Adjustable Robust Maximum Flow Problem with Parametric Ellipsoidal and Polyhedral Uncertainty Set

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***Abstract*—In this paper the Adjustable Robust Maximum Flow Problem (ARMFP) is discussed. The problem is considered as a two-stage optimization problem with two kinds of variables, i.e., adjustable and non-adjustable variables. There is also an assumption that the input parameter, i.e. arc capacities, lie within an uncertainty set. The main challenge in Adjustable Robust Optimization (ARO) is to find whether the robust counterpart of ARMFP be can be formulated into a computationally tractable optimization problem. To this end, a convex continue set is assumed to be the set of the adjustable variables and or the uncertain arc capacities. In this paper it is considered the parametric ellipsoidal and polyhedral uncertainty set. The ARMFP is constructed by defining the maximum flow for the whole network represented by a flow x_{ts} that connect the destination node t back to the source node s. This x_{ts} is assumed to be the adjustable variable. In the case of parametric ellipsoidal uncertainty, the characteristic of ARMFP is analysed using the Theorem of Max-Flow and Min- Cut. In the case of polyhedral uncertainty set, the counterpart is obtained as a linear programming problem. Some examples are presented.**

***Index Terms*—Maximum Flow Problem, Robust Optimization, Adjustable Robust Counterpart, Parametric Ellipsoidal, Polyhedral, Uncertainty Set.**

I. Introduction

THE maximum flow problem (MFP) arises in a wide variety of situations and in several forms. For example, determining the maximum steady state flow of petrol in a pipeline network, cars in a road network, messages in a telecommunication network and electricity in an electrical network. Several ways to solve the problem of maximum flow, including using linear programming. Furthermore, the problem of maximum flow has also become an important problem in daily life. Refers to Ford [1], the maximum flow was first introduced by L.R. Ford and D.R. Fulkerson, in 1956, and continues to be developed to date. However, in reality, there are uncertainty factors that affect the network system. One of the uncertainty factors is the flow capacity on each side in a network that can change. For this reason, optimization techniques are needed that take into account uncertainty in order to obtain an optimal solution that is resistant to data uncertainty.

One area of optimization that is able to solve various problems related to the problem of uncertainty is Robust Optimization (RO) as discussed by Ben-Tal and Nemirovskii in [2]. RO has been widely practiced by previous researchers. Referring to Ben-Tal *et al.* The first step towards achieving the RO method is carried out by A.L. Soyster (see [3]) who discussed how to find a solution that is resistant to data uncertainty in linear programming, In 1995, Mulvey *et al.* ini [4] discussed RO of large-scale systems. Ben-Tal *et al.*. in [5] discuss RO in detail. Gorissen *et al.* in [6] given practical guidance on RO. Until now RO has been growing from year to year. In this framework, the existence of uncertainties in the uncertain optimization model is handled using a methodology so-called Robust Counterpart (RC) Methodology.

This RC is formulated by considering that feasible solutions for all possibilities by using the set of box, ellipsoidal, and polyhedral uncertainty. Referring to Ben-Tal *et al.* in [2] also Chaerani and Roos in [7], the main challenge of RO is to find a set of uncertainties that can be formulated into a computationally tractable optimization problem. Computationally tractable can be analyzed by representing the RC into one of three classes of optimization problems, namely linear programming, conic quadratic, or semidefinit programming. A discusion on some applications of RO in industrial and environmental problem can be seen in Chaerani *et al.* [8]. A recent discussion on a survey of nonlinear robust optimization by Leyffer *et al.* in [9] mentioned that RO is dedicated to solving optimization problems subject to uncertainty: design constraints must be satisfied for all the values of the uncertain parameters within a given uncertainty set. Uncertainty sets may be modeled as deterministic sets (boxes, polyhedra, ellipsoids).

For the last ten years, research on the topic of robust maximum flow problem (RMFP) is still progressive. We conduct a literature study to know what type of approach can be used to model and solve the RMFP. A recent one is a discussion on Heuristic Solutions to robust variants of the minimum-cost integer flow problem, as we can see in Spoljarec [10]. Another discussion on maximum flow-based network interdiction problem considering uncertainties in arc capacities and interdiction resource consumption is presented in Chauhan [11]. An Iterative Security Game for Computing Robust and Adaptive Network Flows is discussed by Ridremont in [12]; Gottchalk in [13] discussed Robust

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flows over time: models and complexity results. The cooperative maximum-flow problem under uncertainty in logistic networks is discussed in Hafezalkotob [14]. Furthermore, in Bertsimas [15], it is shown that robust maximum flow problem can be solved in polynomial time, but the robust minimum cut problem is NP-hard. Minoux in [16] shows that robust network optimization under polyhedral demand uncertainty is NP-hard and also discusses its computational tractability.

Han in [17] presents the maximum flow problem of uncertain network with the arc capacities of the network as uncertain variables. The main purpose of this paper is to solve the maximum flow in an uncertain network by considering the uncertain arc capacities as random or fuzzy variables. Ding *et al.* in [18] discusses the α-maximum flow model with uncertain capacities. Dynamic network design problem under demand uncertainty: an adjustable RO approach is presented in [19]. Moolman in [20] discusses the Maximum Flow and Minimum Cost Maximum Flow Problems: Computing and Applications. The complexity of computing a robust flow is discussed in Disser [21].

RO can be categorised into two types, i.e., the single stage and two-stage models. In RO, the single stage optimization, all decision variables with here and now decisions are considered to be a problem to be resolved immediately. Meanwhile, in the two-stage optimization with the wait and see decision, the decision variables in the second stage are adjusted to the realization of parameter uncertainty. This two-stage RO is known as Adjustable Robust Optimization (ARO) (see Bental *et al.* [22] and [23]). Two-stage RO, state-space representable uncertainty and applications can be seen in Minoux [24]. Atamturk in [25] discusses two-stage robust network flow and design under demand uncertainty. Related work on adjustable RO for maximum flow problem with box uncertainty is discussed in Agustini *et al.* [26].

In this paper, a discussion on Adjustable Robust Counterpart Optimization Model (ARCOM) for UMFP with parametric ellipsoidal uncertainty set and polyhedral uncertainty is presented. The paper is organized as follows. Section III briefly introduces the theory of Max-Flow Min-Cut Theorem, RO, ARO and RMFP. Section IV is devoted to discuss the ARC of MFP and the characteristic of RMF function over the parametric ellipsoidal. In Section IV-B, we discuss a parametric variant of the above ARC-MFP, where the sizes of the uncertainty in c are controlled by a nonnegative scaling parameter. In Section V, we discuss some examples of ARC-MFP. Conclusions can be found in Section VI.

II. Problem Description

In this paper, we discuss the adjustable robust counterpart model for UMFP with an assumption that the uncertain arcs lies in a parametric ellipsoidal and in a polyehedral uncertainty set. In the concept of ARO, the problem considered as two-stage optimization problem. There are two kinds of variables in ARO, i.e., adjustable and non-adjustable variables. For the case of UMFP, the maximum flow is represented by a flow x_{ts} from the destination node t back to the source node s. This x_{ts} is assumed to be the adjustable variable. The UMFP is formulated as a linear programming problem with uncertain arcs capacities. The ARO version for UMFP is called Adjustable Robust Optimization Model for Maximum Flow Problem (ARO-MFP). The main aim is to determine that the ARO-MFP is computationally tractable. We also discuss the case of parametric ellipsoidal uncertainty and its characteristic of ARO-MFP is analysed using the Theorem of Max-Flow and Min-Cut.

To this end, let recall the MFP problem as follows (see Schrijver [27]. Let $G = (\mathcal{V}, \mathcal{A})$ be a directed graph, let $s, t \in \mathcal{V}$ and let $c : \mathcal{A} \to \mathrm{Q}_+$ be a capacity function. The objective in the MFP is to find an $s-t$ flow of maximum value under c. Adding an arc from t to s with $c_{ts} = \infty$, the maximum $t - s$ flow problem can be formulated as

$$\max\{x_{ts} \ : \ Ax = 0, 0 \leq x \leq c\}, \tag{1}$$

where A is the node-arc incidence matrix and x is the vector of flow variables.

Recall the robust maximum flow problem (RMFP) in Chaerani *et al.* [28], the following is a model to handle the maximum flow problem with uncertain arc capacities that belong to a so-called uncertainty set $\mathcal{U}$. We then have to deal with a whole family of maximum flow problems, namely

$$\mathcal{H} = \{\max\{x_{sr} \ : \ Ax = 0, 0 \leq x \leq c\} \ : \ c \in \mathcal{U}\}. \tag{2}$$

Referring to Chaerani *et al.* [28], the major challenge is when and how we can reformulate (2) as a computationally tractable optimization problem. The flow is must be feasible under all possible values of $c \in \mathcal{U}$, and the maximum flow value under this condition is obtained, thus the problem is called the robust maximum flow problem (RMFP), and the flow of maximum value under the uncertain arc capacities is called the robust maximum flow (RMF) value. A natural assumption is used; the network $G = (\mathcal{V}, \mathcal{A})$ is fixed, as well as the nodes s and t. Thus, the uncertainty occurs only in the vector c of arc capacities. We assume $c \in \mathcal{U}$, when $\mathcal{U}$ is the uncertainty set for c. By Robust Linear Optimization methodology (see [5]), the robust counterpart of the RMFP can be stated as

$$\max\{x_{ts} \ : \ Ax = 0, 0 \leq x \leq c, \forall c \in \mathcal{U}\}. \tag{3}$$

Hence, the objective is to find the maximum value of a flow that satisfies $x \leq c$ for all $c \in \mathcal{U}$ where $c_{ts} = \infty$. Certainly, this robust counterpart depends on how we choose the uncertainty set $\mathcal{U}$. In Chaerani *et al.* [28], two different uncertainty sets, namely box and ellipsoidal uncertainty sets are considered.

A case of ellipsoidal uncertainty is considered in [28] is discussed as follows.

$$\mathcal{U} = \{c \ : \ c = c^0 + \mathcal{Q}\zeta, \|\zeta\|_2 \leq 1\}, \tag{4}$$

where $\mathcal{Q}$ is a fixed matrix of size $|\mathcal{A}| \times p$ and $\zeta \in \mathbf{R}^p$ for some p.

Lemma 1: Referring to Chaerani *et al.* [28], the flow x is robust feasible if and only if

$$0 \leq x_a \leq c_a^0 - \|\mathcal{Q}_a\|, \qquad \forall a \in \mathcal{A} \tag{5}$$

where c_a^0 is the nominal capacity on arc a and $\mathcal{Q}_a$ is the row of $\mathcal{Q}$ corresponding to arc a.

Proof: The flow x_a on arc a must satisfy

$$x_a \leq c_a^0 + \mathcal{Q}_a\zeta, \qquad \forall\zeta \ : \ \|\zeta\|_2 \leq 1. \tag{6}$$

This means that

$$x_a \le c_a^0 + \min_{\zeta} \{ \mathcal{Q}_a \zeta \ : \ \|\zeta\|_2 \le 1 \} . \quad (7)$$

The minimum at the right hand side is attained when

$$\zeta = - \frac{\mathcal{Q}_a^T}{\|\mathcal{Q}_a\|_2}, \quad (8)$$

whence the capacity of arc a becomes

$$c_a^0 - \mathcal{Q}_a \frac{\mathcal{Q}_a^T}{\|\mathcal{Q}_a\|_2} = c_a^0 - \frac{\|\mathcal{Q}_a\|^2}{\|\mathcal{Q}_a\|_2} = c_a^0 - \|\mathcal{Q}_a\|_2 . \quad (9)$$

This implies that also in this case, the RMFP is a usual maximum flow problem, with the nominal capacities c_a^0 replaced by $c_a^0 - \|\mathcal{Q}_a\|$, $a \in \mathcal{A}$. So we have proved the next result (see [28]).

Theorem 1: The RMFP with ellipsoidal uncertainty as given by (4), is equivalent to

$$\max\{x_{ts} \ : \ Ax = 0, 0 \le x_a \le c_a^0 - \|\mathcal{Q}_a\| , a \in \mathcal{A}\}. \quad (10)$$

III. Methodology

To formulate the Adjustable Robust Counterpart Optimization model for the maximum flow problem it must be known in advance about Theorem of Max-Flow Min-Cut for of the maximum flow problem, RO, and Adjustable Robust Counterpart Optimization must be known.

A. Max-Flow Min-Cut Theorem

In this subsection, we briefly recall some well known definitions and results about the maximum flow problem including the max flow-min cut theorem (see Schrijver [27]).

Definition 1: For a given network $G = (\mathcal{V}, \mathcal{A})$ and $s, t \in \mathcal{V}$, a function $x \ : \ \mathcal{A} \to \mathbf{R}$ is called an $s - t$ flow if

$$x_a \ge 0 \quad \text{for each} \quad a = (i, j) \in \mathcal{A}, \quad (11)$$

and

$$\sum_{a \in \delta^+(j)} x_a = \sum_{a \in \delta^-(j)} x_a \quad \text{for each} \quad j \in \mathcal{V} \setminus \{s, t\}, \quad (12)$$

where $\delta^+(j)$ and $\delta^-(j)$ denote the sets of arcs leaving j and entering j, respectively. Condition (12) is called the flow conservation law, i.e., the famount of flow entering a node $j \ne s, t$ is equal to the amount of the flow leaving j.

The value of an $s - t$ flow is, by definition, the net flow entering the network. So, the value is given by

$$x_{ts} = \sum_{a \in \delta^+(s)} x_a - \sum_{a \in \delta^-(s)} x_a. \quad (13)$$

As (12) equals the net flow leaving the network, hence we have also

$$x_{ts} = \sum_{a \in \delta^+(t)} x_a - \sum_{a \in \delta^-(t)} x_a. \quad (14)$$

Let $c : \mathcal{A} \to \mathbf{R}_+$ be a capacity function. We say that a flow x is under c (or subject to c) if

$$x_a \le c_a \quad \text{for each} \quad a \in \mathcal{A}. \quad (15)$$

Let $X \subseteq \mathcal{V}$ with $s \in X$ and $t \notin X$. Then the set $\delta^+(X)$

$$\delta^+(X) = \{a = (i, j) \in \mathcal{A} \ : \ i \in X, j \notin X\} \quad (16)$$

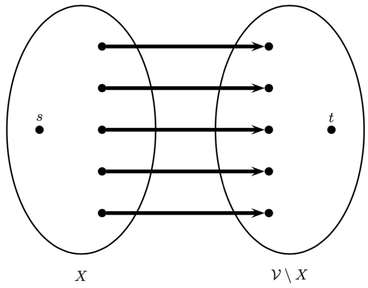

Fig. 1. $s - t$ cut determined by node set X

is called an $s - t$ cut, more specifically, the $s - t$ cut is determined by the nodes set X (see Figure 1). Note that when removing the edges in an $s - t$ cut $\delta^+(X)$ from the network then there is no longer a path from s to t, and hence no flow can be sent through the network. The capacity of a cut $\delta^+(X)$ is defined by

$$c(\delta^+(X)) = \sum_{a \in \delta^+(X)} c_a. \quad (17)$$

The following theorem holds.

Theorem 2 (cf. Schijver [27]): For every flow x and every $s - t$ cut $\delta^+(X)$ one has:

$$x_{ts} \le c(\delta^+(X)). \quad (18)$$

Equality holds if and only if $x_a = c_a$ for each $a \in \delta^+(X)$ and $x_a = 0$ for each $a \in \delta^-(X)$.

Inequality (18) in Theorem 2 is called the weak duality theorem for the max flow problem. Referring to Schijver [27], the following theorem, which is known as the max flow-min cut theorem is known.

Theorem 3: A strong dual to the maximum $s - t$ flow problem (1) is

$$\min_X \left\{ \sum_{a \in \delta^+(X)} c_a \ : \ s \in X \subset \mathcal{V} \setminus \{t\} \right\}, \quad (19)$$

which is the minimum $s - t$ cut problem.

B. Robust Optimization

Referring to Ben-Tal *et al.* in [5], RO is a method to solve optimization problems with data uncertainty and is only known in a set of uncertainties. The general form of the problem of indefinite linear optimization can be formulated as in equation (20) follows:

$$\begin{aligned} &\min \ c^T x \\ &s.t \ \ Ax \le b \\ &\qquad (c, A, b) \in \mathcal{U}. \end{aligned} \quad (20)$$

where $c \in R^n$, $A \in R^{mxn}$, $b \in R^n$, the three decision variables are indefinite coefficients. $\mathcal{U}$ is a notation of the set of uncertainties.

There are three basic assumptions in RO, namely all decision variables state decisions "here and now", decision makers are fully responsible for the consequences of decisions made, if and only if the actual data have been determined in the set of uncertainties $\mathcal{U}$, and constraints on programming problems linear with uncertainty is "hard". In addition, referring to Gorissen *et al.* in [6], in dealing with Linear RO, three things are also assumed. First, the objective function is certainly valuable. If there are uncertainties in the objective function, then the problem can be formulated by replacing the objective function with a single variable function such that uncertainty arises in a constraint function. Second, the right vertex vector b is of certain value. If b is not certain, an extra variable x_{n+1} can be introduced. Third, robustness against $\mathcal{U}$ can be formulated as a constraint-wise problem and the set of uncertainties $\mathcal{U}$ is a closed and convex set.

Assuming that $c \in R^n$ and $b \in R^m$ are of certain value, the Robust Counterpart (RC) formulation of equation (20) is equivalent to equation (21) below.

$$\begin{aligned} \min \;\; & c^T x \\ s.t \;\; & A(\zeta)x \leq b \\ & \forall\ \zeta \in \mathcal{Z}. \end{aligned} \tag{21}$$

Note the uncertain constraint in equation (21) and define the uncertain parameter $a(\zeta) = \bar{a} + P\zeta$ where $\bar{a} \in R^n$ is a nominal value vector and $P \in R^{nxL}$ is a confounding matrix, the set $\mathcal{U}$ is defined as in equation (22).

$$\mathcal{U} = \{a | a = \bar{a} + P\zeta, \quad \zeta \in \mathcal{Z}\} \tag{22}$$

where $\mathcal{Z} \subset R^L$ is an uncertain set of primitive factors, so equation (23) is obtained.

$$(\bar{a} + P\zeta)^T x \leq b, \quad \forall\ \zeta \in \mathcal{Z} \tag{23}$$

The optimal solution from Robust Counterpart is called optimal robust. Furthermore, to reformulate the set of uncertainty $\mathcal{U}$ into a computationally tractable problem, the following theorem applies.

Theorem 4: Referring to Bental *et al.* [2] and Chaerani & Roos [7], assume the set of uncertainty $\mathcal{U}$ is an affine image of the limited set $\mathcal{Z} = \{\zeta\} \subset R^n$, and $\mathcal{U}$ is:

1) The system of linear inequality constraints

$$P\zeta \leq p \tag{24}$$

2) The system of conic quadratic inequality

$$\|P_i\zeta - p_i\|_2 \leq p_i^T\zeta - r_i, i = 1, ..., M \tag{25}$$

3) The systems of linear matrix inequality

$$p_0 + \sum_{i=1}^{dim\zeta} \zeta_i P_i \geq 0 \tag{26}$$

In cases (2) and (3) it is also assumed that the system of the constraints defining $\mathcal{U}$ is strictly feasible. Then, the Robust Counterpart of equation (20) is equivalent to] Linear Programming (LP) problems in the case (1).Conic Quadratic Programming (CQP) problems in cases (2). Semidefinite Programming (SDP) problems in cases (3).

Cite from Gorissen *et al.* in [6], the computational tractability of robust counterpart for different sets of uncertainties can be seen in Table I.

TABLE I
TRACTABILITY FOR CONSTRAINTS WITH UNCERTAINTY SETS

Uncertainty Set	$\mathcal{Z}$	Robust Counterpart	Tractability
Box	$\|\zeta\|_\infty \leq 1$	$a^T x + \|P^T x\|_1 \leq b$	LP
Ellipsoidal	$\|\zeta\|_2 \leq 1$	$a^T x + \|P^T x\|_2 \leq b$	CQP
Polyhedral	$D\zeta + q \geq 0$	$\begin{cases} a^T x + q^T w \leq b \\ D^T w = -P^T x \\ w \geq 0 \end{cases}$	LP

C. Adjustable Robust Counterpart Optimization

Referring to Bental *et al.* in [5] and [22] also in Yanikouglu *et al.* [23], in multistage optimization, the basic paradigm of RO, namely the "here and now" decision, can be relaxed. Some decision variables can be adjusted at a later time according to decision rules, which are a function of (some or all parts of) uncertain data. Adjustable Robust Counterpart (ARC) is given as in equation (27).

$$\begin{aligned} \min_{x,y(.)} \;\; & c^T x \\ s.t \;\; & A(\zeta)x + By(\zeta) \leq b \\ & \forall\ \zeta \in \mathcal{Z}. \end{aligned} \tag{27}$$

where $x \in R^n$ is the first stage decision "here and now" made before $\zeta \in R^L$ is realized, $y \in R^k$ denotes a "wait and see" decision and $B \in R^{mxk}$ which shows a certain matrix coefficient.

In practice, $y(\zeta)$ is often through an approach with affine or linear decision rules $y(\zeta) = y^0 + Q\zeta$ with $y^0 \in R^k$ and $Q \in R^{kxL}$ is the coefficient in the decision rule, which is to be optimized. Thus, the reformulation of equation (27) is as follows:

$$\begin{aligned} \min_{x,y^0,Q} \;\; & c^T x \\ s.t \;\; & A(\zeta) + By^0 + BQ\zeta \leq b \\ & \forall\ \zeta \in \mathcal{Z}. \end{aligned} \tag{28}$$

In the next section, we discuss the formulation of the Adjustable Robust Counterpart Optimization (ARC) model for the maximum flow problem (MFP), the ARC-MFP with parametric ellipsoidal uncertainty set, and the ARC-MFP with polyhedral uncertainty set.

IV. RESULT AND DISCUSSION

A. ARC Optimization Model for the Maximum Flow Problem

Previously, review the model of the maximum flow problem as in equation (1). The first thing to do is determining the parameters of uncertainty. In the case of maximum flow, the flow capacity of each side in the network is an uncertain factor. Therefore, the uncertainty parameter in this maximum flow model is the side capacity of the network. This means that the capacity of node i and j, i.e., c_{ij} is assumed lies within an uncertainty set $\mathcal{U}$. Furthermore, as an uncertainty parameter, c_{ij} can be written into equation (29)

$$c_{ij} = \bar{c}_{ij} + P_{ij}\zeta, \forall\ \zeta \in \mathcal{Z} \tag{29}$$

where $\bar{c}_{ij} \in R^n$ is a nominal value vector of side capacity, $P_{ij} \in R^{nxL}$ is a confounding matrix, and $\zeta \in R^L$ is a primitive uncertainty vector. c_{ij} uncertainty parameters are found in the constraints of the maximum flow model. c_{ij} parameter is a vector of the right hand side of the constraint, so the assumption that the right hand side of the constraint must be of certain valuable can be met by adding an extra variable $\omega_{ij} = 1$ so that the c_{ij} parameter becomes the coefficient of the ω_{ij} variable as in equation (30) as follows:

$$x_{ij} - c_{ij}\omega_{ij} \leq 0, \forall i,j;\ \omega_{ij} = 1 \tag{30}$$

Substitute equation (29) to equation (30) to get the model for the maximum flow problem with uncertainty in the following parameters:

$$\begin{aligned} \max\ & x_{ts} \\ s.t\ \ & \tilde{A}x = 0 \\ & x_{ij} - (\bar{c}_{ij} + P_{ij}\zeta)\omega_{ij} \leq 0,\ \forall\ i,j \\ & \omega_{ij} = 1 \\ & x_{ij} \geq 0,\ \forall\ i,j \end{aligned} \tag{31}$$

The next step is to determine the adjustable decision variable from the maximum flow model in the form of the number of flows from node t to node s (x_{ts}). The x_{ts} variable can adjust to $x_{ts}(\zeta)$ decision rules that depend on ζ and can be defined as follows:

$$x_{ts}(\zeta) = \bar{x}_{ts} + Q\zeta \tag{32}$$

where $\bar{x}_{ts} \in R^n$ is the nominal vector of the amount of flow from node t to node s, $Q \in R^{nxL}$ is a confounding matrix, and $\zeta \in R^L$ is a primitive uncertainty factor. Note that the adjustable variable x_{ts} is found in the objective function of the maximum flow model, so the assumption that the objective function must be of value can be fulfilled by replacing the objective function with a single variable function t and there are uncertainties in a constraint function where $t \leq x_{ts}$ and $t \in R^n$. Substitute equation (32) to equation (31) to obtain an Adjustable Robust Counterpart model for the maximum flow problem as in equation (16) follows:

$$\begin{aligned} \max\ & t \\ s.t\ \ & \bar{x}_{ts} + Q\zeta - t \geq 0 \\ & \tilde{A}x = 0 \\ & x_{ij} - (\bar{c}_{ij} + P_{ij}\zeta)\omega_{ij} \leq 0,\ \forall\ i,j \\ & \omega_{ij} = 1 \\ & x_{ij} \geq 0,\ \forall\ i,j \\ & \zeta \in \mathcal{Z} \end{aligned} \tag{33}$$

B. ARC Optimization Model for Maximum Flow Problem with Parametric Ellipsoidal Uncertainty

Assume that the uncertain parameters and decision variables in the Adjustable Robust Counterpart model for the maximum flow problem are in the set of ellipsoidal uncertainty. Define the set of ellipsoidal uncertainty as follows:

$$\mathcal{Z} = \{\zeta : \|\zeta\|_2 \leq 1\} \tag{34}$$

Robust Counterpart formulation for constraints with the set of ellipsoidal uncertainty as in equation (35) follows:

$$(\bar{a} + P\zeta)^T x \leq b,\ \forall\zeta : \|\zeta\|_2 \leq 1 \tag{35}$$

is equivalent with

$$\begin{aligned} (\bar{a} + P\zeta)^T x &= \bar{a}^T x + \max_{\zeta:\|\zeta\|_2\leq 1} (P^T x)^T \zeta \\ &= \bar{a}^T x + \|P^T x\|_2 \leq b \end{aligned} \tag{36}$$

Assuming the uncertainty is in the set of ellipsoidal uncertainty, the third constraint in equation (33) is equivalent to the following equation:

$$\begin{aligned} x_{ij} - (\bar{c}_{ij} + P_{ij}\zeta)\omega_{ij} &= x_{ij} - \bar{c}_{ij}\omega_{ij} - P_{ij}\omega_{ij}\zeta \\ &= x_{ij} - \bar{c}_{ij}\omega_{ij} - \max_{\zeta:\|\zeta\|_2\leq 1} (P_{ij}\omega_{ij}\zeta) \\ &\leq 0,\ \forall\ i,j \end{aligned} \tag{37}$$

To achieve the best worst condition, choose the unit vector

$$\zeta = \frac{P_{ij}\omega_{ij}}{\|P_{ij}\omega_{ij}\|_2}. \tag{38}$$

By referring to the norm definition, norm$-\mathcal{L}_2$ is the root result of adding up the absolute value of the entry squared.

$$\begin{aligned} \max_{\zeta:\|\zeta\|_2\leq 1} (P_{ij}\omega_{ij}\zeta) &= P_{ij}\omega_{ij}\frac{P_{ij}\omega_{ij}}{\|P_{ij}\omega_{ij}\|_2}, \\ &= \frac{(P_{ij}\omega_{ij})^2}{\sqrt{(P_{ij}\omega_{ij})^2}}, \\ &= \sqrt{(P_{ij}\omega_{ij})^2}, \\ &= \|P_{ij}\omega_{ij}\|_2,\ \forall i,j. \end{aligned} \tag{39}$$

Thus, this implies that the equation (37) can be represented as (40).

$$x_{ij} - \bar{c}_{ij}\omega_{ij} - \|P_{ij}\omega_{ij}\|_2 \leq 0, \forall i,j. \tag{40}$$

In the same way, for the first constraint in equation (33) with the set of ellipsoidal uncertainty, choose the unit vector $\zeta = \frac{Q}{\|Q\|_2}$. This is equivalent to equation (41).

$$\begin{aligned} \bar{x}_{ts} + Q\zeta - t &= \bar{x}_{ts} + (Q\frac{Q}{\|Q\|_2}) - t \\ &= \bar{x}_{ts} + (\frac{Q^2}{\sqrt{Q^2}}) - t \\ &= \bar{x}_{ts} + \sqrt{Q^2} - t, \\ &= \bar{x}_{ts} + \|Q\|_2 - t \geq 0. \end{aligned} \tag{41}$$

Next, substituting equations (40) and (41) into the Optimization model (33), we obtain the Adjustable Robust Counterpart Optimization model with the set of ellipsoidal uncertainty for the maximum flow problem as in equation (42).

$$\begin{aligned} \max\ & t \\ s.t\ \ & \bar{x}_{ts} + \|Q\|_2 - t \geq 0,\ \forall i \\ & \tilde{A}x = 0 \\ & x_{ij} - \bar{c}_{ij}\omega_{ij} - \|P_{ij}\omega_{ij}\|_2 \leq 0, \forall i,j. \\ & \omega_{ij} = 1,\ \forall i,j \\ & x_{ij} \geq 0,\ \forall i,j \end{aligned} \tag{42}$$

Next, we discuss a parametric variant of the above ARC-MFP, where the sizes of the uncertainty perturbation in c are controlled by a nonnegative scaling parameter.

1) Parametric Ellipsoidal Uncertainty for ARC-MFP: Let the uncertainty set $\mathcal{U}_\alpha$ be defined by

$$\mathcal{U}_\alpha = \{c \;:\; c = c^0 + \alpha \mathcal{Q}\zeta, \|\zeta\| \leq 1\}, \tag{43}$$

where c^0 is the nominal value of c, α is a nonnegative scaling parameter. Note that $c^0 + \alpha\mathcal{Q}\zeta$ must be nonnegative for $\forall\zeta$: $\|\zeta\| \leq 1$ to ensure feasibility. Thus, we assume that

$$0 \leq \alpha \leq \alpha_{\max} := \min\left\{\frac{c_a^0}{\|\mathcal{Q}_a\|} \;:\; \|\mathcal{Q}_a\| > 0, a \in \mathcal{A}\right\}. \tag{44}$$

Theorem 5: Let $\mathcal{U}_\alpha$ be the ellipsoidal uncertainty set given by (43) with $0 \leq \alpha \leq \alpha_{\max}$ and let $x_{ts}(\alpha)$ denote the optimal flow value for the robust counterpart. Then $x_{ts}(\alpha)$ is a piecewise monotonically decreasing linear concave function.

Proof: By Theorems 3 and 1, the maximum flow of the ARC-MFP with ellipsoid set $\mathcal{U}_\alpha$, $x_{ts}(\alpha)$ satisfies

$$\begin{aligned} \min\nolimits_X \quad & \sum_{a\in\delta^+(X)} (c_a^0 - \alpha\|\mathcal{Q}_a\|) \\ s.t \quad & s \in X \subseteq \mathcal{V}\setminus\{t\}. \end{aligned} \tag{45}$$

We shall show that $x_{ts}(\alpha)$ is a piecewise linear concave function of α by proving that it is the minimum of a finite family of linear functions. To this end, it is convenient to introduce

$$\mathcal{X} = \{X : s \in X \subseteq \mathcal{V}\setminus\{t\}\} \tag{46}$$

such that (45) can rewritten as follows

$$x_{ts}(\alpha) = \min_X \left\{c^\alpha(\delta^+(X)) \;:\; X \in \mathcal{X}\right\}, \tag{47}$$

where

$$c^\alpha(\delta^+(X)) = \sum_{a\in\delta^+(X)} c_a^0 - \alpha \sum_{a\in\delta^+(X)} \|\mathcal{Q}_a\|. \tag{48}$$

Fixing $X \in \mathcal{X}$ and since $\sum_{a\in\delta^+(X)} \|\mathcal{Q}_a\| \geq 0$, $c^\alpha(\delta^+(X))$ is a monotonically decreasing linear function of α. If $|\mathcal{V}| = n$, then the number of $s-t$ cuts $\mathcal{X}$ is 2^{n-2}. Hence $\mathcal{X}$ is finite. We conclude that $x_{ts}(\alpha)$ is the minimum of a finite set of monotonically decreasing linear functions. This implies that $x_{ts}(\alpha)$ is continuous, concave and monotonically decreasing piecewise linear function.

Next, we discuss some properties of the RMF value function $x_{ts}(\alpha)$.

2) The minimal cuts on a linearity interval and at a breakpoint: The values of α where the slope of $x_{ts}(\alpha)$ changes are called breakpoints of $x_{ts}(\alpha)$ and any interval between two successive break points of $x_{ts}(\alpha)$ is called a linearity interval of $x_{ts}(\alpha)$. For any α in the domain of $x_{ts}(\alpha)$ we denote the set of minimal cut sets by

$$\mathcal{X}_\alpha = \{X \in \mathcal{X} \;:\; x_{ts}(\alpha) = c^\alpha(\delta^+(X))\}. \tag{49}$$

The following theorem shows that the set $\mathcal{X}_\alpha$ is constant on the interior of a linearity interval.

Theorem 6: If $x_{ts}(\alpha)$ is linear on the interval $[\alpha_1, \alpha_2]$, where $\alpha_1 < \alpha_2$ then $\mathcal{X}_\alpha$ is constant for $\alpha \in (\alpha_1, \alpha_2)$.

Proof: Rewrite the RMF value as

$$x_{ts}(\alpha) = \tau - \alpha\sigma, \qquad \alpha \in [\alpha_1, \alpha_2], \tag{50}$$

where

$$\tau = \sum_{a\in\delta^+(X)} c_a^0 \text{ and } \sigma = \sum_{a\in\delta^+(X)} \|\mathcal{Q}_a\|. \tag{51}$$

Consider that for $\beta \in (\alpha_1, \alpha_2)$ such that $X \in \mathcal{X}_\beta$ we have that τ and σ are independent of β. This implies that $\mathcal{X}_\beta$ is independent of β. Since β is arbitrary on the open interval (α_1, α_2), then for any $\alpha \in (\alpha_1, \alpha_2)$ we conclude that $\mathcal{X}_\alpha$ is constant.

At a break point $(\alpha, x_{ts}(\alpha))$, the following holds.

Theorem 7: Let $\mathcal{X}_{\alpha_1}$ and $\mathcal{X}_{\alpha_2}$ be the minimal cuts on two neighboring intervals (α_1, α) and (α, α_2) respectively. Then the minimal cut set at the breakpoint $(\alpha, x_{ts}(\alpha))$ satisfies

$$\mathcal{X}_\alpha \supseteq \mathcal{X}_{\alpha_1} \bigcup \mathcal{X}_{\alpha_2}. \tag{52}$$

Proof: By Theorem 6, the minimal cuts $\mathcal{X}_{\alpha_1}$ and $\mathcal{X}_{\alpha_2}$ are constant on the interval (α_1, α) and (α, α_2) respectively. This implies that at the breakpoint $(\alpha, x_{ts}(\alpha))$, the minimal cuts $\mathcal{X}_a$ contains $\mathcal{X}_{\alpha_1}$ and $\mathcal{X}_{\alpha_2}$. Thus the proof is followed.

C. ARC Optimization Model for the Maximum Flow Problem with Polyhedral Uncertainty

Assume that the uncertain parameters and decision variables in the Adjustable Robust Counterpart model for the maximum flow problem are in the set of polyhedral uncertainty. Define the set of polyhedral uncertainty as (53).

$$\mathcal{Z} = \{\zeta : \delta - D\zeta \geq 0\} \tag{53}$$

Consider that the robust counterpart formulation for constraints with the set of polyhedral uncertainty can be derived as follows

$$(\bar{a} + P\zeta)^T x \leq b, \; \forall\zeta : d - D\zeta \geq 0 \tag{54}$$

which equivalent with

$$\bar{a}^T x + \max_{\zeta: d - D\zeta \geq 0} (P^T x)^T \zeta \leq b, \tag{55}$$

where $\bar{a} \in R^n$ is a nominal value vector, $P \in R^{nxL}$ is a confounding matrix, $\zeta \in R^L$ is a primitive uncertainty vector, $d \in R^m$, and $D \in R^{mxL}$. Consider that using the duality theory, solving (55) is equivalent with solving its dual.

$$\bar{a}^T x + \min_y \left\{d^T y : D^T y = P^T x, y \geq 0\right\} \leq b. \tag{56}$$

This means that $\exists y$ such that

$$\bar{a}^T x + d^T y \leq b, \; D^T y = P^T x, \; y \geq 0 \tag{57}$$

Now, assume that the data uncertainty lies within a polyhedral uncertainty set as in (53) , thus the ARC for problem (33) can be done by reformulating the third constraint of problem (33), as follows.

$$\begin{aligned} x_{ij} - (\bar{c}_{ij} + P_{ij}\zeta)^T \omega_{ij} & \\ = x_{ij} - \bar{c}_{ij}\omega_{ij} & - (P_{ij}^T \omega_{ij})^T \zeta \leq 0, \forall i, j, \end{aligned} \tag{58}$$

which equivalent with

$$x_{ij} - \bar{c}_{ij}\omega_{ij} - \max_{\zeta: d - D\zeta \geq 0} (P_{ij}^T \omega_{ij})^T \zeta \leq 0, \forall i, j \tag{59}$$

Using the duality theory consider that

$$\max_{\zeta: d - D\zeta \geq 0} (P_{ij}^T \omega_{ij})^T \zeta \tag{60}$$

is equivalent with

$$\min_{y_k} \left\{d_k^T y_k : D_k^T y_k = P_{ij}^T \omega_{ij}, y_k \geq 0\right\}. \tag{61}$$

Thus the ARC of the third constraint of problem (33) is

$$x_{ij} - \bar{c}_{ij}\omega_{ij} - d_k^T y_k \leq 0, \quad (62)$$
$$D_k^T y_k = P_{ij}^T \omega_{ij}, y_k \geq 0, \forall i, j \quad (63)$$

In the same way, to determine the ARC of the first constraint to equation (33) with the set of polyhedral uncertainty, consider that the following holds.

$$\begin{aligned} x_{ts} + Q\zeta - t &= x_{ts} + \max_{\zeta: d - D\zeta \geq 0} (Q\zeta) - t \\ &= x_{ts} + \min_{y_x} \left\{ d_x^T y_x : D_x^T y_x = Q, y_x \geq 0 \right\} - t \end{aligned}$$

Thus we have that

$$x_{ts} + d_x^T y_x - t \geq 0, \quad (64)$$
$$D_x^T y_x = Q, y_x \geq 0. \quad (65)$$

Next, substituting equations (62), (63), (64), and (65) into (33), we obtain the Adjustable Robust Counterpart Optimization model with the set of polyhedral uncertainty for the maximum flow problem as in equation (66) below:

$$\begin{aligned} \max \quad & t \\ \text{s.t} \quad & \bar{x}_{ts} + d_x^T y_x - t \geq 0 \\ & \tilde{A}x = 0 \\ & x_{ij} - \bar{c}_{ij}\omega_{ij} - d_k^T y_k \leq 0, \ \forall i, j \\ & D_k^T y_k = P_{ij}^T \omega_{ij} \quad (66) \\ & D_x^T y_x = Q \\ & \omega_{ij} = 1 \\ & y_z, y_k \geq 0 \\ & x_{ij} \geq 0, \ \forall i, j \end{aligned}$$

V. Numerical Simulation

In this section we present some examples of Parametric Ellipsoidal Uncertainty in ARC-MFP, ARC-MFP as a piecewise linear concave function, and a case study of ARC in a network of Energy-Saving Generation Dispatch (ESGD) with ellipsoidal and polyhedral uncertainty set.

A. Parametric Ellipsoidal Uncertainty in ARC-MFP

In this subsection we discuss how the RMF objective function can be a piecewise monotonically decreasing linear concave function of α with n different intervals and $n+1$ break points. We show this by following examples.

Example 1: Consider the network of Figure 2. Taking $\mathcal{Q} = I$, we have a parametric ellipsoid $\mathcal{U}_\alpha$ as

$$\mathcal{U}_\alpha = \{c \ : \ c = c^0 + \alpha\zeta, \|\zeta\| \leq 1\} \quad (67)$$

where α satisfy $0 \leq \alpha \leq 1$ by (44).

The robust arc capacities are then

$$c_a = c_a^0 - \alpha, \qquad \forall a \in \mathcal{A}, \quad (68)$$

hence the RMFP for this example is

$$\max\{x_{71} \ : \ Ax = 0, 0 \leq x_a \leq c_a^0 - \alpha, \forall a \in \mathcal{A}\}. \quad (69)$$

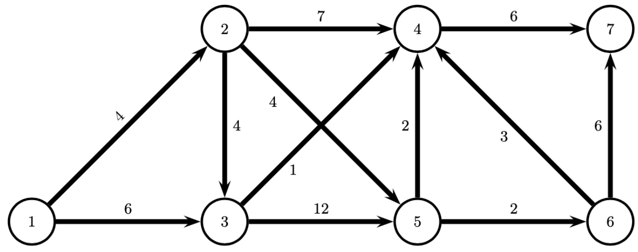

Fig. 2. A maximum flow problem with the nominal arc capacities

In Table II, we present the RMF for $0 \leq \alpha \leq 1$. In Figure 3 we see that the RMF value function $x_{7,1}(\alpha)$ is a piecewise monotonically decreasing linear concave function of α with two different intervals and three break points.

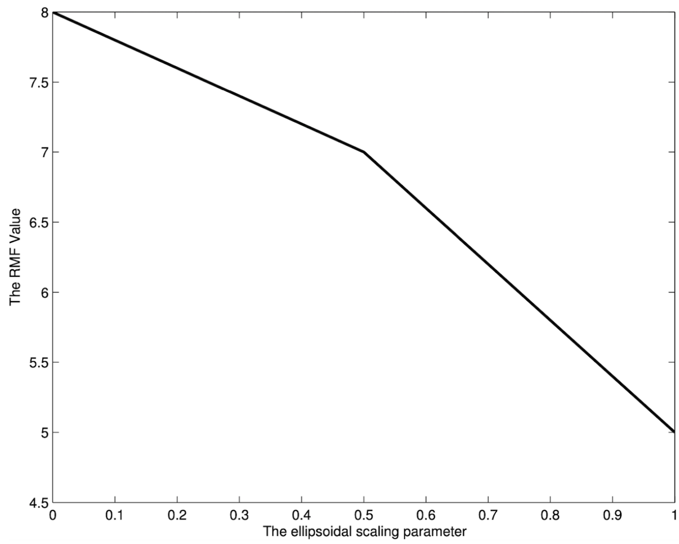

Fig. 3. The RMF $x_{7,1}(\alpha)$ as a piecewise linear concave function of ellipsoidal scalling parameter α.

Example 2: The second example shows that it is possible to have $n+1$ different linearity interval and n breakpoints. The discussion is as follows. Consider a network as shown in Figure 4 with $n \geq 2$. For a given R, define

$$c_k = \frac{R}{\sin\gamma_k}, \quad \text{where} \quad \gamma_k = k(\frac{\pi}{2n}), k = 1, 2, \ldots, n. \quad (70)$$

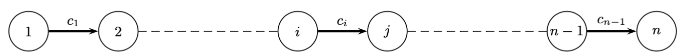

Fig. 4. A simple network for RMFP with $n \geq 2$

The matrix $\mathcal{Q}$ is a diagonal matrix with

$$\mathcal{Q}_{kk} = -\cot\gamma_k, \quad \text{thus} \quad \alpha_{\max} = \min_k \left\{ R\frac{\cos\gamma_k}{\sin^2\gamma_k} \right\}. \quad (71)$$

As an example, for a case with $R = 1$ and $n = 4$, the nominal arc capacities c^0 is

$$c^0 = \begin{pmatrix} 1.0353 \\ 1.1547 \\ 1.4142 \\ 2.0000 \\ 3.8637 \\ \infty \end{pmatrix} \quad (72)$$

TABLE II
THE RMF FOR $x_{v_7,v_1}(\alpha)$ VALUE FOR $\alpha \in [0.0, 1.0]$

Arcs	c^0	α										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
x_{v_1,v_2}	4	3.7929	3.7322	3.6745	3.6163	3.5585	3.5000	3.4000	3.3000	3.2000	3.1000	3.0000
$x_{1,3}$	6	4.2071	4.0678	3.9255	3.7837	3.6415	3.5000	3.2000	2.9000	2.6000	2.3000	2.0000
$x_{2,3}$	4	0.1999	0.1562	0.1111	0.0683	0.0290	0	0	0	0	0	0
$x_{2,4}$	7	3.3709	3.4123	3.4533	3.4899	3.5067	3.5000	3.4000	3.3000	3.2000	3.1000	3.0000
$x_{2,5}$	4	0.2221	0.1636	0.1101	0.0581	0.0229	0	0	0	0	0	0
$x_{3,4}$	1	0.7753	0.7011	0.6320	0.5680	0.5190	0.5000	0.4000	0.3000	0.2000	0.1000	0
$x_{3,5}$	12	3.6317	3.5230	3.4045	3.2840	3.1514	3.0000	2.8000	2.6000	2.4000	2.2000	2.0000
$x_{4,7}$	6	6.0000	5.9000	5.8000	5.7000	5.6000	5.5000	5.2635	5.0197	4.7710	4.5150	4.2567
$x_{5,4}$	2	1.8539	1.7866	1.7146	1.6421	1.5743	1.5000	1.4000	1.3000	1.2000	1.1000	1.0000
$x_{5,6}$	2	2.0000	1.9000	1.8000	1.7000	1.6000	1.5000	1.4000	1.3000	1.2000	1.1000	1.0000
$x_{4,6}$	3	0	0	0	0	0	0	0.0635	0.1197	0.1710	0.2150	0.2567
$x_{6,7}$	6	2.0000	1.9000	1.8000	1.7000	1.6000	1.5000	1.3365	1.1803	1.0290	0.8850	0.7433
$x_{7,1}$	∞	8.0000	7.8000	7.6000	7.4000	7.2000	7.0000	6.6000	6.2000	5.8000	5.4000	5.0000
RMF		8.0000	7.8000	7.6000	7.4000	7.2000	7.0000	6.6000	6.2000	5.8000	5.4000	5.0000

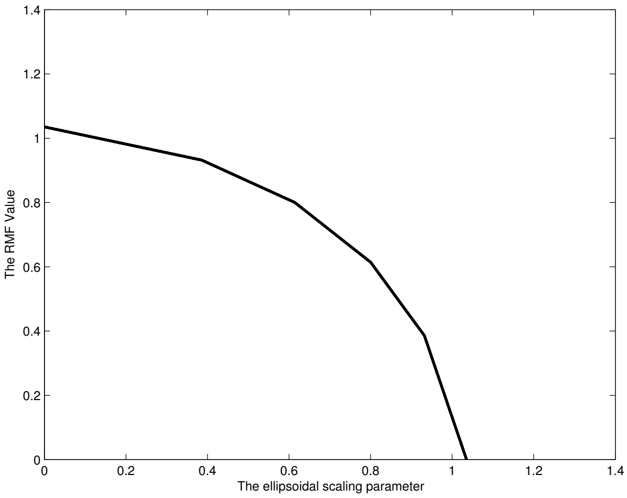

Fig. 5. Optimal value function in Example 2.

and the matrix $\mathcal{Q}$ is

$$\mathcal{Q} = \begin{pmatrix} -0.2679 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5774 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.7321 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3.7321 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In Figure 5, we see that there are four breakpoints and five linearity intervals for $\alpha \in [0, 1.0353]$.

B. A Case Study in a Network of Energy-Saving Generation Dispatch (ESGD)

The data used for the case study are secondary data that already exists referring to Zhang and Cai [29] regarding the network on the issue of Energy-Saving Generation Dispatch (ESGD).

The ESGD problem used has the aim to minimize carbon gas emissions resulting from the use of coal fuel by minimizing the cost function and optimizing the electric current in the distribution system. In this electric power distribution system, electricity will be sent from a power source (generator) through several components such as the transformer and capacitors to the consumer demand. Basically, the electrical distribution system can be described as a network in which some components that play a role in the delivery of electricity.

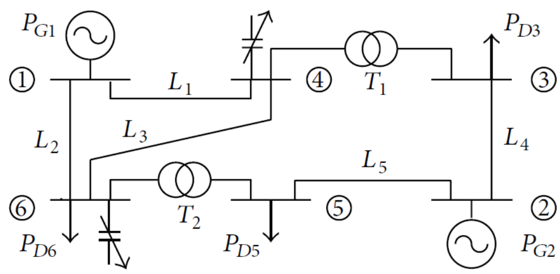

Fig. 6. ESGD Bus Power System from Zhang and Cai [29]

To proceed this problem as a maximum flow problem, a converting step is done to represent the diagram in Figure 6 to a network diagram flow as can be seen in Figure 7.

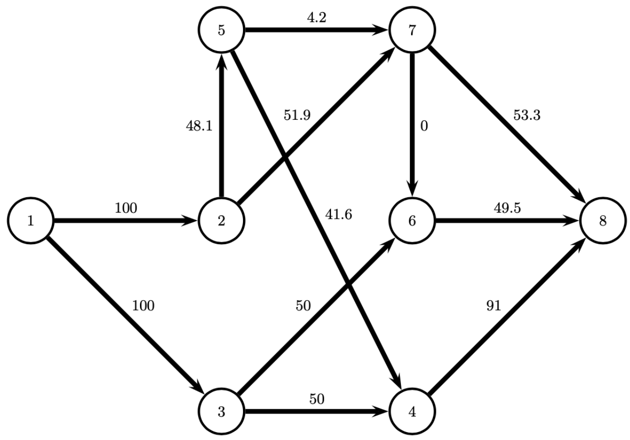

Fig. 7. ESGD Problem as a Network Flow Problem (see Zhang and Cai [29]).

Referring to (73), ESGD is developed by the Chinese government in 2007. Based on the fuel consumption rate from low to high, the power plants are ranked. The power plant will be sorted based on the intensity of its carbon emission from low to high when the fuel consumption rate is the same. The ESGD problem has an objective function on minimizing the operating costs of power generation systems that are considered to represent the fuel costs that are directly proportional to the carbon emissions generated so that both seek to comply with government ESGD schemes.

Referring to Zhang and Cai [29] a power distribution system has been converted into a network. The following is the example for ESGD network as a bus power system as in Figure 6. To this end, a source point is selected first. In this case the point that is representing the generator as power producer, namely point 1 and point 2 are selected as the source points. Next, select the destination points, i.e., the three destination points or demand power points, i.e., the point 3, 5, and 6. The flow in bus power system hence can be followed in order to assemble the network on the diagram of network flow. the points where there is a reduction of power during delivery are become other points. These are the points at which there is a transformer or capacitor. Network diagram flow generated from the diagram bus power system in Figure 6 now has been converted as a network flow in Figure 7.

A variant version of optimization model for ESGD is discussed by Lesmana *et. al* in [30]. The ESGD problem is considered as a min-cost flow model. Differs to [30], in this paper the main focus in this case study study is to find the maximum electric current that can flow on the network such as Figure 7 and the maximum current flow with an uncertainty in the electric power capacity. The maximum flow problem for the ESGD network case study can be written as in equation (73).

$$\begin{aligned} \max \quad & x_{81} \\ s.t \quad & x_{12} + x_{13} - x_{81} = 0 \\ & -x_{12} + x_{25} + x_{27} = 0 \\ & x_{25} + x_{54} + x_{57} = 0 \\ & -x_{13} + x_{36} + x_{34} = 0 \\ & x_{27} - x_{57} + x_{76} + x_{78} = 0 \\ & x_{34} - x_{54} + x_{48} = 0 \\ & x_{36} - x_{76} + x_{68} = 0 \\ & x_{78} - x_{68} - x_{48} + x_{81} = 0 \\ & x_{ij} \leq c_{ij}, \ \forall (i,j) \\ & x_{ij} \geq 0 \end{aligned} \tag{73}$$

The maximum amount of electric current that can flow on the network is 193.2 Ampere. The electric current for each arcs can be seen in Table III for nominal case (without uncertainty).

1) Numerical Simulation Result with Ellipsoidal Uncertainty: The formulation of ARMFP with the ellipsoidal uncertainty set for (73) can be seen in (74). Let the value of P is a random number obtained through Maple software. Thus, the calculation result for equation (74), i.e., the maximum amount of electric current that can flow on the network is 219.1948 Ampere. The optimal amount of electric current on each arcs with the set of ellipsoidal uncertainty can be seen in Table III.

$$\begin{aligned} \max \quad & t \\ s.t \quad & x_{81} + \|Q\| - t \geq 0 \\ & x_{12} + x_{13} - x_{81} - \|Q\| = 0 \\ & x_{12} + x_{25} + x_{27} = 0 \\ & x_{25} + x_{54} + x_{57} = 0 \\ & x_{13} + x_{36} + x_{34} = 0 \\ & x_{27} - x_{57} + x_{76} + x_{78} = 0 \\ & x_{34} - x_{54} + x_{48} = 0 \\ & x_{36} - x_{76} + x_{68} = 0 \\ & x_{78} - x_{68} - x_{48} + x_{81} + \|Q\| = 0 \\ & x_{ij} - c_{ij} - \|P_{ij}\| \leq 0, \ \forall (i,j) \\ & x_{ij} \geq 0 \end{aligned} \tag{74}$$

TABLE III
THE ELECTRIC CURRENT FOR EACH ARCS OF THE NETWORK WITH ELLIPSOIDAL AND POLYHEDRAL UNCERTAINTY SET

Arcs	Optimal (nominal)	Robust Optimal (ellipsoidal)	Robust Optimial (polyhedral)
x_{12}	93.7	104.1759	109.6852
x_{13}	99.5	115.10188	109.2969
x_{25}	41.8	42.8248	23.5330
x_{27}	51.9	61.3511	86.1521
x_{34}	50	58.4124	76.3744
x_{36}	49.5	56.6064	32.9225
x_{48}	91	95.4278	99.9074
x_{54}	41	37.0154	23.5330
x_{57}	0.79	5.8093	0
x_{68}	49.5	56.6064	32.9225
x_{76}	0	0	0
x_{78}	52.7	67.1605	86.1521
x_{81}	193.2	219.1948	218.9821
Q	-	0	0
t	-	219.1948	218.9821

C. Numerical Simulation Result with Polyhedral Uncertainty

The ARMFP with polyhedral uncertainty set can be formulated as (75)

$$\begin{aligned} \max \; & t \\ & x_{81} + d_x y_x - t \geq 0 \\ & x_{12} + x_{13} - x_{81} - d_x y_x = 0 \\ & x_{12} + x_{25} + x_{27} = 0 \\ & x_{25} + x_{54} + x_{57} = 0 \\ & x_{13} + x_{36} + x_{34} = 0 \\ & x_{27} - x_{57} + x_{76} + x_{78} = 0 \\ & x_{34} - x_{54} + x_{48} = 0 \\ & x_{36} - x_{76} + x_{68} = 0 \\ & x_{78} - x_{68} - x_{48} + x_{81} + d_x y_x = 0 \\ & x_{ij} - c_{ij}\omega_{ij} - d_{c_{ij}} y_{c_{ij}} \leq 0, \ \forall (i,j) \\ & D_{c_{ij}} y_{c_{ij}} = P_{ij}\omega_{ij}, \ \forall (i,j) \\ & D_x y_x = Q \\ & \omega_{ij} = 1 \\ & y_x \geq 0 \\ & x_{ij}, y_{c_{ij}} \geq 0, \ \forall (i,j) \end{aligned} \tag{75}$$

Let the value in the variable P, D_x, D_k, d_x, d_k is a random number obtained through Maple software. Based on equation (75) and the value of the variable P, D_x, D_k, d_x, d_k, using Maple 18 software the maximum amount of electric current that can flow on the network is 218,9821 Ampere. The robust optimal of electric current on each arcs with the set of polyhedral uncertainty as in Table III.

VI. Conclusions

The ARC-MFP has been discussed in this paper. It is clearly shown that the ARC-MFP is computationally tractable problem when the uncertainty set is parametric ellipsoidal uncertainty and polyhedral uncertainty set. For the case of parametric ellipsoidal, the characteristic of ARC-MFP objective function is shown to be continuous, concave and a monotonically decreasing piecewise linear function.

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