Solving Two-dimensional Linear and Nonlinear Mixed Integral Equations Using Moving Least Squares and Modified Moving Least Squares Methods

Zahra El Majouti, Rachid El Jid, and Abdelkarim Hajjaj

***Abstract*—In this paper, moving least squares (MLS) and modified moving least squares (MMLS) methods have been employed to estimate the solution of two-dimensional linear and nonlinear Fredholm-Volterra integral equations. The modification means that quadratic base functions can be utilized with the same size of the support domain as linear base functions, resulting in better approximation capability. The proposed methods are meshless because they don't require any background mesh or cell structures and so they are independent of the geometry of the domain. The error estimate of the proposed method is provided. The accuracy and computational efficiency of the methods are illustrated by several numerical tests.**

***Index Terms*—Moving least squares approximation, modified moving least squares approximation, Two-dimensional Fredholm-Volterra integral equation, Gauss-Legendre quadrature, Convergence analysis.**

I. Introduction

TWO-dimensional Fredholm-Volterra integral equations reformulated many varied problems in physics and engineering. They are utilized as mathematical models for many different science applications such as plasma physics [1], approximation of implicit surfaces [2], diffraction theory [3], simulations [4], [5], and computational biomechanics [6], [7]. Some integral equations cannot be solved by the exact methods. Thus, it is desirable to present numerical methods with high performance to solve these equations numerically.

The motivation of this paper is to solve nonlinear Fredholm-Volterra integral equations of the second kind using the moving least squares and modified moving least squares methods. So we consider the following two-dimensional mixed integral equation of the second kind

$$u(x,y)+\int_c^d\int_a^b K_1(x,y,s,v)\Phi_1(s,v,u(s,v))\,\mathrm{d}s\mathrm{d}v \\ +\int_c^y\int_a^x K_2(x,y,s,v)\Phi_2(s,v,u(s,v))\,\mathrm{d}s\mathrm{d}v=g(x,y). \quad (1)$$

The functions $g(x,y)$, $K_1(x,y,s,v)$, $K_2(x,y,s,v)$ are assumed to be given smooth real valued functions on $(x,y)\in[a,b]\times[c,d]$ and $D=\{(x,y,s,v), a\le s\le x\le b, c\le v\le y\le d\}$, respectively and $u(x,y)$ is the solution to be determined. For these types of integral equations, it is usually difficult to obtain analytical solutions and then numerical solutions have to be studied. It is worth remembering that, analytical and numerical analysis of one-dimensional mixed integral equations have been discussed by numerous authors [8], [9], [10].

Actually, the meshless methods have gained more attention, particularly Moving Least Squares method, it has been applied in many branches of modern sciences, such as surface construction [11], function approximation [12], numerical solution of integral equations [8], [9], [10], [13], [14], [15]. Also, further important applications of the meshless moving least square method are the Diffuse Element Method (DEM) presented in [16], the well known Element-Free Galerkin (EFG) method developed in [17], Boundary Node Method (BNM) [18], Hp-cloud method [19] and the Meshless Local Petrov Galerkin (MLPG) method introduced in [20]. The MLS method doesn't depend on the geometry of the domain and it doesn't require domain elements or background cells. The new method does not increase the difficulties for higher dimensional problems, also it is more adaptable and efficient to approximate the unknown function for most classes of mixed integral equations. However, the moment matrix in the MLS method may be singular when the number of points in the local support domain is not enough. In this work, the MMLS method has been applied to overcome this difficulty and finding the best support.

The new idea in this paper is to expand the MMLS method to solve two-dimensional Fredholm-Volterra integral equations, this efficient method prevents a singular moment matrix in the context of MLS based method. The modification is suggested on the quadratic base functions ($m=2$) to be utilized with the same size of the support domain as linear base functions($m=1$). The major advantage of using the MMLS method is that the results converge more quickly to the true solution and give better accuracy than that of MLS approximation. The first use of the modified moving least squares method was proposed in [21] for smoothing and approximating scattered.

The rest of this paper is organized as follows: In Section 2, a brief discussion of the MLS method is outlined. In the next section, the modified MLS approximation is presented. The computational method for solving two-dimensional linear and nonlinear Fredholm-Volterra integral equations is introduced in section 4, then section 5 is devoted to the error estimate of the applied method. Numerical results are

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introduced in section 6, which will be used to verify the theoretical results obtained in section 5. Finally, we conclude the article in Section 7.

II. The classical MLS approximation

The moving least squares (MLS) approximation as a generalization of Shepard's method [22] is developed by Lancaster and Salkauskas [23]. It is one of the meshless methods since it is based on a set of scattered points instead of interpolation on elements. We use this method to approximate two variable functions $X = \{(x_1, y_1), (x_2, y_2), ..., (x_n, y_n)\}$ on the rectangular form $D = [a, b] \times [c, d]$. Let $u : D \to \mathbb{R}$ be a continuous real function and the points (x_i, y_i, u_i), $i = 1, 2, ..., n$ are known. The main point of this meshless method is to estimate a function $u(x, y)$ for every point $(x, y) \in D$ based on the weighted least square.
Let P_q be the space of polynomials of degree $q \ll n$. The MLS approximation $u^\rho(x, y)$ of $u(x, y)$, $\forall(x, y) \in \overline{D}$, can be given as

$$u^\rho(x, y) = p^T(x, y)a(x, y), \quad \forall(x, y) \in \overline{D}, \tag{2}$$

where

$$p(x, y) = [p_0(x, y), p_1(x, y), ..., p_m(x, y)]^T,$$

and $\{p_i(x, y)\}_{i=0}^m$ is a complete basis of P_q of order m and,

$$a(x, y) = [a_0(x, y), a_1(x, y), ..., a_m(x, y)],$$

are unknown coefficients to be determined. In this paper we use monomials and Chebyshev polynomials as a basis. The MLS method presents the approximate function $u^\rho(x, y)$ in a particularized class of differentiable functions which minimize the quantity

$$\begin{aligned} J(x, y) &= \sum_{i=1}^n \omega_i(x, y)(p^T(x_i, y_i)a(x, y) - u_i)^2 \qquad (3) \\ &= [P.a(x, y) - u]^T.W.[P.a(x, y) - u], \end{aligned}$$

where $\omega_i(x, y)$ is the weight function associated with node i, (x_i, y_i) denotes the value of (x, y) at node i, n is the number of nodes in $\overline{D}$ with $w_i(x, y) > 0$ and u_i are the fictitious nodal values, but not the nodal values of the unknown trial function $u^\rho(x, y)$ i.e. $u^\rho(x_i, y_i) \neq u_i$. The matrix P and W are defined as

$$\begin{aligned} P &= [p^T(x_1, y_1), p^T(x_2, y_2), ..., p^T(x_n, y_n)]^T_{n\times(m+1)} \\ W &= diag(\omega_i(x, y)), \quad i = 1, 2, ...n. \end{aligned}$$

A necessary condition for $J(x, y)$ to be minimized is $\nabla J = 0$, which implies the following normal equation

$$\sum_{i=1}^n \omega_i(x, y)p(x_i, y_i)p^T(x_i, y_i)a(x, y) = \sum_{i=1}^n \omega_i(x, y)p(x_i, y_i)u_i. \tag{4}$$

Using the moment matrix

$$\mathbf{A}(\mathbf{x}, y) = \sum_{i=1}^n \omega_i(x, y)p(x_i, y_i)p^T(x_i, y_i),$$

and setting

$$u = [u_1, u_2, ..., u_n]^T,$$

and

$$\begin{aligned} B(x, y) = [&w_1(x, y)p(x_1, y_1), w_2(x, y)p(x_2, y_2), \\ &..., w_n(x, y)p(x_n, y_n)], \end{aligned}$$

(4) becomes as follows

$$A(x, y)a(x, y) = B(x, y)u, \tag{5}$$

and by selecting the nodal points such that A(x) is nonsingular, (5) can be written as

$$a(x, y) = A^{-1}(x, y)B(x, y)u. \tag{6}$$

Substituting (6) into (2) we obtain

$$u^\rho(x, y) = p(x, y)^T A^{-1}(x, y)B(x, y)u = \sum_{i=1}^n \phi_i(x, y)u_i, \tag{7}$$

where

$$\phi_i(x, y) = \sum_{k=1}^m p_k(x, y)[A^{-1}(x, y)B(x, y)]_{ki},$$

$\phi_i(x, y)$ are called the shape functions of the MLS approximation, corresponding to the nodal point (x_i, y_i). If $w_i(x, y) \in C^r(D)$ and $p_k(x, y) \in C^s(D)$, $i = 1, ..., n$, $k = 1, ..., m$ then $\phi_i(x, y) \in C^{min(r,s)}(D)$. The spline weight function is applied in the present work as

$$w_i(x, y) = \begin{cases} 1 - 6\left(\frac{d_i}{\rho_i}\right)^2 + 8\left(\frac{d_i}{\rho_i}\right)^3 - 3\left(\frac{d_i}{\rho_i}\right)^4 & \text{si } 0 \leq d_i \leq \rho_i \\ 0 & \text{si } d_i > \rho_i \end{cases}$$

where $d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ (the Euclidean distance between nodes), ρ_i is the size of the support domain.

III. Modified MLS approximation

The proposed MMLS method will avoid the singular moment matrix in the context of MLS based on meshless methods. This modification allows quadratic base functions to be utilized with the same size of the support domain as linear base functions by adding additional terms based on the coefficients of the polynomial base functions, leading to have a better approximation capability.
The coefficients in the monomial quadratic basis are defined as

$$\begin{aligned} p(x) &= [1, x, y, x^2, xy, y^2]^T, \\ a(x) &= [a_1, a_x, a_y, a_{x^2}, a_{xy}, a_{y^2}]^T. \end{aligned}$$

Therefore, the new functional (3) can be represented as follows

$$\bar{J}(x, y) = \sum_{i=1}^n \omega_i(x, y)[u^\rho(x, y) - u_i]^2 + \nu_{x^2}a_{x^2}^2 + \nu_{xy}a_{xy}^2 + \nu_{y^2}a_{y^2}^2 \tag{8}$$

where $\nu = [\nu_{x^2} \quad \nu_{xy} \quad \nu_{y^2}]$ is a vector of positive weights for the additional constraints.
The modified matrix and the matrix form of (8) are

$$\bar{J}(x, y) = [P.a - u_i]^T.W.[P.a - u_i] + a^T Ma, \quad i = 1, 2, ..., n \tag{9}$$

and

$$M = \begin{bmatrix} O_{3,3} & O_{3,3} \\ O_{3,3} & diag(\nu) \end{bmatrix}$$

where $O_{3,3}$ is the zero matrix and the last three diagonal entries equal to ν.
By minimizing the functional (9), the coefficients $a(x,y)$ will be determined by

$$\bar{A}(x,y)a(x,y) = B(x,y)u_i,$$

where

$$\bar{A} = P^T.W.P + M.$$

The modified approximation can be written as follows

$$\bar{u}^\rho(x,y) = \sum_{i=1}^{n} \bar{\phi}_i(x,y)u_i,$$

with the MMLS shape functions defined by

$$\begin{aligned}\bar{\Phi}(x,y) &= [\bar{\phi}_1(x,y), \bar{\phi}_2(x,y)...\bar{\phi}_n(x,y)]\\ &= p^T(x,y)(P^T.W.P + M)^{-1}B(x,y).\end{aligned}$$

IV. The proposed method

A. 2-D linear Fredholm-Volterra integral equation

Consider the following two-dimensional Fredholm-Volterra integral equation

$$\begin{aligned}u(x,y) &+ \int_c^d \int_a^b K_1(x,y,s,v)u(s,v)\,\mathrm{d}s dv\\ &+ \int_c^y \int_a^x K_2(x,y,s,v)u(s,v)\,\mathrm{d}s\mathrm{d}v = g(x,y),\end{aligned}$$

where $(x,y) \in [a,b] \times [c,d]$, the intervals $[a,x],[c,y]$ are converted respectively to the fixed intervals $[a,b],[c,d]$ by the following linear transformations

$$s(x,\delta) = \frac{x-a}{b-a}\delta + \frac{b-x}{b-a}a, \quad v(y,\beta) = \frac{y-c}{d-c}\beta + \frac{d-y}{d-c}c. \tag{10}$$

Therefore, the equation takes the following form

$$\begin{aligned}u(x,y)+&\int_c^d \int_a^b K_1(x,y,s,v)u(s,v)\,\mathrm{d}s dv\\ &+ \int_c^d \int_a^b \bar{K}_2(x,y,s(x,\delta),v(y,\beta))\\ &\quad u(s(x,\delta),v(y,\beta))\,\mathrm{d}\delta\mathrm{d}\beta = g(x,y),\end{aligned}$$

where

$$\bar{K}_2(x,y,s(x,\delta),v(y,\beta)) = \frac{x-a}{b-a}\frac{y-c}{d-c}K_2(x,y,s(x,\delta),v(y,\beta)).$$

If we replace $u(x,y)$ by $u^\rho(x,y)$ we obtain

$$\begin{aligned}u^\rho(x,y)+&\int_c^d \int_a^b K_1(x,y,s,v)u^\rho(s,v)\,\mathrm{d}s dv\\ &+ \int_c^d \int_a^b \bar{K}_2(x,y,\xi(x,\delta),v(y,\beta))\\ &\quad u^\rho(s(x,\delta),v(y,\beta))\,\mathrm{d}\delta\mathrm{d}\beta = g(x,y),\end{aligned}$$

or equivalently

$$\begin{aligned}\sum_{j=1}^{n} \Big[\phi_j(x,y)+&\int_c^d \int_a^b K_1(x,y,s,v)\phi_j(s,v)\,\mathrm{d}s\,\mathrm{d}v\\ &+ \int_c^d \int_a^b \bar{K}_2(x,y,s(x,\delta),v(y,\beta))\\ &\phi_j(s(x,\delta),v(y,\beta))\mathrm{d}\delta\mathrm{d}\beta\Big]u_j = g(x,y).\end{aligned}$$

Assume that this equation holds at (x_i,y_i)

$$\begin{aligned}\sum_{j=1}^{n} \Big[\phi_j(x_i,y_i)+&\int_c^d \int_a^b K_1(x_i,y_i,s,v)\phi_j(s,v)\,\mathrm{d}s\,\mathrm{d}v\\ &+ \int_c^d \int_a^b \bar{K}_2(x_i,y_i,s(x_i,\delta),v(y_i,\beta))\\ &\phi_j(s(x_i,\delta),v(y_i,\beta))\mathrm{d}\delta\mathrm{d}\beta\Big]u_j = g(x_i,y_i),\end{aligned}$$

where $i = 1,2,...,n$, we compute integrals numerically by using m_1 points quadrature formula with the quadrature points $\{s_k\},\{\delta_k\},\{v_p\},\{\beta_p\}$ and the quadrature weights $\{w_k\},\{w_p\}$. Therefore, the above equation can be written as follows

$$\sum_{j=1}^{n} F_{i,j}\hat{u}_j = g(x_i,y_i), \quad i = 1,2,...,n,$$

where $\hat{u}_j$ are the approximate quantities of u_j and F is a n by n matrix defined by

$$\begin{aligned}F_{i,j} = \phi_j(x_i,y_i)+&\sum_{p=1}^{m_1}\sum_{k=1}^{m_1} K_1(x_i,y_i,s_k,v_p)\phi_j(s_k,v_p)\omega_k\omega_p\\ &+\sum_{p=1}^{m_1}\sum_{k=1}^{m_1} \bar{K}_2(x_i,y_i,s(x_i,\delta_k),v(y_i,\beta_p))\\ &\phi_j(s(x_i,\delta_k),v(y_i,\beta_p))\omega_k\omega_p.\end{aligned}$$

Let's note

$$\hat{u} = [\hat{u}_1,\hat{u}_2,......,\hat{u}_n]^T, \qquad g = [g_1,g_2,....,g_n]^T.$$

Then we have the following linear system of equations

$$F\hat{u} = g. \tag{11}$$

Solving (11), we can approximate $u(x,y)$ as in (7) by

$$u^\rho(x,y) = \sum_{j=1}^{n} \phi_j(x,y)\hat{u}_j, \quad (x,y) \in [a,b] \times [c,d].$$

B. 2-D nonlinear Fredholm-Volterra integral equation

In this section, MLS and MMLS approximations are used to solve two dimensional nonlinear Fredholm-Volterra integral equations of the second kind. Firstly, we transform the intervals $[a,x],[c,y]$ in to $[a,b],[c,d]$ by the last linear transformations (10), then (1) takes the following form

$$\begin{aligned}u(x,y)+&\int_c^d \int_a^b K_1(x,y,s,v)\Psi_1(s,v,u(s,v))\,\mathrm{d}s dv \qquad (12)\\ &+ \int_c^d \int_a^b \bar{K}_2(x,y,s(x,\delta),v(y,\beta))\\ &\Psi_2(s(x,\delta),v(y,\beta),u(s(x,\delta),v(y,\beta))\mathrm{d}\delta\mathrm{d}\beta = g(x,y),\end{aligned}$$

where

$$\bar{K}_2(x,y,s(x,\delta),v(y,\beta)) = \frac{x-a}{b-a}\frac{y-c}{d-c}K_2(x,y,s(x,\delta),v(y,\beta)).$$

We estimate the unknown function $u(x,y)$ as

$$u^\rho(x,y) = \sum_{j=1}^{n} \alpha_j\phi_j(x,y).$$

If in (12) we replace $u(x,y)$ by $u^\rho(x,y)$, we obtain

$$\begin{aligned}\sum_{j=1}^{n} \alpha_j\phi_j(x,y)+&\int_c^d \int_a^b K_1(x,y,s,v)\Psi_1\left(s,v,\sum_{j=1}^{n}\alpha_j\phi_j(s,v)\right)\\ &\mathrm{d}s\,\mathrm{d}v+\int_c^d \int_a^b \bar{K}_2(x,y,s(x,\delta),v(y,\beta))\\ &\Psi_2\left(s(x,\delta),v(y,\beta),\sum_{j=1}^{n}\alpha_j\phi_j(s(x,\delta),v(y,\beta))\right)\\ &\mathrm{d}\delta\mathrm{d}\beta = g(x,y).\end{aligned}$$

If this equation holds at the collocation points (x_i, y_i) we will have

$$\sum_{j=1}^{n}\alpha_j\phi_j(x_i,y_i)+\int_c^d\int_a^b K_1(x_i,y_i,s,v) \tag{13}$$
$$\Psi_1\left(s,v,\sum_{j=1}^{n}\alpha_j\phi_j(s,v)\right)\mathrm{d}s\,\mathrm{d}v$$
$$+\int_c^d\int_a^b \bar{K}_2(x_i,y_i,s(x_i,\delta),v(y_i,\beta))$$
$$\Psi_2\left(s(x_i,\delta),v(y_i,\beta),\sum_{j=1}^{n}\alpha_j\phi_j(s(x_i,\delta),v(y_i,\beta))\right)\mathrm{d}\delta\mathrm{d}\beta$$
$$=g(x_i,y_i).$$

Using a m_1 points quadrature formula with the points $\{s_k\},\{\delta_k\},\{v_p\},\{\beta_p\}$ and weights $\{w_k\},\{w_p\}$ for numerical integration, we obtain

$$\sum_{j=1}^{n}\bar{\alpha}_j\phi_j(x_i,y_i)+\sum_{p=1}^{m_1}\sum_{k=1}^{m_1}w_kw_pK_1(x_i,y_i,s_k,v_p) \tag{14}$$
$$\Psi_1\left(s_k,v_p,\sum_{j=1}^{n}\bar{\alpha}_j\phi_j(s_k,v_p)\right)$$
$$+\sum_{p=1}^{m_1}\sum_{k=1}^{m_1}w_kw_p\bar{K}_2(x_i,y_i,s(x,\delta_k),v(y,\beta_p))$$
$$\Psi_2\left(s(x_i,\delta_k),v(y_i,\beta_p),\sum_{j=1}^{n}\bar{\alpha}_j\phi_j(s(x_i,\delta_k),v(y_i,\beta_p))\right)$$
$$=g(x_i,y_i).$$

The unknowns $\bar{\alpha}_j$ can be found by solving the nonlinear system of algebraic equations which can be solved by any nonlinear solver; in this work we have used the fsolve command of Matlab. So the values of $u(x,y)$ at any point $(x,y)\in[a,b]\times[c,d]$ can be approximated by

$$u^{\rho}(x,y)=\sum_{j=1}^{n}\bar{\alpha}_j\phi_j(x,y)$$

V. Convergence analysis

In this section we will study the error estimation for the proposed method. In [24] Levin presented the error estimates in the uniform norm for a particular weight function in N dimensions, but, he did not obtain error estimates for the derivatives. In [25] Armentano and Duron studied the MLS method for the function and its derivatives in the one dimension obtaining error estimate in L^∞. Armentano in [26] presented the error estimates in L^∞ and L^2 norms for one and N dimensions which generalizes the result given in [25]. Zuppa in [27] proved the error estimates for derivatives of shape function by the condition numbers of the star of nodes in the normal equation. Authors of [28], obtained the error estimates in Sobolev space when $u(x,y)\in C^{m+1}(D)$, and $u(x,y)\in W^{m+1,q}(D)$, respectively. Equation (1) can be represented in abstract form as

$$(I-\mathbf{K})u=g$$

where

$$\mathbf{K}u=\int_0^1\int_0^1 K_1(x,y,s,v)\Psi_1(s,v,u(s,v))\,\mathrm{d}s\,\mathrm{d}v$$
$$+\int_0^y\int_0^x K_2(x,y,s,v)\Psi_2(s,v,u(s,v))\,\mathrm{d}s\,\mathrm{d}v$$

We define the collocation operator $P_n: C(D)\to G_n$ by

$$P_nu(x,y)=\sum_{i=1}^{n}\alpha_i\Phi_i(x,y)\quad (x,y)\in D$$

where $G_n=span\{\Phi_1,\Phi_2,...,\Phi_n\}$ and the coefficients α_i can be determined by solving the linear system

$$P_nu(x_i,y_i)=u(x_i,y_i)\quad i=1,2,...,n$$

we note that (13) can be written in the abstract form

$$(I-P_n\boldsymbol{K})u_n=P_ng. \tag{15}$$

Let the operator $\mathbf{K}_n$ be defined as

$$\mathbf{K}_nu=\sum_{p=1}^{m_1}\sum_{k=1}^{m_1}w_kw_pK_1(x,y,s_k,v_p)\Psi_1\left(s_k,v_p,u(s_k,v_p)\right)$$
$$+\sum_{p=1}^{m_1}\sum_{k=1}^{m_1}w_kw_p\bar{K}_2(x,y,s(x,\delta_k),v(y,\beta_p))$$
$$\Psi_2\left(s(x,\delta_k),v(y,\beta_p),u(s(x,\delta_k),v(y,\beta_p))\right).$$

we can write (14) in the operator form

$$(I-P_n\boldsymbol{K}_n)\hat{u}_n=P_ng. \tag{16}$$

Let $\{T_n,\hat{T}_n\}$ be the operators defined respectively by

$$T_nu=P_n\mathbf{K}u+P_ng,\quad \hat{T}_nu=P_n\mathbf{K}_nu+P_ng$$

So $(15),(16)$ can be written as

$$u_n=T_nu_n,\qquad \hat{u}_n=\hat{T}_n\hat{u}_n.$$

Theorem 1: Suppose P_n is the collocation projection for the shape functions of MLS method corresponding to nodal points $X=\{(x_1,y_1),(x_2,y_2),...,(x_n,y_n)\}$. Assume the family $\{P_n,n\geq 1\}$ is uniformly bounded. If $u\in C^{q+1}(D^*)$ then P_nu converge to u as $n\longrightarrow\infty$ and

$$\|P_nu-u\|_\infty\leq(1+m)Ch_{X,D}^{q+1}|u|_{C^{q+1}}(D^*),$$

where m,C are constants.
We present the following theorem from Vainikko [29] utilised to obtain the error analysis of the proposed method.

Theorem 2: Let T and $\hat{T}$ be continuous over an open set D in Banach space X. Let the equation

$$u=\hat{T}u$$

has an isolated solution $\hat{u}_0$ in D and let the following conditions be satisfied:

- The operator T is Frechet differentiable in some neighborhood of the point $\hat{u}_0$ while the linear operator $I-T^{'}(\hat{u}_0)$ is continuously invertible,
- For some $\eta>0$ and $0<q<1$ the following inequalities are valid (the number $\eta>0$ assumed to be so small that the sphere $\|u-\hat{u}_0\|\leq\eta$ is contained within D).

$$\sup_{\|u-\hat{u}_0\|\leq\eta}\|(I-T^{'}(\hat{u}_0))^{-1}(T^{'}(u)-T^{'}(\hat{u}_0))\|\leq q, \tag{17}$$

$$\alpha=\|(I-T^{'}(\hat{u}_0))^{-1}(T(\hat{u}_0)-\hat{T}(\hat{u}_0))\|\leq\eta(1-q), \tag{18}$$

then the equation $u=Tu$ has in the sphere $\|u-\hat{u}_0\|\leq\eta$ a unique solution u_0. Moreover, the inequality

$$\frac{\alpha}{1+q}\|u_0-\hat{u}_0\|\leq\frac{\alpha}{1-q}$$

is valid.

Theorem 3: let $u_0\in C([0,1]\times[0,1])$ be an isolated solution of

$$u=Ku+g$$

Assume that 1 is not an eigenvalue of the linear operator $T^{'}(u_0)$. Then for sufficiently large n, the operator $(I-\hat{T}_n^{'}(u_0))^{-1}$ is invertible and there exists constant $L>0$ independent of n such that $\|(I-\hat{T}_n^{'}(u_0))^{-1}\|\leq L$.

Theorem 4: let $u_0\in C([0,1]\times[0,1])$ be an isolated solution of

$$u=Ku+g.$$

Assume that 1 is not an eigenvalue of the linear operator $T^{'}(u_0)$. Then for sufficiently large n, the approximate solution $\hat{u}_n$ of (16) is unique in $B(u_0, \eta) = \{u : \|u - u_0\| \leq \eta\}$ for some $\eta > 0$. Moreover, there exists a constant $0 < q < 1$ independent of n such that

$$\frac{\varphi_n}{1+q} \leq \|u_0 - \hat{u}_n\| \leq \frac{\varphi_n}{1-q},$$

where $\varphi_n = \|(I - \hat{T}_n^{'}(u_0))^{-1}(\hat{T}_n(u_0) - T(u_0))\|$.

Proof: Applying theorem 3 we have $(I - \hat{T}_n^{'}(u_0))^{-1}$ exists and it is uniformly bounded i.e, there exists a constant $L > 0$ such that

$$\|(I - \hat{T}_n^{'}(u_0))^{-1}\| \leq L.$$

Assume that $\|P_n\| < p$ and $\|\mathbf{K}_n'\| < M$, then

$$\begin{aligned}\|\hat{T}_n^{'}(u) - \hat{T}_n^{'}(u_0)\| &= \|P_n\mathbf{K}_n^{'}u - P_n\mathbf{K}_n^{'}u_0\| \\ &\leq \|P_n\|\|\mathbf{K}_n^{'}u - \mathbf{K}_n^{'}u_0\| \\ &\leq pM\eta, \quad \forall u \in B(u_0, \eta).\end{aligned}$$

Thus, we obtain

$$\sup_{\|u-\hat{u}_0\|\leq\eta} \|(I - \hat{T}_n^{'}(u_0))^{-1}(\hat{T}_n^{'}(u) - \hat{T}_n^{'}(u_0))\| \leq LpM\eta \leq q,$$

where $0 < q < 1$, which demonstrates (17) for η sufficiently small. Also we have

$$\begin{aligned}\varphi_n &= \|(I - \hat{T}_n^{'}(u_0))^{-1}(\hat{T}_n(u_0) - T(u_0))\| \\ &\leq \|(I - \hat{T}_n^{'}(u_0))^{-1}\|\|(\hat{T}_n(u_0) - T(u_0))\| \\ &\leq L\|(\hat{T}_n(u_0) - T(u_0))\|.\end{aligned}$$

Here, we will prove that $\|(\hat{T}_n(u_0) - T(u_0))\| \longrightarrow 0$ as $n \longrightarrow \infty$. Now consider

$$\begin{aligned}&\|(\hat{T}_n(u_0) - T(u_0))\| = \|P_n\mathbf{K}_nu_0 + P_nf - \mathbf{K}u_0 - g\| \\ &\leq \|P_n\{\mathbf{K}_nu_0 + g - \mathbf{K}u_0 - g\}\| + \|(P_n - I)(\mathbf{K}u_0 + g)\| \\ &\leq p\|\mathbf{K}_n - \mathbf{K}\|\|u_0\| + \|P_nu_0 - u_0\| \longrightarrow 0 \quad as\, n \longrightarrow \infty \quad (19)\end{aligned}$$

For sufficient large n we have $\beta_n \leq \eta(1 - q)$. Since (18) is satisfied. Then from theorem 2 we have

$$\frac{\varphi_n}{1+q}\|\hat{u}_n - u_0\| \leq \frac{\varphi_n}{1-q}$$

■

We complete the error estimate by the following theorem.

Theorem 5: let $u_0 \in C^{q+1}(D^*)$ be an isolated solution of the equation $u = Kz + g$ and u_n be the discrete MLS collocation method of u_0. Then we have

$$\begin{aligned}\|\hat{u}_n - u_0\|_{L^\infty(D)} &\leq \frac{Lp}{1-q}\|K_n - K\|_{L^\infty(D)}\|u_0\|_{L^\infty(D)} \\ &+ \frac{L}{1-q}(1+m)Ch_{X,D}^{q+1}|u_0|_{C^{q+1}(D^*)},\end{aligned}$$

Proof: We have

$$\|\hat{u}_n - u_0\| \ \leq \frac{\varphi_n}{1-q}$$

Utilizing theorems 1, 3, and Eq. (19) we obtain

$$\begin{aligned}\|\hat{u}_n - u_0\|_{L^\infty(D)} &\leq \frac{\varphi_n}{1-q} \\ &\leq \frac{\|(I - \hat{T}_n^{'}(u_0))^{-1}(\hat{T}_n(u_0) - T(u_0))\|_{L^\infty(D)}}{1-q} \\ &\leq \frac{L}{1-q}\|(\hat{T}_n(u_0) - T(u_0))\|_{L^\infty(D)} \\ &\leq \frac{L}{1-q}\|P_N\|_{L^\infty(D)}\|K_n - K\|_{L^\infty(D)}\|u_0\|_{L^\infty(D)} \\ &+ \frac{L}{1-q}\|P_nu_0 - u_0\|_{L^\infty(D)} \\ &\leq \frac{Lp}{1-q}\ \|K_n - K\|_{L^\infty(D)}\|u_0\|_{L^\infty(D)} \\ &+ \frac{L}{1-q}(1+m)Ch_{X,D}^{q+1}|u_0|_{C^{q+1}(D^*)}.\end{aligned}$$

■

VI. Numerical results

To show the validity of the method as a numerical tool. Linear and nonlinear Fredholm-Volterra integral equations are solved. For numerical implementation we put $h_X = \frac{1}{n-1}$, then in computations of the MLS method we put for linear case $\rho_i = 2 \times h_X$, for quadratic case $\rho_i = 2.5 \times h_X$, for degree 3 case $\rho_i = 3 \times h_X$, and for degree 4 case $\rho_i = 4 \times h_X$, where h_X is the distance between two consecutive nodes. When we use the MMLS method, we take for quadratic case $\rho_i = 2 \times h_X$. Also, we use the 5-points Gauss-Legendre quadratic rule for numerical integration and spline weight functions for approximating integrals in the scheme. Furthermore, for computing shape function in MMLS method, we take $\nu_e = 10^{-9}$; with $e = 1; 2; 3$ as weights of additional coefficients for MMLS, it should be pointed out that, this value was selected experimentally. Accuracy of the numerical solutions can be worked out by measuring the $\|e\|_\infty$ and $e(x, y)$ norms which are defined by

$$\begin{aligned}\|e\|_\infty &= \max|u_{ex}(x, y) - \hat{u}(x, y)|, \quad (x, y) \in D, \\ e(x, y) &= |u_{ex}(x, y) - \hat{u}(x, y)|, \qquad (x, y) \in D.\end{aligned}$$

where $\hat{u}$ is the approximate solution of the exact solution u_{ex}. The rate convergence presented in this work is defined as

$$Ratio = \frac{\ln(\|e_n\|_\infty) - \ln(\|e_{n'}\|_\infty)}{\ln(h_X) - \ln(h_{X'})},$$

$(\|e_n\|_\infty, \|e_{n'}\|_\infty)$ are the maximum errors of the previous and the current row respectively. The "Fsolve" command is employed to solve the nonlinear system of algebraic equations. All calculations are done by Matlab.

A. *Example 1: 2-D linear*

As the first example, consider the two-dimensional linear Fredholm-Volterra integral equation

$$\begin{aligned}&u(x, y) + \int_0^1\int_0^1 cos(x - s)\exp(v)u(s, v)\,\mathrm{d}s\,\mathrm{d}v \\ &+ \frac{1}{2}\int_0^y\int_0^x sin(x - s)\exp(v - y)u(s, v)\,\mathrm{d}s\,\mathrm{d}v = g(x, y).\end{aligned}$$

where

$$\begin{aligned}g(x, y)) &= sin(x)\exp(-y) + \frac{y}{4}\exp(-y)\Big(sin(x) - xcos(x)\Big) \\ &+ \frac{1}{4}\Big(2sin(x) - cos(2 - x) + cos(x)\Big), \quad 0 \leq x, y \leq 1.\end{aligned}$$

The function $g(x, y)$ has been chosen such that the exact solution of the integral equation is $u_{ex}(x, y) = sin(x)\exp(-y)$. The integral equation is solved on the square domain $D = [0, 1] \times [0, 1]$. Numerical results are presented in Table I in terms of $\|e\|_\infty$ at different numbers of n, and Figure 1 is depicting the absolute error with $(n = 43 \times 43)$ points. As we expected, from Figure 1, the error near the boundary increases which effects on the global error. As we can see, the results gradually converge to the exact values as the number of nodes increases, and the obtained results by the MMLS method are better than the results given by the MLS method. According to the Theorem 5, the ratio of error remains approximately constant for the linear case $O(h_X^2)$ and for the quadratic case $O(h_X^3)$ as $N \to \infty$.

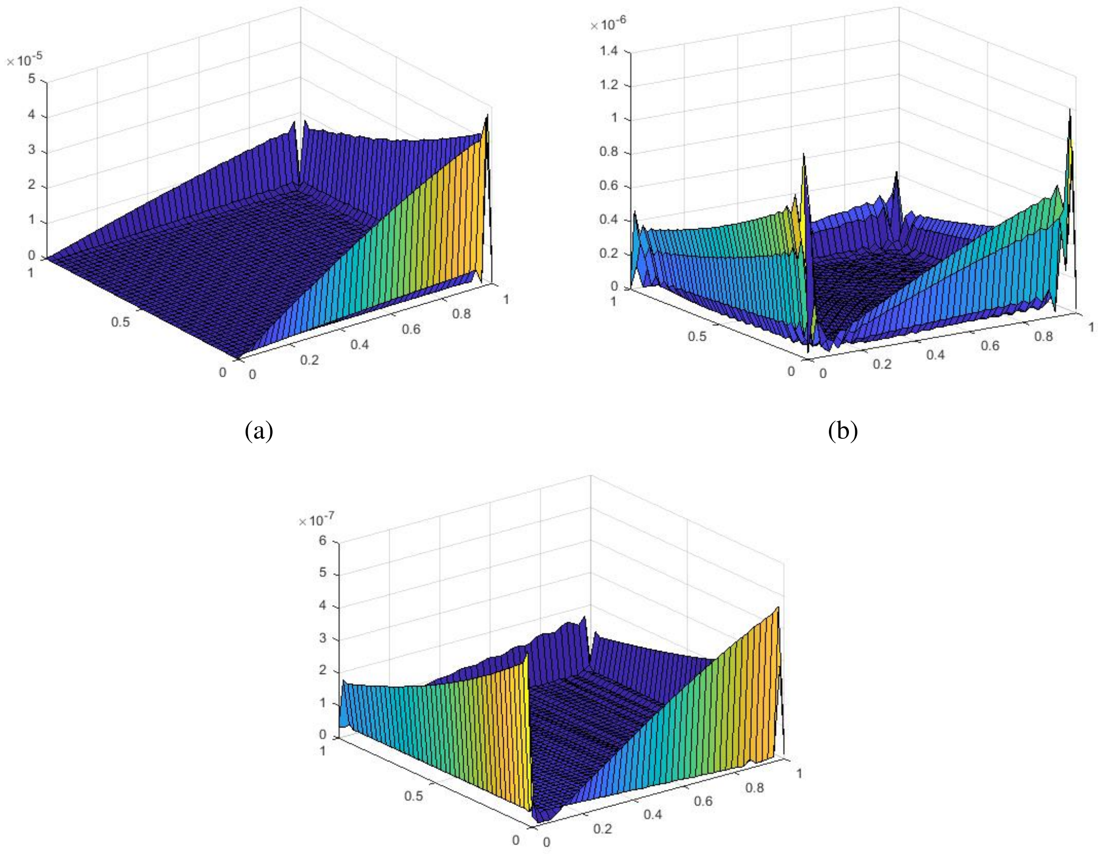
B. *Example 2: 1-D nonlinear*

Consider the following nonlinear Volterra-Fredholm integral equation by using the shifted Chebyshev polynomials as basis

$$\begin{aligned}&u(x) - \int_0^x sin(x - s)cos(u(x))\,\mathrm{d}s \\ &- \frac{1}{8}\int_0^1 (x - s)u(x)\,\mathrm{d}s = g(x), \quad x \in [0, 1],\end{aligned}$$

TABLE I
MAXIMUM ERRORS USING DIFFERENT VALUES OF N FOR EXAMPLE 1

n	h_X	MLS approximation error				MMLS approximation error	
		Linear basis	Ratio	Quadratic basis	Ratio	Quadratic basis	Ratio
3×3	0.50	1.43×10^{-2}	-	7.47×10^{-3}	-	2.83×10^{-3}	-
5×5	0.25	4.41×10^{-3}	1.69	1.30×10^{-3}	2.52	4.77×10^{-4}	2.56
9×9	0.12	1.22×10^{-3}	1.85	1.73×10^{-4}	2.90	6.36×10^{-5}	2.90
19×19	0.05	2.54×10^{-4}	1.93	1.55×10^{-5}	2.97	7.08×10^{-6}	2.70
37×37	0.02	6.46×10^{-5}	1.97	1.94×10^{-6}	2.99	9.17×10^{-7}	2.94
43×43	0.02	4.75×10^{-5}	1.99	1.21×10^{-6}	3.06	5.36×10^{-7}	3.48



(a) (b) (c)

Fig. 1. Approximation error :(a) Linear basis(MLS), (b) Quadratic basis(MLS), (c) Quadratic basis(MMLS) of Example 1

where

$$g(x) = \frac{49}{48} - \frac{17x}{16} - \frac{1}{2}\Big(sin(x)(xcos(1) + sin(1)) - xsin(1)cos(x)\Big).$$

The exact solution of this equation is $u_{ex}(x) = 1 - x$.

TABLE II
MAXIMUM ERRORS USING DIFFERENT VALUES OF DEGREE BASIS M FOR EXAMPLE 2

n	h_X	$m=1$	$m=2$	$m=3$	$m=4$
5	0.25	8.65×10^{-4}	3.56×10^{-5}	3.65×10^{-6}	1.37×10^{-6}
9	0.12	2.89×10^{-4}	2.88×10^{-6}	1.35×10^{-6}	3.95×10^{-8}
15	0.07	1.36×10^{-4}	1.12×10^{-6}	1.55×10^{-7}	2.65×10^{-9}
21	0.05	9.51×10^{-5}	2.50×10^{-7}	4.59×10^{-8}	4.43×10^{-10}
25	0.04	7.61×10^{-5}	1.30×10^{-7}	2.78×10^{-8}	2.78×10^{-10}

Table II shows $\|e\|_\infty$ at the different numbers of nodes that are regularly employed in the segment. According to the table, when the degree basis increases, the numerical results show the higher performance of the MLS method with the Chebyshev basis, and also the results converge to the exact values as the number of nodes increases.

C. *Example 3: 2-D nonlinear*

Consider the following two-dimensional nonlinear Fredholm-Volterra integral equation defined as :

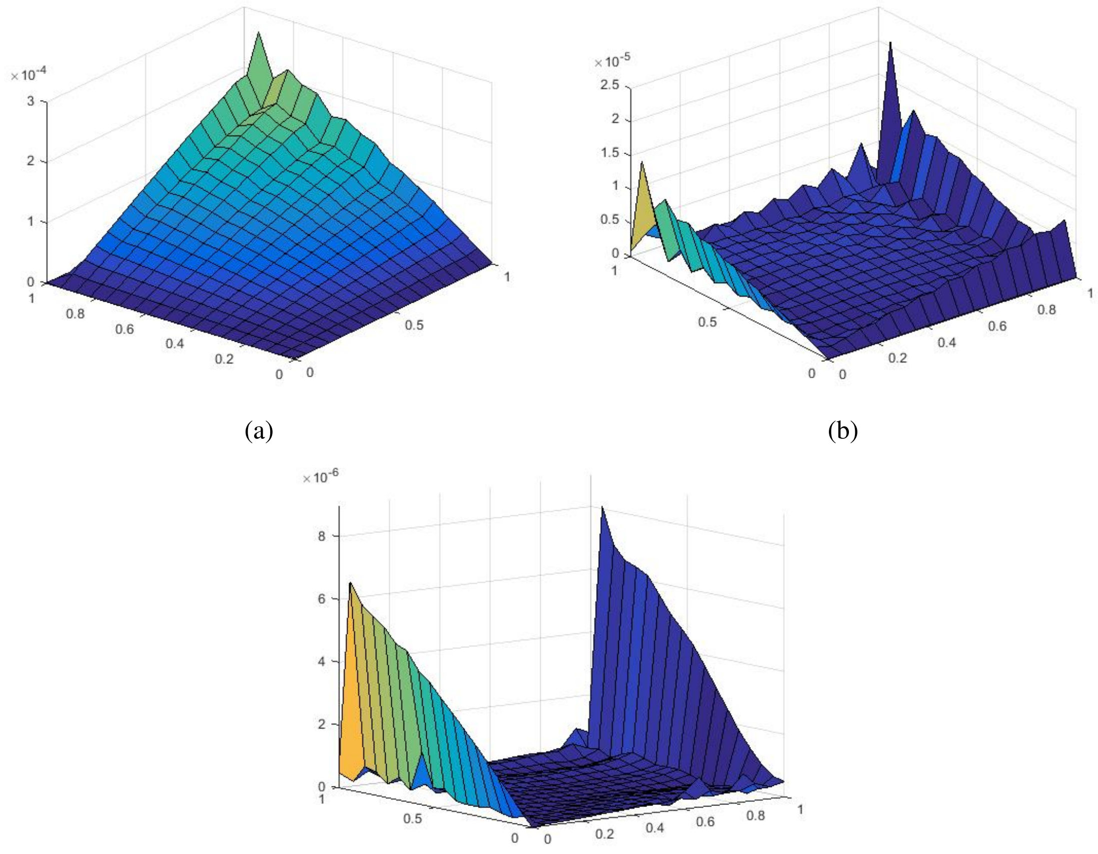
$$u(x,y) - \int_0^1\int_0^1 (v-s)sin(x-y)u(s,v)\,\mathrm{d}s\,\mathrm{d}v - \int_0^y\int_0^x (xs^2 + cos(v))u^2(s,v)\,\mathrm{d}s\,\mathrm{d}v = g(x,y), \quad 0 \le x,y \le 1.$$

Where

$$g(x,y) = xsin(y)\Big(1 - \frac{1}{9}x^2 sin^2(y)\Big) + \frac{1}{10}x^6\Big(\frac{1}{2}sin(2y) - y\Big) + \frac{1}{6}sin(x-y)\Big(2 - 3sin(1) + cos(1)\Big).$$

TABLE III
MAXIMUM ERRORS USING DIFFERENT VALUES OF N FOR EXAMPLE 3

n	h_X	MLS approximation error				MMLS approximation error	
		Linear basis	Ratio	Quadratic basis	Ratio	Quadratic basis	Ratio
3×3	0.50	2.23×10^{-2}	-	1.08×10^{-2}	-	7.51×10^{-3}	-
4×4	0.33	1.09×10^{-2}	1.76	3.89×10^{-3}	2.51	1.55×10^{-3}	3.89
5×5	0.25	6.59×10^{-3}	1.74	1.56×10^{-3}	3.17	4.93×10^{-4}	3.98
9×9	0.12	1.91×10^{-3}	1.78	1.89×10^{-4}	3.04	6.35×10^{-5}	2.95
12×12	0.09	8.43×10^{-4}	2.56	8.04×10^{-5}	2.68	2.80×10^{-5}	2.57
18×18	0.05	2.66×10^{-4}	2.64	2.07×10^{-5}	3.11	8.07×10^{-6}	2.43



(a) (b) (c)

Fig. 2. Approximation error :(a) Linear basis(MLS), (b) Quadratic basis(MLS), (c) Quadratic basis(MMLS) of Example 3

The exact solution is $u_{ex}(x, y) = xsin(y)$. Table III shows the maximum errors for different values of m and n that are regularly employed in the unit square, and the absolute error for ($n = 18 \times 18$) is graphically shown in figure 2. We can see that when the values of n increases, the maximum errors decrease, the results of MMLS approximation converge more quickly to the true solution than that of MLS approximation. The ratio of error remains constant for the linear case (≈ 2) and the quadratic case (≈ 3). So the numerical results show that the proposed method will be of $O(h_X^{q+1})$ as it is expected in Theorem 5.

D. *Example 4: 2-D nonlinear*

As the final example, we consider the following two-dimensional nonlinear Fredholm-Volterra integral equation

$$u(x,y) - \int_0^1 \int_0^1 \exp(-x-y-4) u^2(s,v)\,\mathrm{d}s\,\mathrm{d}v \\ - \frac{1}{4}\int_0^y \int_0^x (ys+xv)u(s,v)\,\mathrm{d}s\,\mathrm{d}v = g(x,y),\ x \in [0,1].$$

where

$$g(x,y) = \frac{1}{2}\exp(y) + xy - \frac{1}{72}\exp(-x-y-4)\Big(35 \\ + 9\exp(2)\Big) - \frac{x^2}{48}\Big(4y^3x - 3y + \exp(y)(9y-6) + 6\Big).$$

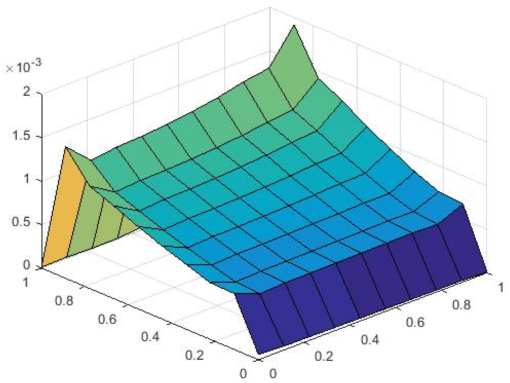
The analytic solution of this problem is $u_{ex}(x, y) = \frac{1}{2}exp(y) + xy$. The maximum errors using MLS and MMLS approximation are shown in table IV, the absolute error with ($n = 10 \times 10$) points is plotted in figure 3. The results, presented in table V, support the maximum errors using different values of ν when $n = 5 \times 5$, this value was selected experimentally, we can see that MMLS approximation gives better accuracy when we put ν as small positive numbers. As shown, from table IV, the faster method is modified moving least squares method, also as n (the number of nodes) increases, the error term decreases in both MLS and MMLS approximation. Apparently, the method provides accurate numerical solutions for mixed integral equations.

TABLE IV
MAXIMUM ERRORS USING DIFFERENT VALUES OF N FOR EXAMPLE 4

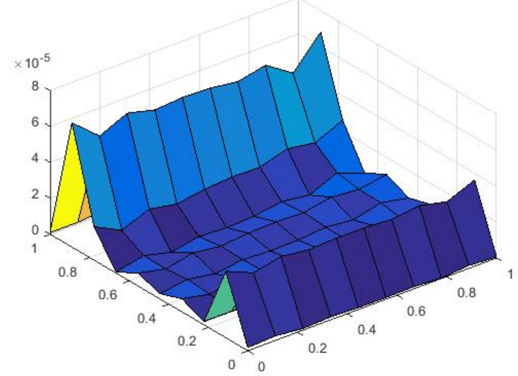
n	h_X	MLS approximation error		MMLS approximation error
		Linear basis	Quadratic basis	Quadratic basis
3×3	0.50	2.44×10^{-2}	1.03×10^{-3}	5.39×10^{-4}
5×5	0.25	7.99×10^{-3}	6.20×10^{-4}	6.31×10^{-5}
6×6	0.20	5.60×10^{-3}	3.60×10^{-4}	2.19×10^{-5}
10×10	0.11	1.91×10^{-3}	6.91×10^{-5}	4.68×10^{-6}

TABLE V
MAXIMUM ERRORS USING DIFFERENT VALUES OF ν AND $n = 5 \times 5$ FOR EXAMPLE 4

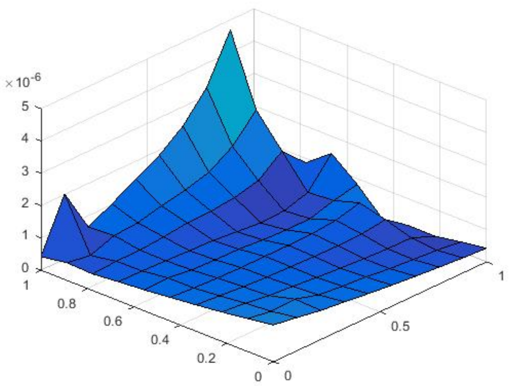
ν	10^{-6}	10^{-8}	10^{-9}	10^{-10}
$\|e\|_\infty$	1.42×10^{-4}	6.34×10^{-5}	6.31×10^{-5}	6.31×10^{-5}



(a)



(b)



(c)

Fig. 3. Approximation error :(a) Linear basis(MLS), (b) Quadratic basis(MLS), (c) Quadratic basis(MMLS) of Example 4

VII. CONCLUSION

The linear and nonlinear Fredholm-Volterra integral equations are usually difficult to solve analytically. As a result, it is required to obtain approximate solutions. In this work, we presented the moving least squares and modified moving least squares methods for solving two-dimensional linear and nonlinear Fredholm-Volterra integral equations. The numerical examples show that the approximation of the MMLS gave more accurate results than that of the classical MLS. The most advantage of the MMLS method is the ability to get an approximation for cases when classical MLS with quadratic base functions fail because of a singular moment matrix. The efficiency of the obtained solutions can be improved by taking more nodes in the rectangular domain. The absolute and maximum errors have been used to measure the accuracy of the method, also the rate convergence is examined with some examples. The achieved results show the validity and accuracy of the new technique and have confirmed the theoretical error estimates. The MMLS approximation is a powerful tool for solving mixed integral equations and it can be easily extended to three-dimensional problems due to the easy adaption of the MMLS method for the 3-D space.

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