma 2.1 and Equation (1), we have

$$\begin{aligned}
r_{ij}(S_G) &= ((L_{H_i} + I_{n_i})^{-1} \otimes I_{n_i})_{ii} + ((L_{H_i} + I_{n_i})^{-1} \\
&\quad \otimes I_{n_i})_{jj} - 2((L_{H_i} + I_{n_i})^{-1} \otimes I_{n_i})_{ij},
\end{aligned}$$

as stated in (ii).
(iii) For any $i \in V(G)$, $j \in V(H_l)$ $(l = 1, 2)$, by Lemma 2.1 and Equation (1), we have

$$\begin{aligned}
r_{ij}(S_G) &= (L_G^{\#})_{ii} + ((L_{H_i} + I_{n_i})^{-1} \otimes I_{n_i})_{jj} \\
&\quad -2(L_G^{\#}B)_{ij},
\end{aligned}$$

as stated in (iii).
Next we compute the Kirchhoff index of S_G.
By Lemma 2.3, we have

$Kf(S_G)$

$$\begin{aligned}
&= (n + 2m)tr(N) - 1^TN1 \\
&= (n + 2m)\Big(tr(L_G^{\#}) + tr(\frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B) \\
&\quad + tr(\frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B)\Big) - 1^TN1 \\
&= (n + 2m)\left(\frac{1}{n}Kf(G) + m + \frac{1}{2}tr(B^TL_G^{\#}B)\right) \\
&\quad -1^TN1 \\
&= (n + 2m)\Big(\frac{1}{n}Kf(G) + m + \frac{1}{2}\sum_{i<j, \{i,j\} \in E(G)} \\
&\quad [(L_G^{\#})_{ii} + (L_G^{\#})_{jj} + 2(L_G^{\#})_{ij}]\Big) - 1^TN1 \\
&= (n + 2m)\Big(\frac{1}{n}Kf(G) + m + \frac{1}{2}\sum_{i<j, \{i,j\} \in E(G)} \\
&\quad [(2L_G^{\#})_{ii} + (2L_G^{\#})_{jj} - r_{ij}(G)]\Big) - 1^TN1 \\
&= (n + 2m)\Big(\frac{1}{n}Kf(G) + m + tr(D_GL_G^{\#}) \\
&\quad -\frac{n-1}{2}\Big) - 1^TN1.
\end{aligned}$$

By Lemma 2.2, $L_G^{\#}1 = 0$, then

$$\begin{aligned}
1^TN1 &= 2 \times 1^T(\frac{1}{2}I_m + \frac{1}{4}B^TL_G^{\#}B)1 + \\
&\quad \frac{1}{2} \times 1^T(B^TL_G^{\#}B)1.
\end{aligned}$$

Note that $B\mathbf{1} = \pi$, where $\pi^T = (d_1, d_2, ..., d_n)$, then

$$1^TN1 = m + \pi^TL_G^{\#}\pi.$$

Plugging the above equation into $Kf(S_G)$, we obtain the required result in (iv).

IV. The resistance distance and Kirchhoff index of $G\tilde{\circ} \wedge_{i=1}^n H_i$

In this section, we focus on determing the resistance distance and Kirchhoff index of the generalized corona $G\tilde{\circ} \wedge_{i=1}^n H_i$ whenever G and $H_i (i = 1, 2, ..., n)$ are arbitrary graphs.

Theorem 3.1 Let G be a graph with n vertices and m edges. Let H_i be a graph with t_i vertices for $i = 1, 2, ..., n$. Then $G\tilde{\circ} \wedge_{i=1}^n H_i$ have the resistance distance and Kirchhoff index as follows:
(i) For any $i, j \in V(G)$, we have

$$\begin{aligned}
r_{ij}(G\tilde{\circ} \wedge_{i=1}^n H_i) &= (L_G^{\#})_{ii} + (L_G^{\#})_{jj} - 2(L_G^{\#})_{ij} \\
&= r_{ij}(G).
\end{aligned}$$

(ii) For any $i, j \in V(H_k) (k = 1, 2, \ldots, n)$, we have
$r_{ij}(G\tilde{\circ} \wedge_{i=1}^n H_i)$

$$\begin{aligned}
&= ((L_{H_i} + I_{n_i})^{-1} \otimes I_{n_i})_{ii} + ((L_{H_k} + I_{n_j})^{-1} \otimes I_{n_i})_{jj} \\
&\quad -2((L_{H_k} + I_{n_i})^{-1} \otimes I_{n_i})_{ij}.
\end{aligned}$$

(iii) For any $i \in V(G)$, $j \in V(H_k) (k = 1, 2, \ldots, n)$, we have
$r_{ij}(G\tilde{\circ} \wedge_{i=1}^n H_i)$

$$= (L_G^{\#})_{ii} + ((L_{H_k} + I_{n_i})^{-1} \otimes I_{n_i})_{jj} - 2(L_G^{\#})_{ij}.$$

(iv) The Kirchhoff index of $G\tilde{\circ}\wedge_{i=1}^{n} H_i$ is
$Kf(G\tilde{\circ}\wedge_{i=1}^{n} H_i)$

$$\begin{aligned} &= (n+\sum_{i=1}^{n} t_i)\left(\frac{1}{n}Kf(G)+n\sum_{i=1}^{n}\sum_{t_i}^{j=1}\frac{1}{\mu_j(H_i)+1}\right.\\ &\left. +tr(C^T L_G^{\#} C)\right)-\sum_{i=1}^{n} t_i-\delta^T L_G^{\#}\delta, \end{aligned}$$

where $\mu_j(H_i)(j=1,2,\ldots,t_i)$ denote the Laplacian eigenvalues of H_i, C equals (1), $\delta^T=(t_1,t_2,...,t_n)$.

Proof Let A and B_i be the adjacency matrices of G and H_i, respectively, for $i=1,2,\cdots,n$. Let $V=diag(t_1,t_2,\cdots,t_n)$. With a suitable labeling for vertices of $G\tilde{\circ}\wedge_{i=1}^{n} H_i$, the Laplacian matrix of $G\tilde{\circ}\wedge_{i=1}^{n} H_i$ can be written as follows:

$$L_{G\tilde{\circ}\wedge_{i=1}^{n} H_i}=\begin{pmatrix} V+D_G-A & -C\\ -C^T & \Delta+I-Q\end{pmatrix},$$

where $\Delta=diag(D(H_1),D(H_2),\cdots,D(H_n))$ and

$$Q=\begin{pmatrix} B_1 & 0 & 0 & ... & 0\\ 0 & B_2 & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & B_n\end{pmatrix},$$

$$C=\begin{pmatrix} 1_{t_1}^T & 0 & 0 & ... & 0\\ 0 & 1_{t_2}^T & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & 1_{t_n}^T\end{pmatrix}. \tag{2}$$

Let $T_i=L_{H_i}+I_{t_i}(i=1,2,\ldots,n)$, then

$$T=\Delta+I-Q=\begin{pmatrix} T_1 & 0 & 0 & ... & 0\\ 0 & T_2 & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & T_n\end{pmatrix}.$$

First we begin with the computation of $\{1\}$-inverse of $L_{G\tilde{\circ}\wedge_{i=1}^{n} H_i}$. By Lemma 2.4, we have

$$\begin{aligned} H &= V+D_G-A-CT^{-1}C^T\\ &= \begin{pmatrix} t_1+d_1 & 0 & ... & 0\\ 0 & t_2+d_2 & ... & 0\\ 0 & 0 & ... & 0\\ 0 & 0 & ... & t_n+d_n\end{pmatrix}-A\\ &\quad -C\begin{pmatrix} T_1^{-1} & 0 & 0 & ... & 0\\ 0 & T_2^{-1} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & T_n^{-1}\end{pmatrix}C^T\\ &= \begin{pmatrix} t_1+d_1 & 0 & ... & 0\\ 0 & t_2+d_2 & ... & 0\\ 0 & 0 & ... & 0\\ 0 & 0 & ... & t_n+d_n\end{pmatrix}-A\\ &\quad -\begin{pmatrix} 1_{t_1}^T T_1^{-1}1_{t_1} & 0 & \cdots & 0\\ 0 & 1_{t_2}^T T_2^{-1}1_{t_2} & \cdots & 0\\ 0 & 0 & & 0\\ 0 & 0 & \cdots & 1_{t_n}^T T_n^{-1}1_{t_n}\end{pmatrix}\\ &= D_G-A=L_G. \end{aligned}$$

So $H^{\#}=L_G^{\#}$.
According to Lemma 2.4, we calculate $-H^{\#}BD^{-1}$ and $-D^{-1}B^TH^{\#}$.

$-H^{\#}BD^{-1}$

$$\begin{aligned} &= -L_G^{\#}\begin{pmatrix} -1_{t_1}^T & 0 & 0 & ... & 0\\ 0 & -1_{t_2}^T & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & -1_{t_n}^T\end{pmatrix}\\ &\quad \begin{pmatrix} T_1^{-1} & 0 & 0 & ... & 0\\ 0 & T_2^{-1} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & T_n^{-1}\end{pmatrix}\\ &= -L_G^{\#}\begin{pmatrix} -1_{t_1}^T & 0 & 0 & ... & 0\\ 0 & -1_{t_2}^T & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & -1_{t_n}^T\end{pmatrix}\\ &= L_G^{\#}C \end{aligned}$$

and
$-D^{-1}B^TH^{\#}=-(H^{\#}BD^{-1})^T=C^TL_G^{\#}$.
We are ready to compute the $D^{-1}B^TH^{\#}BD^{-1}$.
$D^{-1}B^TH^{\#}BD^{-1}$

$$\begin{aligned} &= \begin{pmatrix} T_1^{-1}1_{t_1} & 0 & 0 & ... & 0\\ 0 & T_2^{-1}1_{t_2} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & T_n^{-1}1_{t_n}\end{pmatrix}\\ &\quad L_G^{\#}\begin{pmatrix} 1_{t_1}^T & 0 & 0 & ... & 0\\ 0 & 1_{t_2}^T & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & 1_{t_n}^T\end{pmatrix}\\ &\quad \begin{pmatrix} T_1^{-1} & 0 & 0 & ... & 0\\ 0 & T_2^{-1} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & T_n^{-1}\end{pmatrix}\\ &= \begin{pmatrix} 1_{t_1} & 0 & 0 & ... & 0\\ 0 & 1_{t_2} & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & 1_{t_n}\end{pmatrix}L_G^{\#}\\ &\quad \begin{pmatrix} 1_{t_1}^T & 0 & 0 & ... & 0\\ 0 & 1_{t_2}^T & 0 & ... & 0\\ 0 & 0 & ... & ... & 0\\ 0 & 0 & 0 & ... & 1_{t_n}^T\end{pmatrix}\\ &= C^TL_G^{\#}C. \end{aligned}$$

Based on Lemma 2.4, the following matrix

$$N=\begin{pmatrix} L_G^{\#} & L_G^{\#}C\\ C^TL_G^{\#} & T^{-1}+C^TL_G^{\#}C\end{pmatrix} \tag{3}$$

is a symmetric $\{1\}$-inverse of $L_{G\tilde{\circ}\wedge_{i=1}^{n} H_i}$.

(i) For any $i,j\in V(G)$, by Lemma 2.1 and Equation (2), we have

$$\begin{aligned} r_{ij}(G\tilde{\circ}\wedge_{i=1}^{n} H_i) &= (L_G^{\#})_{ii}+(L_G^{\#})_{jj}-2(L_G^{\#})_{ij}\\ &= r_{ij}(G), \end{aligned}$$

as stated in (i).

(ii) For any $i,j\in V(H_k)(k=1,2,\ldots,n)$, by Lemma 2.1 and Equation (2), we have
$r_{ij}(G\tilde{\circ}\wedge_{i=1}^{n} H_i)$

$$\begin{aligned} &= ((L_{H_i}+I_{n_i})^{-1}\otimes I_{n_i})_{ii}+((L_{H_j}+I_{n_j})^{-1}\otimes I_{n_i})_{jj}\\ &\quad -2((L_{H_i}+I_{n_i})^{-1}\otimes I_{n_i})_{ij}, \end{aligned}$$

as stated in (ii).

(iii) For any $i \in V(G)$, $j \in V(H_k)$ $(k = 1, 2, \ldots, n)$, by Lemma 2.1 and Equation (2), we have

$$r_{ij}(G\tilde{\circ} \wedge_{i=1}^{n} H_i) = (L_G^{\#})_{ii} + ((L_{H_k} + I_{n_j})^{-1} \otimes I_{n_i})_{jj} - 2(L_G^{\#})_{ij},$$

as stated in (iii).

Next we compute the Kirchhoff index of $G\tilde{\circ} \wedge_{i=1}^{n} H_i$.

By Lemma 2.3, we have

$Kf(G\tilde{\circ} \wedge_{i=1}^{n} H_i)$

$$\begin{aligned} &= (n + \sum_{i=1}^{n} t_i) tr(N) - 1^T N 1 \\ &= (n + \sum_{i=1}^{n} t_i) \Big(tr(L_G^{\#}) + tr(T^{-1}) \\ &\quad + tr(C^T L_G^{\#} C) \Big) - 1^T N 1 \\ &= (n + \sum_{i=1}^{n} t_i) \left(\frac{1}{n} Kf(G) + \sum_{i=1}^{n} tr(L_{H_i} + I_{t_i})^{-1} \right. \\ &\quad \left. + tr(C^T L_G^{\#} C) \right) - 1^T N 1. \end{aligned}$$

Note that the eigenvalues of $(L_{H_i} + I_{t_i})$ $(i = 1, 2, ..., n)$ are $\mu_1(H_i) + 1, \mu_2(H_i) + 1, ..., \mu_{t_i}(H_i) + 1$, then

$$tr(T^{-1}) = \sum_{i=1}^{n} \sum_{j=1}^{t_i} \frac{1}{\mu_j(H_i) + 1}. \tag{4}$$

By Lemma 2.2, $L_G^{\#} 1 = 0$, then

$$1^T N 1 = 1^T T^{-1} 1 + 1^T C^T L_G^{\#} C 1.$$

and

$1^T T^{-1} 1$

$$\begin{aligned} &= \begin{pmatrix} 1_{t_1}^T & 1_{t_2}^T & \cdots & 1_{t_n}^T \end{pmatrix} \\ &\quad \begin{pmatrix} T_1^{-1} & 0 & 0 & ... & 0 \\ 0 & T_2^{-1} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & T_n^{-1} \end{pmatrix} \begin{pmatrix} 1_{t_1} \\ 1_{t_2} \\ \cdots \\ 1_{t_n} \end{pmatrix} \\ &= \sum_{i=1}^{n} 1_{t_i}^T (L_{H_i} + I_{t_i})^{-1} 1_{t_i} = \sum_{i=1}^{n} t_i \end{aligned} \tag{5}$$

and

$$\begin{aligned} 1^T C^T &= \begin{pmatrix} 1_{t_1}^T & 1_{t_2}^T & \cdots & 1_{t_n}^T \end{pmatrix} \\ &\quad \begin{pmatrix} 1_{t_1} & 0 & 0 & ... & 0 \\ 0 & 1_{t_2} & 0 & ... & 0 \\ 0 & 0 & ... & ... & 0 \\ 0 & 0 & 0 & ... & 1_{t_n} \end{pmatrix} \\ &= (t_1, t_2, ..., t_n) = \delta^T. \end{aligned} \tag{6}$$

Plugging $(4), (5)$ and (6) into $Kf(G\tilde{\circ} \wedge_{i=1}^{n} H_i)$, we obtain the required result in (iv).

V. Conclusion

In this paper, we obtain the closed-form formulas for resistance distance and Kirchhoff index of the diamond hierarchical graphs when G is an arbitrary graph. And the resistance distance and Kirchhoff index of the generalized corona $G\tilde{\circ} \wedge_{i=1}^{n} H_i$ whenever G and H_i are arbitrary graphs are given. The resistance distance and Kirchhoff index of the corona of G_1 and G_2 have already obtained in Ref. [10]. It is easily known that the result of Theorem 3.1 generalize the result of Theorem 1 in [10].

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