A Fractional Boundary Value Problem with φ-Riemann-Liouville Fractional Derivative

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***Abstract*—The fractional boundary value problem with Riemann-Liouville derivative with respect to a Kernel function $\varphi(t)$ was investigated by us. We used a technique called monotone iteration. The positive solutions for the fractional problem were found by us. Moreover, the iterative format was established.**

***Index Terms*—Monotone iterative technique; Positive solution; Riemann-Liouville derivative; A Kernel function $\varphi(t)$.**

I. Introduction

BEcause of the extensive use of fractional differential equation in the natural sciences, more and more scholars are engaged in this aspect of research (see [1-21,32]). Now let's look at a simple but realistic example from mechanics, in this model, the author used the fractional derivatives successfully. Here we will display the behavior of certain materials under external forces. The laws of Hooke and Newton is usually used to deal with this kind of problem in mechanics. Relationship between stress $\sigma(t)$ and strain $\varepsilon(t)$ is our interesting topic. When we consider the viscous liquids, we usually choose the following tool

$$\nu(t) = \xi D'\zeta(t) \tag{1}$$

here ξ is called the material constant.

$$\nu(t) = ED^0\zeta(t), \tag{2}$$

as you can see from the relationship above, Hooke's law is a correct method for simulating the stress-strain relationship of elastic solids.

Now, if we control the strain, we build a model, let $\zeta(t) = t$ when $t \in [0, T]$ for $T > 0$. Then we can have

$$\nu(t) = Et$$

for an elastic solid and

$$\nu(t) = \xi = const$$

with regard to a viscous liquid. We can reduce these equations to the following form

$$\psi_k = \frac{\sigma(t)}{\varepsilon(t)}t^k, \tag{3}$$

here $\psi_0 = E$ and $\psi_1 = \eta$. In practical matters, when $0 < k < 1$, the equation (3) represent the viscoelastic materials.

In the references, we can find many definitions about fractional derivatives, such as Riesz-Caputo derivative [26], Hadamard derivative [24], Caputo derivative [22], Atangana-Baleanu fractional derivative [25], Riemann-Liouville derivative [23]. Many other forms of generalizations of fractional derivatives are also discussed in the recent articles, for example in [27], ψ-Hilfer fractional derivative existed in a integro-differential equation was considered. The Katugampola fractional derivative was introduced in [28].

φ-Caputo fractional derivative exists in the following fractional initial value problem was discussed in Almeida et al. [29],

$$^{C}D_{a^+}^{\alpha,\varphi}x(t) = g(t, x(t)), \qquad t \in [c, d],$$

$$x(c) = x_c, \quad x_{\psi}^{[l]}(c) = x_c^l, \quad l = 1, 2, \cdots, m-1.$$

In [30], the φ-Riemann-Liouville fractional derivative exists in the following fractional problem was considered.

$$D_{0^+}^{\alpha,\varphi}v(\tau) + f(\tau, v(\tau)) = 0, \qquad \tau \in (0, 1),$$

$$v(0) = 0, \quad v(1) = 0,$$

where $1 < \alpha \leq 2$. In this article, Seemab et al obtained the positive solutions for the above differential equation. Unfortunately, the first order derivative doesn't exist in the $f(x, v(x))$ of the fractional differential equation of Seemab et al [30].

In [31], the Caputo operator about the new function ψ lies in a fractional boundary value problems was discussed.

$$^{C}D_{c^+}^{\beta,\,\psi}v(\tau) = f(\tau, v(\tau)), \qquad \tau \in [c, d],$$

$$v_{\psi}^{[l]}(c) = v_c^l, \quad l = 0, 1, \cdots, n-2; \quad v_{\psi}^{[n-1]}(d) = v_d,$$

here $n-1 < \beta \leq n$ $(n = [\beta]+1)$, $f \in C[a, b] \times R \times R \to R$ and $y_a^k, y_b \in R$, $(k = 0, 1, \cdots, n-2)$, $y \in C^{n-1}[a, b]$ so as to $^{C}D_{a^+}^{\alpha,\,\psi}y$ can exist. Moreover, $^{C}D_{a^+}^{\alpha,\,\psi}y$ is continuous on the interval $[a, b]$. The authors got the unique solution in this article.

Sun [33] gave the solution for the Sturm-Liouville-like problem

$$(\phi_p(v'(\tau)))' + q(\tau)f(\tau, v(\tau), v'(\tau)) = 0, \ \tau \in (0, 1),$$

$$v(0) - \alpha v'(\xi) = 0,$$

$$v(1) + \beta v'(\eta) = 0.$$

The author utilized a technique called monotone iteration. In this article, Sun didn't require the above equation has lower and upper solutions. The positive solutions were obtained and the iterative schemes were established at the same time. Unfortunately, the derivative of this above equation in this article is only of integer order.

On the other hand, most of the existing literature does not consider the computational methods of the solution. Well, here's the problem: When the solutions exist, how can we compute them? Based on the above results, in this paper, the Riemann-Liouville derivative with respect to a Kernel

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function $\varphi(t)$ exists in the following fractional problem was discussed.

$$D_{0^+}^{\alpha,\varphi}v(t)+g(t,v(t),v'(t))=0, \qquad t\in(0,1), \tag{4}$$

$$v(0)=0, \quad v'(0)=0, \quad v(1)=0, \tag{5}$$

where $2<\alpha\le 3$. Moreover, the function $g:[0,1]\times[0,\infty)\times(-\infty,\infty)\to[0,\infty)$ is continuous and the following function which is increasing strictly satisfies $\varphi:[0,1]\to[0,1],\ \varphi\in C^2[0,1],\ \varphi'(x)\neq 0'\ x\in[0,1]$ and $(\alpha-1)\varphi'(t)<1$. We got the positive solution which is monotone and iterative for the above differential equation.

Let's take a look at the layout of this article. In Section 1, we give a descriptive introduction. Some theorems and definitions about φ-Riemann-Liouville fractional integral and derivative are presented in section 2. At last, the iterative positive solution for (4), (5) is considered.

II. Preliminaries

We now present some definitions, notations and results of Riemann-Liouville integral and derivative with respect to a Kernel function $\varphi(t)$.

Definition 2.1 [29] Now suppose that $n-1<\beta<n$. we gave a function $g\in[c,d]$ which is integrable and $\varphi\in C^n[c,d],\ \varphi'(t)\neq 0$ an increasing differentiable function. Here's the definition for φ-Riemann-Liouville fractional integral of g

$$I_{c^+}^{\beta,\varphi}g(x)=\frac{1}{\Gamma(\beta)}\int_c^x\varphi'(s)(\varphi(x)-\varphi(s))^{\beta-1}g(s)ds.$$

Definition 2.2 [29] Now suppose that $n-1<\beta<n$. φ is given just as definition 2.1. Assume that $g:[a,b]\to R$ is a function which is integrable. Here's the definition for Riemann-Liouville derivative with respect to a Kernel function $\varphi(t)$ as follows

$$\begin{aligned}D_{a^+}^{\beta,\varphi}g(x)&=\left(\frac{1}{\varphi'(x)}\frac{d}{dt}\right)^nI_{a^+}^{n-\beta,\varphi}g(x)\\&=\frac{1}{\Gamma(n-\beta)}\left(\frac{1}{\varphi'(x)}\frac{d}{dt}\right)^n\\&\quad\int_a^x\varphi'(s)(\varphi(x)-\varphi(s))^{n-\beta-1}g(s)ds.\end{aligned}$$

Let $\alpha,\beta>0$, then the relation

$$I_{a^+}^{\alpha,\varphi}I_{a^+}^{\beta,\varphi}h(x)=I_{a^+}^{\alpha+\beta,\varphi}h(x)$$

holds.

Definition 2.3 [29] Let $\beta>0$. $\varphi\in C^n[c,d],\ \varphi'(t)>0$ and $\varphi'(t)\neq 0,\ t\in[c,d]$. Suppose $h\in C^{n-1}[c,d]$, here's the definition for φ-Caputo fractional derivative of h

$${}^CD_{c^+}^{\beta,\varphi}h(x)=D_{c^+}^{\beta,\varphi}\left[h(x)-\sum_{k=0}^{n-1}\frac{h_\varphi^{[k]}(c)}{k!}(\varphi(x)-\varphi(c))^k\right],$$

here $n=[\beta]+1$ if $\beta\notin N,\ n=\beta$ if $\beta\in N$.

Theorem 2.1 [29] Let $h:[c,d]\to R$. The following results are valid.

1. Suppose $x\in C[c,d]$, we have ${}^CD_{c^+}^{\beta,\varphi}I_{c^+}^{\beta,\varphi}h(x)=h(x)$
2. Suppose $x\in C^{n-1}[c,d]$, just we get

$$I_{c^+}^{\beta,\varphi}\,{}^CD_{c^+}^{\beta,\varphi}h(x)=h(x)-\sum_{k=0}^{n-1}\frac{h_\varphi^{[k]}(c)}{k!}(\varphi(x)-\varphi(c))^k.$$

Lemma 2.1 Suppose g is continuous which is defined on $[0,1]$ and $2<\beta\le 3$. The following fractional problem

$$D_{0^+}^{\beta,\varphi}v(t)+g(t)=0, \qquad t\in(0,1), \tag{6}$$

$$v(0)=0, \quad v'(0)=0, \quad v(1)=0, \tag{7}$$

has a solution which is unique as following

$$v(t)=\int_0^1G(t,s)\varphi'(s)g(s)ds, \tag{8}$$

here

$$G(t,s)=\frac{\Lambda(t)}{\Gamma(\beta)}\begin{cases}(\varphi(1)-\varphi(s))^{\beta-1}\\ -\dfrac{1}{\Lambda(t)}(\varphi(t)-\varphi(s))^{\beta-1}, \\ \qquad\qquad\qquad 0\le s\le t\le 1,\\ (\varphi(1)-\varphi(s))^{\beta-1}, \quad 0\le t\le s\le 1,\end{cases} \tag{9}$$

with $\Upsilon(t)=\varphi(t)-\varphi(0)$ and $\Lambda(t)=\dfrac{(\Upsilon(t))^{\beta-1}}{(\Upsilon(1))^{\beta-1}}$.

Proof: First, we presume the fractional problems (6), (7) has a solution $v(t)$. Then, from Theorem 2.1, we can get

$$\begin{aligned}v(t)&=c_1(\varphi(t)-\varphi(0))^{\beta-1}+c_2(\varphi(t)-\varphi(0))^{\beta-2}\\&\quad+c_3(\varphi(t)-\varphi(0))^{\beta-3}-I_{0^+}^{\beta,\varphi}g(t).\end{aligned}$$

$c_2=c_3=0$ can be obtained just from the relations $v(0)=0,\ \ v'(0)=0$. Thus,

$$v(t)=c_1(\varphi(t)-\varphi(0))^{\beta-1}-I_{0^+}^{\beta,\varphi}g(t).$$

Similar to [30], we have (8), (9) hold. ■

Lemma 2.2 We can conclude that the function $G(t,s)$ described in (9) matches the following relationship.

1. $G(t,s)>0$ for all $t,s\in(0,1)$;
2. For $s\in(0,1)$, $\max\limits_{t\in[0,1]}G(t,s)\le\dfrac{(\varphi(1)-\varphi(s))^{\beta-1}}{\Gamma(\beta)(\Upsilon(1))^{\beta-1}}$;
3. For $s\in(0,1)$, there exists a positive function ω which can make the following relationship hold:

$$\min_{t\in[\frac14,\frac34]}G(t,s)\ge\omega(s)\max_{t\in[0,1]}G(t,s); \tag{10}$$

4. For $s\in(0,1)$,

$$\max_{t\in[0,1]}\frac{\partial G(t,s)}{\partial t}\le\frac{(\varphi(1)-\varphi(s))^{\beta-1}}{(\Upsilon(1))^{\beta-1}\Gamma(\beta-1)}\max_{t\in[0,1]}\varphi'(t). \tag{11}$$

Proof: The proof of the properties 1.2.3 are given in [30], now we prove the property 4. From (9), we have

$$\frac{\partial G(t,s)}{\partial t}=\begin{cases}\dfrac{\Lambda'(t)}{\Gamma(\beta)}(\varphi(1)-\varphi(s))^{\beta-1}\\ -\dfrac{1}{\Gamma(\beta-1)}(\varphi(t)-\varphi(s))^{\beta-2}\varphi'(t),\\ \qquad\qquad\qquad 0\le s\le t\le 1,\\ \dfrac{\Lambda'(t)}{\Gamma(\beta)}(\varphi(1)-\varphi(s))^{\beta-1}, \quad 0\le t\le s\le 1.\end{cases} \tag{12}$$

$2<\beta\le 3,\ \varphi'(t)>0$ imply that

$$\begin{aligned}\frac{\partial G(t,s)}{\partial t}&\le\frac{\Lambda'(t)}{\Gamma(\beta)}(\varphi(1)-\varphi(s))^{\beta-1}\\&=\frac{(\varphi(t)-\varphi(0))^{\beta-2}(\varphi(1)-\varphi(s))^{\beta-1}}{(\Upsilon(1))^{\beta-1}\Gamma(\beta-1)}\varphi'(t)\\&\le\frac{(\varphi(1)-\varphi(s))^{\beta-1}}{(\Upsilon(1))^{\beta-1}\Gamma(\beta-1)}\max_{t\in[0,1]}\varphi'(t).\end{aligned}$$

■

III. MAIN RESULTS

We define the norm

$$\|v\| = \{\max_{\tau\in[0,1]} |v(\tau)|,\ \max_{\tau\in[0,1]} |v'(\tau)|\}.$$

Thus we have $E = C^1[0,1]$ is a Banach space in the above norm case.

Let $K \subset E$ defined by

$$K = \{v \in E:\ v(t) \geq 0,\quad 0 \leq t \leq 1\}. \tag{13}$$

For $v \in K$, the operator A is given by the following relation

$$(Av)(t) = \int_0^1 G(t,s)\varphi'(s)g(s,v(s),v'(s))ds. \tag{14}$$

Obviously, if $v(t)$ satisfies the relation $v = Av$, then we can say the fractional problem (4), (5) has a solution $v(t)$.

Lemma 3.1 [30] Operator relation $A:\ K \to K$ described by (14) is a completely continuous operator.

Denote

$$B = \min\left\{\frac{b\Gamma(\beta+1)}{\varphi(1)-\varphi(0)},\quad \frac{b\beta\Gamma(\beta-1)}{(\varphi(1)-\varphi(0))\max\limits_{0\leq t\leq 1}\varphi'(t)}\right\}.$$

Theorem 3.1 Suppose that we can find a number $b > 0$, satisfies

(H_1) $g(t,\ \mu_1,\ \nu_1) \leq g(t,\ \mu_2,\ \nu_2)$, $\quad if$ $0 \leq t \leq 1,\quad 0 \leq \mu_1 \leq \mu_2 \leq b,\quad 0 \leq |\nu_1| \leq |\nu_2| \leq b$;

(H_2) $\max\limits_{0\leq t\leq 1} g(t,b,b) \leq B$;

(H_3) $g(t,\ 0,\ 0) \not\equiv 0\ if\ 0 \leq t \leq 1$.

Thus there is a positive solution $\nu^* \in K$ for equation (4),(5). Moreover, the ν^* satisfies

$$0 < \nu^* \leq b,\ 0 < |(\nu^*)'| \leq b.$$

Furthermore,

$$\lim_{n\to\infty} \nu_n = \lim_{n\to\infty} A^n\nu_0 = \nu^*$$

$$\lim_{n\to\infty} (\nu_n)' = \lim_{n\to\infty} (A^n\nu_0)' = (\nu^*)'$$

here

$$\nu_0(t) = b(\Upsilon(t))^{\beta-1} = b(\varphi(t)-\varphi(0))^{\beta-1},\quad 0 \leq t \leq 1.$$

Proof: Set

$$K_b = \{v \in K|\ \|v\| < b\}.$$

Then we are going to explain $A:\ \overline{K_b} \to \overline{K_b}$.

Let $v \in \overline{K_b}$, just we have

$$0 \leq v(t) \leq \max_{0\leq t\leq 1} |v(t)| \leq \|v\| \leq b, \tag{15}$$

$$0 \leq |v'(t)| \leq \max_{0\leq t\leq 1} |v'(t)| \leq \|v\| \leq b. \tag{16}$$

We can get the following relationship just from conditions (H_1) and (H_2).

$$\begin{aligned} 0 &\leq g(t,v(t),v'(t)) \leq g(t,b,b) \leq \max_{0\leq t\leq 1} g(t,b,b) \\ &\leq B. \end{aligned} \tag{17}$$

So, from (17) and lemma 2.2, the following relations hold.

$$\begin{aligned} \Big|(Au)(t)\Big| &= \Big|\int_0^1 G(t,s)\varphi'(s)g(s,v(s),v'(s))ds\Big| \\ &\leq \int_0^1 G(t,s)\varphi'(s)\Big|g(s,v(s),v'(s))\Big|ds \\ &\leq \int_0^1 G(t,s)\varphi'(s)\Big|g(s,b,b)\Big|ds \\ &\leq \int_0^1 \max_{t\in[0,1]} G(t,s)\varphi'(s)\Big|g(s,b,b)\Big|ds \\ &\leq B\int_0^1 \frac{(\varphi(1)-\varphi(s))^{\beta-1}}{(\varphi(1)-\varphi(0))^{\beta-1}\Gamma(\beta)}\varphi'(s)ds \\ &\leq \frac{b\Gamma(\beta+1)}{\varphi(1)-\varphi(0)}\frac{\varphi(1)-\varphi(0)}{\Gamma(\beta+1)} \\ &= b. \end{aligned} \tag{18}$$

$$\begin{aligned} \Big|(Au)'(t)\Big| &= \Big|\int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi'(s)g(s,v(s),v'(s))ds\Big| \\ &\leq \int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi'(s)\Big|g(s,v(s),v'(s))\Big|ds \\ &\leq \int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi'(s)\Big|g(s,b,b)\Big|ds \\ &\leq \int_0^1 \max_{t\in[0,1]}\frac{\partial G(t,s)}{\partial t}\varphi'(s)\Big|g(s,b,b)\Big|ds \\ &\leq B\int_0^1 \frac{(\varphi(1)-\varphi(s))^{\beta-1}\max\limits_{0\leq t\leq 1}\varphi'(t)}{(\varphi(1)-\varphi(0))^{\beta-1}\Gamma(\beta-1)}\varphi'(s)ds \\ &\leq \frac{b\beta\Gamma(\beta-1)}{(\varphi(1)-\varphi(0))\max\limits_{0\leq t\leq 1}\varphi'(t)} \\ &\quad \frac{(\varphi(1)-\varphi(0))\max\limits_{0\leq t\leq 1}\varphi'(t)}{\beta\Gamma(\beta-1)} \\ &= b. \end{aligned} \tag{19}$$

This means $A:\ \overline{K_b} \to \overline{K_b}$. Let

$$\nu_0(t) = b(\Upsilon(t))^{\beta-1},\qquad 0 \leq t \leq 1.$$

Then

$$\nu_0'(t) = b((\Upsilon(t))^{\beta-1})' = b(\beta-1)(\Upsilon(t))^{\beta-2}\varphi'(t),\ 0 \leq t \leq 1.$$

$2 < \beta \leq 3,\ \varphi : [0,1] \to [0,1]$ and $(\beta-1)\varphi'(t) < 1$ imply that

$$0 \leq \nu_0(t) \leq b,\quad 0 \leq |\nu_0'(t)| \leq b.$$

We define $\nu_1 = A\nu_0$, thus $\nu_1 \in \overline{K_b}$, we write

$$\nu_{n+1} = A\nu_n = A^{n+1}\nu_0,\quad (n = 0,1,2,\cdots). \tag{20}$$

We can conclude that $\nu_n \in A\overline{K_b} \subseteq \overline{K_b},\quad n = 0,1,2,\cdots$ just from $A:\ \overline{K_b} \to \overline{K_b}$.

There exists a sequentially compact set $\{\nu_n\}_{n=0}^{\infty}$ precisely because A is a completely continuous operator, therefore,

$$\begin{aligned}\nu_1(t) &= A\nu_0(t) = \int_0^1 G(t,s)\varphi'(s)g(s,\nu_0(s),\nu_0'(s))ds \\ &\le \int_0^1 G(t,s)\varphi'(s)\Big|g(s,\nu_0(s),\nu_0'(s))\Big|ds \\ &\le \int_0^1 G(t,s)\varphi'(s)\Big|g(s,b,b)\Big|ds \\ &\le B\int_0^1 \frac{\Lambda(t)}{\Gamma(\beta)}(\varphi(1)-\varphi(s))^{\beta-1}\varphi'(s)ds \\ &\le \frac{b\Gamma(\beta+1)}{\varphi(1)-\varphi(0)}(\Upsilon(t))^{\beta-1}\frac{\varphi(1)-\varphi(0)}{\Gamma(\beta+1)} \\ &= b(\Upsilon(t))^{\beta-1} = \nu_0(t).\end{aligned} \tag{21}$$

$$\begin{aligned}|\nu_1'(t)| &= |(A\nu_0)'(t)| \\ &= \Big|\int_0^1 \frac{\partial G(t,s)}{\partial t}\varphi'(s)g(s,\nu_0(s),\nu_0'(s))ds\Big| \\ &\le \int_0^1 \Big|\frac{\partial G(t,s)}{\partial t}\Big|\varphi'(s)\Big|g(s,\nu_0(s),\nu_0'(s))\Big|ds \\ &\le \int_0^1 \Big|\frac{\partial G(t,s)}{\partial t}\Big|\varphi'(s)\Big|g(s,b,b)\Big|ds \\ &\le B\int_0^1 \frac{\Lambda'(t)}{\Gamma(\beta)}(\varphi(1)-\varphi(s))^{\beta-1}\varphi'(s)ds \\ &= \frac{b\Gamma(\beta+1)}{\varphi(1)-\varphi(0)}\int_0^1 \frac{((\Upsilon(t))^{\beta-1})'}{\Gamma(\beta)(\varphi(1)-\varphi(0))^{\beta-1}} \\ &\quad (\varphi(1)-\varphi(s))^{\beta-1}\varphi'(s)ds \\ &= b((\Upsilon(t))^{\beta-1})' = \nu_0'(t).\end{aligned} \tag{22}$$

Then we have

$$\nu_1(t) \le \nu_0(t), \quad |\nu_1'(t)| \le |\nu_0'(t)|, \quad 0 \le t \le 1.$$

Thus,

$$\nu_2(t) = A\nu_1(t) \le A\nu_0(t) = \nu_1(t), \quad 0 \le t \le 1,$$

$$|\nu_2'(t)| = |(A\nu_1)'(t)| \le |(A\nu_0)'(t)| = |\nu_1'(t)|, \qquad 0 \le t \le 1.$$

We can find

$$\begin{aligned}\nu_{n+1} &\le \nu_n, \quad |\nu_{n+1}'(t)| \le |\nu_n'(t)|, \\ &0 \le t \le 1, \quad n = 1,2,\cdots.\end{aligned}$$

just by mathematical induction. Thus, there exists $\nu^* \in \overline{K_b}$ satisfies $\nu_n \to \nu^*$. If we choose $n \to \infty$ in (20), we have $A\nu^* = \nu^*$ because A is a continuous operator.

Suppose $g(t,\ 0,\ 0) \not\equiv 0,\ 0 \le t \le 1$, then (4),(5) has no zero solution. Therefore, (4),(5) has a solution ν^* which is positive on $[0,1]$. ■

Denote

$$B_k = \min\left\{\frac{b_k\Gamma(\beta+1)}{\varphi(1)-\varphi(0)},\quad \frac{b_k\beta\Gamma(\beta-1)}{(\varphi(1)-\varphi(0))\max\limits_{0\le t\le 1}\varphi'(t)}\right\}.$$

Corollary 3.1 Except for $(H_1),\ (H_3)$ hold, we can find numbers $b_1, b_2, \cdots, b_n$ which satisfy $0 < b_1 < b_2 < \cdots < b_n$, moreover,

$$(H_2)' \ \max_{0\le t\le 1} g(t,b_m,b_m) \le B_m, \quad m = 1,2,\cdots,n,$$

specially,

$$\lim_{l\to+\infty}\max_{0\le t\le 1} g(t,l,b_m) = 0, \quad m = 1,2,\cdots,n.$$

Thus there are 2n positive solution $\nu_m^* \in K,\ m = 1,2,\cdots,n$ for equation (4),(5). Moreover, the ν_m^* satisfy $0 < \nu_m^* \le b_m,\ 0 < |(\nu_m^*)'| \le b_m$ and $\lim\limits_{n\to\infty}\nu_{m_n} = \lim\limits_{n\to\infty}A^n\nu_{m_0} = \nu_m^*$, where

$$\nu_{m_0}(t) = b_m(\Upsilon(t))^{\beta-1} = b_m(\varphi(t)-\varphi(0))^{\beta-1}, \quad 0 \le t \le 1.$$

Proof: The iterative schemes in Corollary 3.1 are

$$\nu_{m_1} = A\nu_{m_0}, \quad \nu_{m_2} = A\nu_{m_1} = A^2\nu_{m_0},$$

$$\nu_{m_{n+1}} = A\nu_{m_n} = A^n\nu_{m_0}, \quad m = 1,2,\cdots,\ n = 1,2,\cdots.$$

■

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