Existence of Positive Solutions for a State-dependent Hybrid Functional Differential Equation

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Abstract—In this paper, we study the existence of positive solutions for an initial value problem (IVP) of a state-dependent hybrid functional differential equation. The continuous dependence of the unique solution will be proved. Some special cases and examples will be given.

Index Terms—Hybrid differential equations, state-dependent, quadratic integral equation, existence of positive solutions, continuous dependence, fixed point theorem.

I. INTRODUCTION

In the last years, quadratic perturbations of nonlinear equations have received a lot of attention from specialists. This type of equations are called hybrid equations. Many specialists work on the theory of hybrid equations, see for example [9], [22], [30].

Usually, the equations of deviating arguments are with deviation depends only on the time, however, when the deviation of the arguments depend upon both the state variable ξ and also the time t is incredibly important theoretically and practically. This type of equations is named self-reference or state-dependent equations.

Equations which have state-dependent delays attract the attention of specialists since it have plenty of application models, like the two-body problem of classical electrodynamics, also have many applications in the class of problems that have past memories, for example in hereditary phenomena, see [15-18]. Several papers have devoted to this category of equations, see for instance [1], [7]-[16], [19], [20] and [21]-[29].

One of the first studies in this field was introduced by Eder [11], the author studied the problem

\[ \xi'(t) = \xi(\xi(t)) \quad t \in A \subset R. \]

with the condition \( \xi(t_0) = \xi_0. \)

A generalization of Eder’s results was introduced by Fečkan [19], the author worked on the problem

\[ \xi'(t) = \psi(\xi(\xi(t))), \quad t \in A \subset R \]

where \( \psi \in C^1(R). \)

The problem

\[ \xi'(t) = \psi(t, \xi(\xi(t))), \quad t \in [a, b] \]
\[ \xi(0) = \xi_0 \]

was studied by Buică [8] where \( \psi \) satisfied Lipshitz condition.

EL-Sayed and Ebead [16] relaxed the assumptions of Buică and generalized their results, they studied the delay-refereed problem

\[ \xi(t) = \psi(t, \int_0^t \eta(s, \xi(\xi(s))) \, ds), \quad t \in [0, T] \]

where \( \eta \) satisfies Carathéodory condition.

El-Sayed and Ebead [14] and [15] studied the delay-refereed problem

\[ \frac{d}{dt} \xi(t) = \psi(t, \xi(t)), \quad a.e. \ t \in [0, T], \]
\[ \xi(0) = \xi_0 \]

under the two cases:

(i) \( \eta(t, \xi) \leq t. \)

(ii) \( \eta(t, \xi) \leq \xi. \)

Here we shall study the IVP of a state-dependent hybrid functional differential equation

\[ \frac{d}{dt} \left( \xi(t) - \lambda(t) \right) = \psi(t, \xi(\xi(t))), \quad a.e. \ t \in [0, T] \]
\[ \xi(0) = A(0). \]

Our aim in this work is to prove the existence of positive solution of (1)-(2). The continuous dependence of the unique solution on the functions \( \lambda, \psi \) will be proved. To illustrate our results some examples will be given.

This paper consists of six sections. Section 1 is the introduction and a brief survey of the topic. In Section 2, we introduce the main theorem (existence). We prove the uniqueness theorem in Section 3. Continuous dependence of the solution has been proved in Section 4. Some examples was introduced by Section 5. Finally, in Section 6; as an application to our work, we give a numerical example, we use successive approximation method to give an estimation for the solution of a problem of this type.

II. EXISTENCE OF SOLUTIONS

First, We provide proof of the existence of positive solutions \( \xi \in C[0, T] \) for the state-dependent equation

\[ \xi(t) = \lambda(t) + \eta(t, \xi(\xi(t))) \int_0^t \psi(s, \xi(\xi(s))) \, ds, \quad t \in [0, T], \]

under the following assumptions:
Let the assumptions (1) imply that
\[ \rho(t) = 0, \quad \forall t \in [0, T]. \]
This assumption implies that \( \theta(t) \leq t \).

There exists a solution \( L \in (0, 1) \) for the equation
\[ k_2 M_1 T L^2 - L + (c + M_1 M_2 + k_1 M_2) T = 0, \]
where \( M_1 = A_1 + k_3 T, \) \( M_2 = A_2 + k_2 T. \)

(7) \( L T + |\xi(0)| \leq T. \)

**Theorem 2.1.** Let the assumptions (1) – (7) be satisfied, then the state-dependent equation (3) has a positive solution \( \xi \in C[0, T]. \)

**Proof.** Define the set \( S_L \) by
\[ S_L = \{ \xi \in C[0, T] : |\xi(t_2) - \xi(t_1)| \leq L |t_2 - t_1| \} \subset C[0, T]. \]

Now define the operator \( H \) associated with (3) by:
\[ H \xi(t) = \lambda(t) + \eta(t, \xi(\theta(t))) \int_0^t \psi(s, \xi(\theta(s))) ds, \quad t \in [0, T]. \]

First, we show that \( H \xi \) is uniformly bounded on \( S_L. \) Let \( \xi \in C[0, T], \) then
\[ |H \xi(t)| \leq |\lambda(t)| + |\eta(t, \xi(\theta(t)))| \int_0^t |\psi(s, \xi(\theta(s)))| ds. \]

Using Assumptions (2) we can get
\[ |\eta(t_2, \xi(\theta(t_2))) - \eta(t_1, \xi(\theta(t_1)))| \leq k_1 |t_2 - t_1| + k_2 |\xi(\theta(t_2)) - \xi(\theta(t_1))| \]
\[ \leq k_1 |t_2 - t_1| + k_2 L |\theta(t_2) - \theta(t_1)| \]
\[ \leq k_1 |t_2 - t_1| + k_2 L^2 |t_2 - t_1|. \]

Hence
\[ |\lambda(t_2) - \lambda(t_1)| \leq c |t_2 - t_1| < \delta, \]
then
\[ |H \xi(t_2) - H \xi(t_1)| \leq |\lambda(t_2) - \lambda(t_1)| + |\eta(t_2, \xi(\theta(t_2))) - \eta(t_1, \xi(\theta(t_1)))| \int_0^t |\psi(s, \xi(\theta(s)))| ds. \]

This proves that \( H \xi \) is uniformly bounded on \( S_L. \)

Next, we show that \( H : S_L \rightarrow S_L \) and \( \{ H \xi \} \) is equi-
continuous on the set \( S_L. \)

Let \( x \in S_L \) and \( t_1, t_2 \in [0, T] \) with \( |t_2 - t_1| < \delta, \) then
\[ |H \xi(t_2) - H \xi(t_1)| \leq |\lambda(t_2) - \lambda(t_1)| + |\eta(t_2, \xi(\theta(t_2))) - \eta(t_1, \xi(\theta(t_1)))| \int_0^t |\psi(s, \xi(\theta(s)))| ds. \]
\[ |\eta(t_2, \xi(\varphi(t_2))) - \eta(t_1, \xi(\varphi(t_1)))| = \int_{t_1}^{t_2} |\psi(s, \xi(\varphi(s)))| \, ds. \]

From (4)-(8) we can get
\[ |H \eta(t_2) - H \eta(t_1)| \leq c(t_2 - t_1) + (k_2 \xi(\varphi(t_2))) + A_2 \]
\[ \int_{t_1}^{t_2} (k_3 \xi(\varphi(s))) + A_1 \, ds \]
\[ + (k_1 + k_2 L^2) |t_2 - t_1| + \int_{t_1}^{t_2} (k_3 \xi(\varphi(s))) + A_1 \, ds \]
\[ \leq c(t_2 - t_1) + (k_2 T + A_2) (k_3 T + A_1) |t_2 - t_1| \]
\[ + (k_1 + k_2 L^2) (k_3 T + A_1) |t_2 - t_1| \]
\[ \leq c(t_2 - t_1) + M_2 M_1 |t_2 - t_1| \]
\[ + (k_1 + k_2 L^2) M_1 T |t_2 - t_1| \]
\[ = (c + M_2 M_1 + (k_1 + k_2 L^2) M_1 T) |t_2 - t_1| \]
\[ = L |t_2 - t_1|. \]

Then we proved that \( H : S_L \rightarrow S_L \) and \( \{ H \xi \} \) is equi-continuous on \( S_L \). Applying Arzela-Ascoli Theorem ([24] page (54)), we deduce that \( H \) is compact.

Next, we show that \( H \) is continuous. Let \( \{ \eta_n \} \subset S_L \) be such that \( x_n \rightarrow x \) (i.e., \( \| \eta_n(x_n) - \eta(t) \| \leq \epsilon \) ) this implies that \( |\eta_n(\xi(\varphi(t))) - \eta(\xi(\varphi(t)))| \leq \epsilon, \) for any \( \epsilon > 0, \) we have
\[ |\eta_n(\xi(\varphi(t))) - \eta(\xi(\varphi(t)))| \leq \epsilon \]
\[ \eta_n(\xi(\varphi(t))) \rightarrow \eta(\xi(\varphi(t))) \text{ in } S_L. \]

Now from assumptions (2-4) and Lebesgues dominated convergence Theorem ([10] page(151)), we get
\[ \lim_{n \rightarrow \infty} \eta(t, \eta_n(\xi(\varphi(t)))) = \eta(t, \xi(\varphi(t))) \]
and
\[ \lim_{n \rightarrow \infty} \int_0^t \psi(s, \eta_n(\xi(\varphi(s)))) \, ds = \int_0^t \psi(s, \xi(\varphi(s))) \, ds, \]

hence
\[ \lim_{n \rightarrow \infty} (H \eta_n)(t) = \lambda(t) + \lim_{n \rightarrow \infty} \eta(t, \eta_n(\xi(\varphi(t)))) \]
\[ \lim_{n \rightarrow \infty} \int_0^t \psi(s, \eta_n(\xi(\varphi(s)))) \, ds \]
\[ = \lambda(t) + \eta(t, \xi(\varphi(t))) \int_0^t \psi(s, \xi(\varphi(s))) \, ds \]
\[ = (H \xi)(t). \]

This proves that \( H \) is continuous. Now according to ([23] page (482)), there exist solutions \( \xi \in C[0, T] \) of (3).

Also, from our assumptions, we can get
\[ \xi(t) = \lambda(t) + \eta(t, \xi(\varphi(t))) \int_0^t \psi(s, \xi(\varphi(s))) \, ds \geq 0, \]
\[ t \in [0, T]. \]

**Theorem 2.2.** Let the assumptions (1) – (7) be satisfied, then (1)-(2) has positive solutions \( \xi \in C[0, T]. \)

**Proof.** Let \( \xi \) be a solution of (1)-(2). Integrate (1), then substitute by (2), we get (3). Let \( \xi \) be a solution of (3). Differentiate (3) we obtain (1) and the initial value (2). This proves the equivalence between (1)-(2) and (3). Then (1)-(2) has positive solutions \( \xi \in C[0, T]. \)

The next corollary relax the assumptions and generalize the results in [8] and [19].

**Corollary 2.1.** Using to Theorem 2.1 with \( \eta(t, \xi) = 1, \) then the state-dependent equation
\[ \xi(t) = \lambda(t) + \int_0^t \psi(s, \xi(\varphi(s))) \, ds, \quad t \in [0, T] \]
has solution \( \xi \in C[0, T]. \) Consequently if \( \lambda(t) = \xi_0, \) then
\[ \xi(t)' = \psi(t, \xi(\varphi(t))) \text{ a.e. } t \in (0, T] \]
\[ \xi(0) = \xi_0 \]
has positive solutions \( \xi \in C[0, T]. \)

**III. UNIQUENESS OF THE SOLUTION**

Here we introduce the uniqueness theorem of (1)-(2). For this aim, we assume that:
\[(1') |\psi(t, \xi) - \psi(t, y)| \leq k_3 \| \xi - y \| \]
\[(2') \sup \psi(t, 0) \leq A_1, \quad t \in [0, T]. \]

**Theorem 3.3.** Let the assumptions (1)-(3), (5)-(7), (1') and (2') be satisfied, if
\[ (M_2 k_3 + M_1 k_2) (L + 1) T < 1, \]
then The solution of (1) and (2) is unique.

**Proof.** From assumptions (1') and (2') we can deduce assumption (4)
\[ |\psi(t, \xi)| \leq k_3 \| \xi \| + |\psi(t, 0)| \leq k_3 \| \xi \| + A_1, \]
thus using Theorem 2.1, (3) has at least a solution. Now, if 
(3) has two solutions $\xi$ and $y$, then
\[
|\xi(t) - y(t)| = |\eta(t, \xi(\vartheta(t)))| \int_{t}^{t'} \psi(s, \xi(\vartheta(s)))ds \\
- |\eta(t, y(\vartheta(t)))| \int_{t}^{t'} \psi(s, y(\vartheta(s)))ds | \leq |\eta(t, \xi(\vartheta(t)))| \\

||\xi(t) - y(t)|| = ||\eta(t, \xi(\vartheta(t)))|| \int_{t}^{t'} \psi(s, \xi(\vartheta(s)))ds \\
- |\eta(t, y(\vartheta(t)))| \int_{t}^{t'} \psi(s, y(\vartheta(s)))ds | \leq |\eta(t, \xi(\vartheta(t)))| \\

\leq |\eta(t, \xi(\vartheta(t)))| - |\eta(t, y(\vartheta(t)))| \\
+ |\xi(t) - y(t)|| \int_{t}^{t'} \psi(s, \xi(\vartheta(s)))ds \\
+ |\eta(t, \xi(\vartheta(t)))| - |\eta(t, y(\vartheta(t)))| \\
\leq L||\xi(t) - y(t)|| \\
+ |\xi(t) - y(t)|| \int_{t}^{t'} \psi(s, \xi(\vartheta(s)))ds \\
\leq L||\xi - y|| + ||\xi - y|| \\
= (L + 1)||\xi - y||,
\]
thus we have
\[
||\xi - y|| \leq (M_{2} k_{3} + M_{1} k_{2})(L + 1)||\xi - y|| \\
\]
and
\[
(1 - (M_{2} k_{3} + M_{1} k_{2})(L + 1)T)||\xi - y|| \leq 0,
\]
since $(M_{2} k_{3} + M_{1} k_{2})(L + 1)T < 1$, then we get $\xi = y$ and the solution of (3) is unique. Consequently the solution of (1)-(2) is unique.

**Corollary 3.2.** According to Theorem 3.3, let $\eta(t, \xi) = 1$ and $\lambda(t) = \xi_0$, then (9) and (10) has a unique positive solution $\xi \in C(0, T)$.
Now, as in [17] and [18], we introduce the next theorem.

**Theorem 3.4.** According to Theorem 3.3, we can approximate the numerical solution of (3) by
\[
\xi(t) = \lim_{n \to \infty} \xi_n(t)
\]
where $\xi_n(t)$ is constructed by
\[
\xi_n(t) = \lambda(t) + \eta(t, \xi_{n-1}(\vartheta(t))) \\
\int_{0}^{t} \psi(s, \xi_{n-1}(\vartheta(s)))ds,
\]
where $\xi_0(t) = \lambda(t)$.

**IV. CONTINUOUS DEPENDENCE**

A. Continuous dependence on the function $\lambda$

**Definition 4.1.** The solution of (1) and (2) continuously depends on $\lambda$ if $\forall \epsilon_1 > 0 \exists \delta_1(\epsilon_1) > 0$ such that:
\[
||\lambda - \lambda^*|| \leq \delta_1 \implies ||\xi - \xi^*|| \leq \epsilon_1
\]
where
\[
\frac{d}{dt} \left\{ \xi^*(t) - \lambda^*(t) \right\} = \psi(s, \xi^*(\xi^*(s))), \quad a.e. \quad t \in [0, T]
\]
with the initial data
\[
\xi^*(0) = \lambda^*(0).
\]

**Theorem 4.5.** According to Theorem 3.3, the solution of (1) and (2) continuously depends on $\lambda$.

**Proof.** Assume that the functions $\xi$ and $\xi^*$ are the two unique solutions of (1)-(2) and (11)-(12) respectively, thus we have
\[
||\xi(t) - \xi^*(t)|| = ||\lambda(t) + \eta(t, \xi(\vartheta(t))) \int_{0}^{t} \psi(s, \xi(\vartheta(s)))ds \\
- \lambda^*(t) - \eta(t, \xi^*(\vartheta(t))) \int_{0}^{t} \psi(s, \xi^*(\vartheta(s)))ds ||
\]
\[
= ||\lambda(t) - \lambda^*(t) + \eta(t, \xi(\vartheta(t))) \int_{0}^{t} \psi(s, \xi(\vartheta(s)))ds \\
+ \eta(t, \xi(\vartheta(t))) - \eta(t, \xi^*(\vartheta(t))) \int_{0}^{t} \psi(s, \xi^*(\vartheta(s)))ds ||
\]
\[
\leq ||\lambda(t) - \lambda^*(t)|| + ||\eta(t, \xi(\vartheta(t)))|| \\
+ ||\eta(t, \xi(\vartheta(t))) - \eta(t, \xi^*(\vartheta(t))) || \\
\int_{0}^{t} \psi(s, \xi^*(\vartheta(s)))ds) ||
\]
\[
\leq ||\lambda(t) - \lambda^*(t)|| + ||\eta(t, \xi(\vartheta(t)))|| \\
+ ||\eta(t, \xi(\vartheta(t))) - \eta(t, \xi^*(\vartheta(t))) || \\
\int_{0}^{t} \psi(s, \xi^*(\vartheta(s)))ds) ||
\]
\[
\leq \delta_1 + (k_{2} k_{3} + M_{1} k_{2})(L + 1)T||\xi - y|| \\
+ |k_{2}||\xi(\vartheta(t)) - \xi^*(\vartheta(t))|| \\
+ |k_{2}||\xi(\vartheta(t)) - \xi^*(\vartheta(t))|| \\
\int_{0}^{t} \psi(s, \xi^*(\vartheta(s)))ds) ||
\]
\[
\leq \delta_1 + (k_{2} k_{3} + M_{1} k_{2})(L + 1)||\xi - \xi^*|| t
\]

\[
+ k_{2} M_{1}(L + 1)||\xi - \xi^*|| t
\]
Proof. (1) and (2) continuously depends on the function $\psi$.

Thus,

$$\|\xi - \xi^*\| (1 - (k_2 M_1 + k_2 M_3) (L + 1) T) \leq \delta_1$$

and

$$\|\xi - \xi^*\| \leq \frac{\delta_1}{1 - (k_2 M_1 + k_2 M_3) (L + 1) T} = \epsilon_1$$

since $(k_2 M_1 + k_2 M_3) (L + 1) T < 1$, then the solution of (3) continuously depends on the functions $\lambda$. Consequently the solution of (1) and (2) continuously depends on $\lambda$.

B. Continuous dependence on the function $\psi$

**Definition 4.2.** The solution of (1) and (2) continuously depends on the function $\psi$, if $\forall \epsilon_2 > 0 \exists \delta_2(\epsilon_2) > 0$ such that:

$$\|\psi - \psi^*\| \leq \delta_2 \implies \|x - x^*\| \leq \epsilon_2$$

where

$$d\left\{ \begin{array}{l}
x^*(t) - \lambda(t) \\
\eta(t, x^*(\psi(t)))
\end{array} \right\} = \psi^* (s, x^*(\psi^*(s))), \ a. e. \ t \in [0, T]$$

with the initial data

$$x^*(0) = \lambda(0).$$

**Theorem 4.6.** According to Theorem 3.3, the solution of (1) and (2) continuously depends on the function $\psi$.

Proof. Assume that $\xi$ and $\xi^*$ are the two unique solutions of (1)-(2) and (13)-(14) respectively, then we get

$$\|\xi(t) - \xi^* (t)\|$$

$$= |\lambda(t) + \eta(t, \xi(\psi(t)))| \int_0^t \psi^* (s, \xi^*(\psi^*(s))) ds$$

$$= |\lambda(t) - \eta(t, x^*(\psi^*(\psi(t))))| \int_0^t \psi^* (s, x^*(\psi^*(\psi(s)))) ds|$$

$$= |\eta(t, \xi(\psi(t)))| \left( \int_0^t \psi^* (s, \xi^*(\psi^*(s))) ds \right)$$

$$+ \int_0^t \psi^* (s, \xi^*(\psi^*(s))) ds$$

$$\leq |\eta(t, \xi(\psi(t)))| [ \int_0^t \psi^* (s, \xi^*(\psi^*(s))) - \psi(s, \xi^* (\psi^*(s))) ds ]$$

$$+ |\eta(t, \xi(\psi(t))) - \eta(t, x^*(\psi^*(\psi(t))))| \int_0^t \psi^* (s, \xi^*(\psi^*(s))) ds$$

$$\leq (k_2 |\xi(t(\psi(t)))| + A_2)$$

$$\left( \int_0^t \psi(s, \xi^*(\psi^*(s))) - \psi^* (s, \xi^*(\psi^*(\psi(s)))) ds \right)$$

$$\left( \int_0^t \psi(s, \xi^*(\psi^*(s))) + A_1 ds \right)$$

$$+ (k_2 |\xi(t(\psi(t))) + A_2) \int_0^t \psi^* (s, \xi^*(\psi^*(\psi(s)))) ds$$

$$\leq \left( k_2 (L + 1) \|\xi - \xi^*\| + M_2 T \delta_2 \right)$$

Thus,

$$\|\xi - \xi^*\| (1 - (k_2 M_1 + k_2 M_3) (L + 1) T) \leq M_2 T \delta_2$$

and

$$\|\xi - \xi^*\| \leq \frac{M_2 T \delta_2}{1 - (k_2 M_1 + k_2 M_3) (L + 1) T} = \epsilon_2$$

since $(k_2 M_1 + k_2 M_3) (L + 1) T < 1$, then the solution of (1) and (2) continuously depends on $\psi$.

V. Examples

**Example 5.1.** Consider the state-dependent problem

$$d\left\{ \begin{array}{l}
x(t) - \frac{1 + 2t}{32} \\
\frac{1}{32} \left( \sin (5(t + 1)) + e^{-t} \right)
\end{array} \right\}$$

$$= \frac{1}{5 - t} \left| \sin (5(t + 1)) + e^{-t} \right|$$

with the initial data

$$x(0) = \frac{1}{32}.$$
Example 5.2. Consider the state-dependent problem

\[
\frac{d}{dt} \left\{ \xi(t) - \frac{1}{15}(1 - t^2) \right\} = \frac{1}{8} \left( t + |\sin (\xi(\alpha(t)))| \right)
\]

(17)

with the initial data

\[
\xi(0) = \frac{1}{15}
\]

(18)

where \( t \in (0, \frac{1}{2}] \) and \( \alpha \in (0, 1] \). Here: \( \vartheta(t) = \alpha t \),

\[
\lambda(t) = \frac{1}{15}(1 - t^2),
\]

\[
\eta(t, \xi(\vartheta(t))) = \left( \frac{1}{4} t + \frac{1}{9} \xi(\alpha t) \right),
\]

\[
\psi(t, \xi(\vartheta(t))) = \frac{t + |\sin (\xi(\alpha t))|}{8},
\]

thus we have: \( c = \frac{1}{15}, A_1 = \frac{1}{15}, A_2 = \frac{1}{8}, k_1 = \frac{1}{4}, k_2 = \frac{1}{15}, k_3 = \frac{1}{8}, M_1 = \frac{1}{8}, M_2 = \frac{1}{72} \).

\( L \simeq 0.1079 < 1 \) and \( L T + |\xi(0)| = 0.121 \leq T = \frac{1}{2} \).

Also we have

\[
(k_2 M_1 + k_3 M_2) T (L + 1) \simeq 0.20 < 1.
\]

Now form Theorem 3.3, the solution of (17)-(18) is unique.

VI. APPLICATION

Finally, we introduce an application of Theorem 3.4 in the next numerical example.

Example 6.3. Consider the state-dependent problem

\[
\frac{d}{dt} \left\{ \xi(t) - \left( \frac{1}{2} t - \frac{5}{512} t^3 \right) \right\} = \frac{1}{2} \left( t - |\xi(t)| \right)
\]

(19)

where \( t \in (0, 1] \) with the initial data

\[
\xi(0) = 0.
\]

(20)

with

\[
\xi(t) = \left( \frac{1}{2} t - \frac{5}{512} t^3 \right)
\]

\[
+ \frac{1}{24} \left( t + |\xi(t)| \right) \int_0^t \frac{1}{2} \left( s - |\xi(s)| \right) ds,
\]

t \in [0, 1].

The exact solution of (19)-(20) is \( \xi(t) = \frac{1}{2} t \). Here: \( \vartheta(t) = t \),

\[
\lambda(t) = \left( \frac{1}{2} t - \frac{5}{512} t^3 \right),
\]

\[
\eta(t, \xi(\vartheta(t))) = \frac{1}{24} \left( t + |\xi(t)| \right),
\]

\[
\psi(t, \xi(\vartheta(t))) = \frac{1}{2} \left( t - |\xi(t)| \right),
\]

then we have \( c = \frac{271}{512}, A_1 = \frac{1}{2}, A_2 = \frac{1}{72}, k_1 = k_2 = \frac{1}{72}, k_3 = \frac{1}{8}, M_1 = \frac{1}{8}, M_2 = \frac{1}{72} \).

\( L \simeq 0.673 < 1 \) and \( L T + |\xi(0)| \approx 0.673 \leq T = 1 \). Also we have

\[
(k_2 M_1 + k_3 M_2) T (L + 1) \simeq 0.139 < 1.
\]

Now using Theorem 3.3, the solution of (19)-(20) is unique.

| \hline
| \hline
| \hline

\begin{tabular}{|c|c|c|c|}
\hline
| Solution \& Approximation | Error 1 | Error 2 |
\hline
| First approximation | Second approximation |
\hline
| 0.1 | 0.199999999999999996 | 0.0000000000000001 |
| 0.2 | 0.199999999999999996 | 0.0000000000000001 |
| 0.3 | 0.199999999999999996 | 0.0000000000000001 |
| 0.4 | 0.199999999999999996 | 0.0000000000000001 |
| 0.5 | 0.199999999999999996 | 0.0000000000000001 |
| 0.6 | 0.199999999999999996 | 0.0000000000000001 |
| 0.7 | 0.199999999999999996 | 0.0000000000000001 |
| 0.8 | 0.199999999999999996 | 0.0000000000000001 |
| 0.9 | 0.199999999999999996 | 0.0000000000000001 |
\hline
\end{tabular}

Now, we use the method of successive approximation (Picard Method) to estimate the solution of (19)-(20).

\[
\xi_n(t) = \left( \frac{1}{2} t - \frac{5}{512} t^3 \right) + \frac{1}{24} \left( t + \xi_{n-1}(\xi_{n-1}(t)) \right)
\]

\[
\int_0^t \frac{1}{2} \left( s - \xi_{n-1}(\xi_{n-1}(s)) \right) ds
\]

(21)

\[
\xi_0(t) = \left( \frac{1}{2} t - \frac{5}{512} t^3 \right).
\]

(22)

and

\[
\xi(t) = \lim_{n \rightarrow \infty} \xi_n(t).
\]

(19)-(20) was programmed using Matlab software and the results are shown in Table 1.

Table 1 shows the values of the solution obtained in the first and the second iterations for different values of \( t \). We also calculated the error (\( \text{Exact solution} - \text{Approximated solution} \)) at every step.

Columns 4 and 5 in Table 1 give the errors corresponding to the first and the second iterations respectively.

We deduce that the error will be very small when the number of iterations increase. So, when \( n \rightarrow \infty \) the approximated solution and the exact solution will coincide.

Here we calculated the error for the first and the second iterations only since the error is very small in the next iterations.

VII. CONCLUSION

In this paper, we have proved the existence, the uniqueness, and the continuous dependence of the unique solution of a state-dependent hybrid differential equation under suitable assumptions. Here we have generalized the results in [7], [8], [17], and [18]. Some examples, to illustrate the obtained results, have been given. Also, the method of successive approximation has been used to estimate the solution of a given example.

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REFERENCES

