Solvability of Fractional Functional Boundary Value Problems with p-Laplacian Operator at Resonance

Tingting Xue*, Xing Chen, and Xiaolin Fan

***Abstract*—We concern with the existence of solutions of fractional p-Laplacian equations with functional boundary value conditions at resonance. Applying the extension of continuation theorem, some new existence results on this problem are obtained. Finally, this article also gives examples to verify the main results. The work of this paper is to extend some current results to a completely nonlinear situation.**

***Index Terms*—Riemann-Liouville fractional derivative, functional boundary conditions, p-Laplacian operator, resonance, continuation theorem.**

I. Introduction

FRACTIONAL differential equations are studied by many scholars because of its wide application background (see [1-5]). Ameen and Novati [5] considered the following fractional epidemic model:

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x\left(t\right)=-\beta x\left(t\right)y\left(t\right),t\geq t_{0},\\ {}^{C}D_{0+}^{\alpha}y\left(t\right)=\beta x\left(t\right)y\left(t\right)-\gamma y\left(t\right),t\geq t_{0},\\ {}^{C}D_{0+}^{\alpha}z\left(t\right)=\gamma y\left(t\right),t\geq t_{0},\\ x\left(t_{0}\right)=N_{1}\geq 0,y\left(t_{0}\right)=N_{2}\geq 0,z\left(t_{0}\right)=N_{3}\geq 0, \end{cases}$$

where $0<\alpha\leq 1$, $\beta>0$, $\gamma>0$, ${}^{C}D_{0+}^{\alpha}$ is a Caputo fractional derivative. The specific significance of the remaining parameters can be found in literature [6].

In recent years, there has been great interest in functional boundary value problems (see [7-17]). For example, Zou and Cui [12] first discussed the following fractional functional boundary value problems (FFBVPs for short):

$$\begin{cases} D_{0+}^{\alpha}u\left(t\right)=f\left(t,u\left(t\right),D_{0+}^{\alpha-1}u\left(t\right),D_{0+}^{\alpha-2}u\left(t\right)\right),\ t\in[0,1],\\ I_{0+}^{3-\alpha}u(t)\big|_{t=0}=0,\Phi_{1}\left[D_{0+}^{\alpha-1}u\left(t\right)\right]=0,\Phi_{2}\left[D_{0+}^{\alpha-2}u\left(t\right)\right]=0, \end{cases}$$

where $2<\alpha<3$, D_{0+}^{α} is a Riemann-Liouville fractional derivative, $\Phi_{1},\Phi_{2}:C\left[0,1\right]\rightarrow\mathbb{R}$ are continuous, linear functional. In particular, the functional conditions here can be similarly represented as the following form:

$$\Phi_{1}\left[D_{0+}^{\alpha-1}u\left(t\right)\right]=\int_{0}^{1}D_{0+}^{\alpha-1}u\left(t\right)d\eta_{1}\left(t\right)=0,$$

$$\Phi_{2}\left[D_{0+}^{\alpha-2}u\left(t\right)\right]=\int_{0}^{1}D_{0+}^{\alpha-2}u\left(t\right)d\eta_{2}\left(t\right)=0.$$

It is worth mentioning that η_1, η_2 are Riemann-Stieltjes measures. Some existence results were obtained by giving some sufficient conditions as follows:

$$\begin{aligned} &(A1)\,\Phi_{1}\left(1\right)\Phi_{2}\left(1\right)\neq 0;\\ &(A2)\,\Phi_{1}\left(1\right)=\Phi_{2}\left(t\right)=0,\Phi_{2}\left(1\right)\neq 0;\\ &(A3)\,\Phi_{1}\left(1\right)=\Phi_{2}\left(1\right)=0,\Phi_{2}\left(t\right)\neq 0;\\ &(A4)\,\Phi_{1}\left(1\right)\neq 0,\Phi_{2}\left(1\right)=\Phi_{2}\left(t\right)=0;\\ &(A5)\,\Phi_{1}\left(1\right)=\Phi_{2}\left(1\right)=\Phi_{2}\left(t\right)=0. \end{aligned}$$

However, since the condition $(A6)\,\Phi_{1}\left(1\right)=0$, $\Phi_{2}\left(t\right)$, $\Phi_{2}\left(1\right)\neq 0$ was not considered in [12], Kosmatov and Jiang [13] continued to research the solvability of the following FFBVPs under the condition $(A6)$:

$$\begin{cases} D_{0+}^{\alpha}u\left(t\right)=f\left(t,u\left(t\right),D_{0+}^{\alpha-2}u\left(t\right),D_{0+}^{\alpha-1}u\left(t\right)\right),t\in(0,1),\\ u(0)=0,B_{1}(u)=B_{2}(u)=0. \end{cases}$$

Note that the conditions $B_{1}(u)=B_{2}(u)=0$ can be expressed as the following concrete form:

$$\begin{aligned} B_{i}\left(u\right)=&\int_{0}^{1}u\left(t\right)d\xi_{i1}\left(t\right)+\int_{0}^{1}D_{0+}^{\alpha-2}u\left(t\right)d\xi_{i2}\left(t\right)\\ &+\int_{0}^{1}D_{0+}^{\alpha-1}u\left(t\right)d\xi_{i3}\left(t\right)=0,i=1,2, \end{aligned}$$

Where $\xi_{ij}\left(t\right),i=1,2,j=1,2,3$, are Riemann-Stieltjes measures. Obviously, the boundary conditions here are more general than [12]. Here, Mawhin's continuous theorem (see [19]) is applied.

Based on the above literature, this paper will study the following FFBVPs with p-Laplacian operator:

$$\begin{cases} D_{0+}^{\beta}\phi_{p}\left(D_{0+}^{\alpha}u\left(t\right)\right)=f\left(t,u\left(t\right),D_{0+}^{\alpha-2}u\left(t\right),\right.\\ \qquad\qquad\left.D_{0+}^{\alpha-1}u\left(t\right),D_{0+}^{\alpha}u\left(t\right)\right),\quad t\in(0,1),\\ u\left(0\right)=D_{0+}^{\alpha}u\left(0\right)=0,\mathrm{T}_{1}\left(u\right)=\mathrm{T}_{2}\left(u\right)=0, \end{cases}\tag{1}$$

where $2<\alpha\leq 3$, $0<\beta\leq 1$, $3<\alpha+\beta\leq 4$, D_{0+}^{α} is a Riemann-Liouville fractional derivative, $f\in C([0,T]\times\mathbb{R}^{4},\mathbb{R})$ and $\mathrm{T}_{1},\mathrm{T}_{2}:C[0,1]\rightarrow\mathbb{R}$ are linear bounded functional and satisfy the resonance condition $\mathrm{T}_{1}\left(t^{\alpha-1}\right)\mathrm{T}_{2}\left(t^{\alpha-2}\right)=\mathrm{T}_{1}\left(t^{\alpha-2}\right)\mathrm{T}_{2}\left(t^{\alpha-1}\right)$. $\phi_{p}\left(\cdot\right)$ is a p-Laplacian operator, $\phi_{p}\left(s\right)=|s|^{p-2}s$, $p>1$, $\phi_{p}\left(0\right)=0$.

The p-Laplacian operator originated from the research of turbulent flow in porous medium. Leibenson [18] recommended the p-Laplacian equation as below:

$$\left(\phi_{p}\left(x'\left(t\right)\right)\right)'=f\left(t,x\left(t\right),x'\left(t\right)\right).$$

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Tingting Xue is corresponding author, with School of Mathematics and Physics, Xinjiang Institute of Engineering, Urumqi, 830000 PR China; e-mail: xuett@cumt.edu.cn.

Xing Chen is with School of Mathematics and Physics, Xinjiang Institute of Engineering, Urumqi, 830000 PR China.

Xiaolin Fan is with School of Mathematics and Physics, Xinjiang Institute of Engineering, Urumqi, 830000 PR China.

Later, many scholars have developed a strong interest in the p-Laplacian operator and obtained some excellent results, as shown in [20-21, 24-30]. Since the p-Laplacian operator is nonlinear, the continuous theorem of Mawhin is no longer applicable to study the p-Laplacian problems. Applying the extended continuous theorem, Jiang [21] discussed fractional p-Laplacian problems as follows:

$$\begin{cases} D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}u(t)\right) = -f\left(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha}u(t)\right), t \in (0,1), \\ u(0) = D_{0+}^{\alpha}u(0) = 0, u(1) = \int_0^1 h(t)u(t)dt, \end{cases}$$

where $0 < \beta \leq 1,\ 1 < \alpha \leq 2,\ \int_0^1 h(t)t^{\alpha-1}dt = 1$. And inspired by [21], this paper studies a more general functional boundary value problem (1). In summary, the results of this paper will further expand and enrich the work of [12,13,21].

II. Preliminaries

Here, some relevant definitions and lemmas are shown as follows, for more details, please see [20-23].

Definition 2.1 ([20]). Let X, Z be real Banach spaces with norms $\|\cdot\|_X, \|\cdot\|_Z$, respectively. A continuous operator $M : X \cap \mathrm{dom}M \to Z$ is said to be quasi-linear if

(i) $\mathrm{Im}M := M(X \cap \mathrm{dom}M)$ is a closed subset of Z,

(ii)$\mathrm{Ker}M := \{x \in X \cap \mathrm{dom}M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,
where $\mathrm{dom}M$ denotes the domain of the operator M.

Definition 2.2 ([21]). Let $X_1 = \mathrm{Ker}M$ and X_2 be the complement space of X_1 in X. Then $X = X_1 \oplus X_2$. Let $P : X \to X_1$ be projector and $\Omega \in X$ be an open and bounded set with the origin $\theta \in \Omega$. Suppose that $N_\lambda : \overline{\Omega} \to Z,\ \lambda \in [0,1]$ is a continuous and bounded operator. Denote N_1 by N. Let $\sum_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$. N_λ is said to be M-quasi-compact in $\overline{\Omega}$ if there exists a vector subspace Z_1 of Z satisfying $\dim Z_1 = \dim X_1$ and two operators Q and R such that for $\lambda \in [0,1]$,

(a) $\mathrm{Ker}Q = \mathrm{Im}M$,

(b) $QN_\lambda x = \theta,\ \lambda \in (0,1) \Leftrightarrow QNx = \theta$,

(c) $R(\cdot, 0)$ is the zero operator and $R(\cdot,\lambda)\big|_{\sum_\lambda} = (I-P)\big|_{\sum_\lambda}$,

(d) $M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda$,
where $Q : Z \to Z_1,\ QZ = Z_1$ is continuous, bounded and satisfies $Q(I-Q) = 0$ and $R : \overline{\Omega} \times [0,1] \to X_2$ is continuous and compact.

Lemma 2.1 ([21]). Let X and Z be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Z$, respectively, and $\Omega \in X$ be an open and bounded nonempty set. Suppose

$$M : X \cap \mathrm{dom}M \to Z$$

is a quasi-linear operator and that $N_\lambda : \overline{\Omega} \to Z,\ \lambda \in [0,1]$ is M-quasi-compact. In addition, if the following conditions hold:

$(C_1)\ Mx \neq N_\lambda x,\ \forall x \in \partial\Omega \cap \mathrm{dom}M,\ \lambda \in (0,1)$,

$(C_2)\ \deg\{JQN,\ \Omega \cap \mathrm{Ker}M,\ 0\} \neq 0$,
then the abstract equation $Mx = Nx$ has at least one solution in $\mathrm{dom}M \cap \overline{\Omega}$, where $N = N_1,\ J : \mathrm{Im}Q \to \mathrm{Ker}M$ is a homeomorphism with $J(\theta) = \theta$.

Definition 2.3([22]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.4([22]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}y(s)ds,$$

provided the right-hand side is pointwise defined on $(0,\infty)$, where $n = [\alpha] + 1$.

Lemma 2.2([22]). If $n-1 < \alpha \leq n,\ u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation $D_{0+}^{\alpha}u(t) = 0$ has the solution

$$u(t) = c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},\ c_i \in \mathbb{R},\ n = [\alpha]+1.$$

Lemma 2.3([22]). Let $n-1 < \alpha \leq n$, if $D_{0+}^{\alpha}u(t) \in C(0,1) \cap L^1(0,1)$, then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n},$$

for some $c_i \in \mathbb{R},\ i = 1,2,\cdots,n,\ n = [\alpha]+1$.

Lemma 2.4([22]). Assume $u \in C[0,1],\ 0 \leq p \leq q$, then

$$D_{0+}^{p}I_{0+}^{q}u(t) = I_{0+}^{q-p}u(t).$$

Lemma 2.5([22]). Let $\alpha \geq 0$,
(i) if $\lambda > -1$, and $\lambda \neq \alpha - i,\ i = 1,2,\cdots,[\alpha]+1$, then

$$D_{0+}^{\alpha}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha},$$

(ii) $D_{0+}^{\alpha}t^{\alpha-i} = 0, i = 1,2,\cdots,[\alpha]+1$.

Lemma 2.6([23]). Assume $a, b \in R$, then

(i) $(|a|+|b|)^p \leq |a|^p + |b|^p,\ 0 < p \leq 1$,

(ii) $(|a|+|b|)^p \leq 2^{p-1}(|a|^p + |b|^p),\ p > 1$.

III. Main result

For convenience, this article makes $X = \left\{u | u, D_{0+}^{\alpha-2}u, D_{0+}^{\alpha-1}u, D_{0+}^{\alpha}u \in C[0,1]\right\}$, the norm of which is as follows:

$$\|u\|_X = \max\left\{\|u\|_\infty, \left\|D_{0+}^{\alpha-2}u\right\|_\infty, \left\|D_{0+}^{\alpha-1}u\right\|_\infty, \left\|D_{0+}^{\alpha}u\right\|_\infty\right\}.$$

And, let $Y = C[0,1]$ with norm $\|y\|_Y = \|y\|_\infty$, where $\|y\|_\infty = \max_{t\in[0,1]} |y(t)|$. It is easy to see that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two Banach spaces. In order to prove the theorem, the hypothesis of this paper are given below.

(A_0) Functionals $\mathrm{T}_i : X \to \mathbb{R}$, $i = 1,2$, satisfy the relations $\mathrm{T}_1\left(t^{\alpha-2}\right) = \gamma_1, \mathrm{T}_1\left(t^{\alpha-1}\right) = \gamma_2, \mathrm{T}_2\left(t^{\alpha-2}\right) = k\gamma_1, \mathrm{T}_2\left(t^{\alpha-1}\right) = k\gamma_2$, where $\gamma_1, \gamma_2, k \in \mathbb{R}, \gamma_1^2 + \gamma_2^2 \neq 0$.

(A_1) Functionals $\mathrm{T}_i : X \to \mathbb{R}$ are linear bounded with the respective norms $\|\mathrm{T}_i\|$, $i = 1,2$.

(A_2) Functional

$$F(y) = (\mathrm{T}_2 - k\mathrm{T}_1)\left(I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta} y\right)\right)\right), \ \frac{1}{p} + \frac{1}{q} = 1,$$

is increasing.

(H_1) There exists $M_0 > 0$, such that if $\left|D_{0+}^{\alpha-2} u(t)\right| + \left|D_{0+}^{\alpha-1} u(t)\right| > M_0$, then $F(Nu) \neq 0$.

(H_2) There exist nonnegative functions $a, b, c, d, e \in C[0,1]$ such that

$$|f(t,u,v,w,z)| \leq a(t) + b(t)|u|^{p-1} + c(t)|v|^{p-1} + d(t)|w|^{p-1} + e(t)|z|^{p-1}, t \in [0,1], u, v, w, z \in \mathbb{R},$$

with

$$\frac{A_1^{p-1} L\|b\|_\infty + L\|c\|_\infty + L\|d\|_\infty + \|e\|_\infty}{\Gamma(\beta+1)} < 1, \quad (2)$$

where $A_1 = \frac{1}{\Gamma(\alpha+1)} + \frac{2}{\Gamma(\alpha)} + \frac{7}{2\Gamma(\alpha-1)}$, $L = \max\left\{1, 2^{p-2}\right\}$.

(H_3) There exists $M_1 > 0$, such that if $|c| > M_1$, then one of these is established:

$$cQN\left(c\left(\gamma_2 t^{\alpha-2} - \gamma_1 t^{\alpha-1}\right)\right) > 0, \quad (3)$$

$$cQN\left(c\left(\gamma_2 t^{\alpha-2} - \gamma_1 t^{\alpha-1}\right)\right) < 0. \quad (4)$$

(H_4) There exists $M_0{}' > 0$ such that if $\left|D_{0+}^{\alpha-1} u(t)\right| > M_0{}'$, then $F(Nu) \neq 0$.

(H_5) There exist nonnegative functions $a, b, c, d, e \in C[0,1]$ such that

$$|f(t,u,v,w,z)| \leq a(t) + b(t)|u|^{p-1} + c(t)|v|^{p-1} + d(t)|w|^{p-1} + e(t)|z|^{p-1}, t \in [0,1], u, v, w, z \in \mathbb{R},$$

with

$$C_2 L_q \left(\frac{\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty}{\Gamma(\beta+1)}\right)^{q-1} \times \left[1 + \frac{\|\mathrm{T}_1\|_\infty \left(|\gamma_1|\Gamma(\alpha) + |\gamma_2|\Gamma(\alpha-1)\right)}{\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha-1)}\right] < 1, \quad (5)$$

where $M_\alpha = \max\left\{\left(|r_1| + |r_2|\right), \left(|r_1|\Gamma(\alpha) + |r_2|\Gamma(\alpha-1)\right)\right\}$, $C_2 = M_\alpha/\gamma_1\Gamma(\alpha) + 1$, $L_q = \max\left\{1, 2^{q-2}\right\}$.

This paper defines the operators $M : X \cap \mathrm{dom}M \to Y$, $N_\lambda : X \to Y$, and their specific forms are as follows:

$$\begin{aligned} Mu(t) &= D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha} u(t)\right), \\ N_\lambda u(t) &= \lambda f\left(t, u(t), D_{0+}^{\alpha-2} u(t), D_{0+}^{\alpha-1} u(t), D_{0+}^{\alpha} u(t)\right), \quad t, \lambda \in [0,1], \end{aligned} \quad (6)$$

where

$$\mathrm{dom}M = \left\{u \in X \,\middle|\, D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha} u\right) \in Y, \ u(0) = D_{0+}^{\alpha} u(0) = 0, \mathrm{T}_1(u) = \mathrm{T}_2(u) = 0\right\}. \quad (7)$$

Then FFBVPs (1) can be converted to the operator equation $Mu = Nu$, $u \in \mathrm{dom}M, N = N_1$. For the purpose of proving the theorem in this paper, some relevant lemmas are shown as follows.

Lemma 3.1. M is a quasi-linear operator, and

$$\mathrm{Ker}M = \left\{u \in X \,\middle|\, u(t) = c\left(\gamma_2 t^{\alpha-2} - \gamma_1 t^{\alpha-1}\right), \ c \in \mathbb{R}\right\}, \quad (8)$$

$$\mathrm{Im}M = \{y \in Y \,|\, F(y) = 0\}. \quad (9)$$

Proof. The condition (A_0) and Lemma 2.2 imply that (8) holds. Clearly, $\mathrm{dim}\mathrm{Ker}M = 1$ and $\mathrm{Ker}M$ is linearly homeomorphic to $\mathbb{R}$.

If $y \in \mathrm{Im}M$, there exists a function $u \in \mathrm{dom}M$ with $D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha} u(t)\right) = y(t)$. Lemma 2.3 and conditions $u(0) = D_{0+}^{\alpha} u(0) = 0$ show that

$$u(t) = I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta} y\right) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, c_1, c_2 \in \mathbb{R}.$$

Functional boundary condition $\mathrm{T}_i(u) = 0$, $i = 1,2$ implies that

$$\mathrm{T}_1(u) = \mathrm{T}_1\left(I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta} y\right)\right) + c_1\gamma_2 + c_2\gamma_1 = 0,$$

$$\mathrm{T}_2(u) = \mathrm{T}_2\left(I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta} y\right)\right) + c_1 k\gamma_2 + c_2 k\gamma_1 = 0.$$

Obviously, $(\mathrm{T}_2 - k\mathrm{T}_1)\left(I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta} y\right)\right) = 0$, i.e. $F(y) = 0$.

Conversely, suppose $y \in Y$ and satisfies $F(y) = 0$. Let

$$u(t) = I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta} y\right) - \frac{\mathrm{T}_1\left(I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta} y\right)\right)}{\gamma_1^2 + \gamma_2^2}\left(\gamma_1 t^{\alpha-2} + \gamma_2 t^{\alpha-1}\right),$$

then, we have $u \in \mathrm{dom}M$ and $Mu(t) = D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha} u(t)\right) = y(t)$. So, $y \in \mathrm{Im}M$, and $\mathrm{Im}M \subset Y$ is closed. Hence, M is a quasi-linear operator. □

Define operators $P : X \to \mathrm{Ker}M$ and $Q : Y \to \mathbb{R}$ as follows:

$$Pu(t) = \frac{\gamma_2 D_{0+}^{\alpha-2} u(0) - \gamma_1 D_{0+}^{\alpha-1} u(0)}{\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha-1)}\left(\gamma_2 t^{\alpha-2} - \gamma_1 t^{\alpha-1}\right), \quad t \in [0,1], \quad (10)$$

and $Qy(t) = c$. Among them, $F(y - c) = 0$. Obviously, P is a projector and $\mathrm{Ker}Q = \mathrm{Im}M$.

Lemma 3.2. $Q : Y \to Y_1$ is continuous, bounded and $Q(I - Q)y = Q(y - Qy) = 0$, $y \in Y$, $QY = Y_1$, where $Y_1 = \mathbb{R}$.

Proof. For $y \in Y$, the hypothetical condition (A_2) implies that the function $F(y - c)$ is continuous, decreasing on c. We take $a_1 = \min_{t\in[0,1]} y(t)$, $a_2 = \max_{t\in[0,1]} y(t)$. Then $F(y - a_1) > 0$, $F(y - a_2) < 0$. Therefore, there is a unique constant $c \in [a_1, a_2]$, which satisfies $F(y - c) = 0$. So,

Q is well defined. If $y_1, y_2 \in Y$, then $Qy_1 = c_1,\ Qy_2 = c_2$. Since ϕ_q is strictly increasing, if $c_2 - c_1 > \|y_2 - y_1\|_\infty$, then

$$\begin{aligned}0 &= F(y_2 - c_2) = (\mathrm{T}_2 - k\mathrm{T}_1)\left(I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(y_2 - c_2)\right)\right)\right)\\ &= (\mathrm{T}_2 - k\mathrm{T}_1)\left(I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(y_1 - c_1 + (y_2 - y_1) - (c_2 - c_1))\right)\right)\right)\\ &< (\mathrm{T}_2 - k\mathrm{T}_1)\left(I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(y_1 - c_1)\right)\right)\right) = F(y_1 - c_1) = 0.\end{aligned}$$

Obviously, it is contradictory. Conversely, if $c_2 - c_1 < -\|y_2 - y_1\|_\infty$, then

$$\begin{aligned}0 &= F(y_2 - c_2) = F(y_1 - c_1 + (y_2 - y_1) - (c_2 - c_1))\\ &> F(y_1 - c_1) = 0.\end{aligned}$$

Clearly, it is also contradictory. Therefore,

$$|c_2 - c_1| \le \|y_2 - y_1\|_\infty.$$

So, Q is continuous. In addition, if $\Omega \subset Y$ is bounded, then $Q(\Omega)$ is bounded. In other words, Q is bounded. According to the definition of Q, $Q(I - Q)y = Q(y - Qy) = 0,\ y \in Y$ and $QY = Y_1$. □

Lemma 3.3. The operator $R : X \times [0,1] \to X_2$ is as follows:

$$\begin{aligned}R(u,\lambda)(t) =& I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)\\ &- \frac{\mathrm{T}_1\left(I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)}{\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha - 1)}\\ &\times\left(\gamma_1\Gamma(\alpha)t^{\alpha-2} + \gamma_2\Gamma(\alpha - 1)t^{\alpha-1}\right),\end{aligned}$$

where $\mathrm{Ker}M \oplus X_2 = X$. Then $R : \overline{\Omega} \times [0,1] \to X_2$ is continuous and compact, where $\Omega \subset X$ is bounded.

Proof. The continuity of T_1, Q, f imply that $R(u,\lambda)$ is continuous. Thus, there exist two constants $k_1, k_2 > 0$ such that $\left|f\left(t, u(t), D_{0+}^{\alpha-2}u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha}u(t)\right)\right| \le k_1, |Qf| \le k_2$ for $u \in \overline{\Omega}$. Then, one has

$$\begin{aligned}|R(u,\lambda)| &\le \left[1 + \frac{\|\mathrm{T}_1\|_\infty(|\gamma_1|\Gamma(\alpha) + |\gamma_2|\Gamma(\alpha - 1))}{\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha - 1)}\right]\\ &\quad\times\left\|I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)\right\|_X\\ &\le \left[1 + \frac{\|\mathrm{T}_1\|_\infty(|\gamma_1|\Gamma(\alpha) + |\gamma_2|\Gamma(\alpha - 1))}{\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha - 1)}\right]\\ &\quad\times\frac{1}{\Gamma(\alpha + 1)}\phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right),\end{aligned}$$

$$\begin{aligned}\left|D_{0+}^{\alpha-2}R(u,\lambda)\right| &= \left|\int_0^t (t - s)\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u(s)\right)\right)ds\right.\\ &\quad - \frac{\mathrm{T}_1\left(I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)}{\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha - 1)}\\ &\quad\left.\times\Gamma(\alpha)\Gamma(\alpha - 1)(\gamma_1 + \gamma_2 t)\right|\\ &\le \left[\frac{1}{2} + \frac{\|\mathrm{T}_1\|_\infty(|\gamma_1| + |\gamma_2|)}{\alpha(\gamma_1^2(\alpha - 1) + \gamma_2^2)}\right]\\ &\quad\times\phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right),\end{aligned}$$

$$\begin{aligned}\left|D_{0+}^{\alpha-1}R(u,\lambda)\right| &= \left|\int_0^t \phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u(s)\right)ds\right.\\ &\quad - \frac{\mathrm{T}_1\left(I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)}{\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha - 1)}\\ &\quad\left.\times\Gamma(\alpha)\Gamma(\alpha - 1)\gamma_2\right|\\ &\le \left[1 + \frac{\|\mathrm{T}_1\|_\infty|\gamma_2|}{\alpha(\gamma_1^2(\alpha - 1) + \gamma_2^2)}\right]\\ &\quad\times\phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right),\end{aligned}$$

$$\left|D_{0+}^{\alpha}R(u,\lambda)\right| = \left|\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right| \le \phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right).$$

Therefore, R is bounded in $\overline{\Omega} \times [0,1]$. For $(u,\lambda) \in \overline{\Omega} \times [0,1]$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we get

$$\begin{aligned}&|R(u,\lambda)(t_2) - R(u,\lambda)(t_1)|\\ &\le \left|I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)(t_2) - I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)(t_1)\right|\\ &\quad + \frac{\|\mathrm{T}_1\|_\infty\phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right)}{\Gamma(\alpha + 1)(\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha - 1))}\\ &\quad\times\left[|\gamma_1|\Gamma(\alpha)\left(t_2^{\alpha-2} - t_1^{\alpha-2}\right) + |\gamma_2|\Gamma(\alpha - 1)\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)\right].\end{aligned}$$

Considering

$$\begin{aligned}&\left|I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)(t_2) - I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)(t_1)\right|\\ &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2 - s)^{\alpha-1}\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)ds\right.\\ &\quad\left. - \int_0^{t_1}(t_1 - s)^{\alpha-1}\left(\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)\right)ds\right|\\ &\le \frac{\phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right)}{\Gamma(\alpha)}\left|\int_0^{t_1}\left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right]ds + \int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}ds\right|\\ &\le \frac{(t_2^{\alpha} - t_1^{\alpha})}{\Gamma(\alpha + 1)} \times \phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right),\end{aligned}$$

combined with the above results, we obtain

$$\begin{aligned}&|R(u,\lambda)(t_2) - R(u,\lambda)(t_1)| \le \frac{\phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right)}{\Gamma(\alpha + 1)} \times [(t_2^{\alpha} - t_1^{\alpha})\\ &+ \frac{\|\mathrm{T}_1\|_\infty\left(|\gamma_1|\Gamma(\alpha)\left(t_2^{\alpha-2} - t_1^{\alpha-2}\right) + |\gamma_2|\Gamma(\alpha - 1)\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)\right)}{(\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha - 1))}\Bigg].\end{aligned}$$

Thus, $\left\{R(u,\lambda)\,\middle|\,(u,\lambda) \in \overline{\Omega} \times [0,1]\right\}$ is equicontinuous. At the same time, we can also work out

$$\begin{aligned}&\left|D_{0+}^{\alpha-2}R(u,\lambda)(t_2) - D_{0+}^{\alpha-2}R(u,\lambda)(t_1)\right|\\ &\le \left|\int_0^{t_2}(t_2 - s)\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)ds\right.\\ &\quad\left. - \int_0^{t_1}(t_1 - s)\phi_q\left(I_{0+}^{\beta}(I - Q)N_\lambda u\right)ds\right|\\ &\quad + \frac{\|\mathrm{T}_1\|_\infty\phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right)}{\Gamma(\alpha + 1)(\gamma_1^2\Gamma(\alpha) + \gamma_2^2\Gamma(\alpha - 1))}\\ &\quad\times\Gamma(\alpha)\Gamma(\alpha - 1)|\gamma_2|(t_2 - t_1)\\ &\le \phi_q\left(\frac{k_1 + k_2}{\Gamma(\beta + 1)}\right) \times \left[\frac{(t_2^2 - t_1^2)}{2} + \frac{\|\mathrm{T}_1\|_\infty|\gamma_2|(t_2 - t_1)}{\alpha(\gamma_1^2(\alpha - 1) + \gamma_2^2)}\right],\end{aligned}$$

and

$$\begin{aligned}&\left|D_{0+}^{\alpha-1}R\left(u,\lambda\right)\left(t_2\right)-D_{0+}^{\alpha-1}R\left(u,\lambda\right)\left(t_1\right)\right|\\&\leq\left|\int_0^{t_2}\phi_q\left(I_{0+}^{\beta}\left(I-Q\right)N_\lambda u\right)ds\right.\\&\quad\left.-\int_0^{t_1}\phi_q\left(I_{0+}^{\beta}\left(I-Q\right)N_\lambda u\right)ds\right|\\&\leq\left(t_2-t_1\right)\phi_q\left(\frac{k_1+k_2}{\Gamma\left(\beta+1\right)}\right).\end{aligned}$$

Thus, $\left\{D_{0+}^{\alpha-2}R\left(u,\lambda\right)\middle|(u,\lambda)\in\overline{\Omega}\times[0,1]\right\}$ and $\left\{D_{0+}^{\alpha-1}R\left(u,\lambda\right)\middle|(u,\lambda)\in\overline{\Omega}\times[0,1]\right\}$ are equicontinuous. Finally, we will verify $\left\{D_{0+}^{\alpha}R\left(u,\lambda\right)\middle|(u,\lambda)\in\overline{\Omega}\times[0,1]\right\}$ is equicontinuous. If $(u,\lambda)\in\overline{\Omega}\times[0,1]$, $t_1,t_2\in[0,1]$, $t_1<t_2$, then

$$\begin{aligned}&\left|D_{0+}^{\alpha}R\left(u,\lambda\right)\left(t_2\right)-D_{0+}^{\alpha}R\left(u,\lambda\right)\left(t_1\right)\right|\\&=\left|\phi_q\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^{t_2}\left(t_2-s\right)^{\beta-1}\left(I-Q\right)N_\lambda u\left(s\right)ds\right)\right.\\&\left.-\phi_q\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^{t_1}\left(t_1-s\right)^{\beta-1}\left(I-Q\right)N_\lambda u\left(s\right)ds\right)\right|.\end{aligned}$$

Since

$$\begin{aligned}&\left|\frac{1}{\Gamma\left(\beta\right)}\int_0^{t}\left(t-s\right)^{\beta-1}\left(I-Q\right)N_\lambda u\left(s\right)ds\right|\\&\leq\frac{\left(k_1+k_2\right)}{\Gamma\left(\beta+1\right)},(u,\lambda)\in\overline{\Omega}\times[0,1],\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{1}{\Gamma\left(\beta\right)}\int_0^{t_2}\left(t_2-s\right)^{\beta-1}\left(I-Q\right)N_\lambda u\left(s\right)ds\right.\\&\left.-\frac{1}{\Gamma\left(\beta\right)}\int_0^{t_1}\left(t_1-s\right)^{\beta-1}\left(I-Q\right)N_\lambda u\left(s\right)ds\right|\\&=\frac{1}{\Gamma\left(\beta\right)}\left|\int_0^{t_1}\left[\left(t_2-s\right)^{\beta-1}-\left(t_1-s\right)^{\beta-1}\right]\left(I-Q\right)N_\lambda u\left(s\right)ds\right.\\&\qquad\left.+\int_{t_1}^{t_2}\left(t_2-s\right)^{\beta-1}\left(I-Q\right)N_\lambda u\left(s\right)ds\right|\\&\leq\frac{\left(k_1+k_2\right)}{\Gamma\left(\beta+1\right)}\left(t_2^{\beta}-t_1^{\beta}\right),\end{aligned}$$

and considering that ϕ_q is uniformly continuous in $[-(k_1+k_2)/\Gamma(\beta+1),(k_1+k_2)/\Gamma(\beta+1)]$, we obtain that $\left\{D_{0+}^{\alpha}R\left(u,\lambda\right)\middle|(u,\lambda)\in\overline{\Omega}\times[0,1]\right\}$ is equicontinuous, too. Applying the Arzelà-Ascoli theorem, we can get that $R:\overline{\Omega}\times[0,1]\to X_2$ is compact. □

Lemma 3.4. If $\Omega\subset X$ is an open and bounded set, then the operator N_λ is M-quasi-compact in $\overline{\Omega}$.

Proof. Clearly, $\mathrm{Im}P=\mathrm{Ker}M$, $\dim\mathrm{Ker}M=\dim\mathrm{Im}Q$, $Q(I-Q)=0$, $\mathrm{Ker}Q=\mathrm{Im}M$, $R(\cdot,0)=0$, meanwhile, the condition (b) of Definition 2.2 is satisfied. For $u\in\sum_\lambda=\{u\in\overline{\Omega}:Mu=N_\lambda u\}$, we obtain $QN_\lambda u=0$ and $N_\lambda u=D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}u\left(t\right)\right)$. The conditions $D_{0+}^{\alpha}u\left(0\right)=u\left(0\right)=D_{0+}^{\alpha}R\left(u,\lambda\right)\left(0\right)=R\left(u,\lambda\right)\left(0\right)=0$ imply that

$$u\left(t\right)=I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}N_\lambda u\right)\right)+\frac{D_{0+}^{\alpha-1}u\left(0\right)}{\Gamma\left(\alpha\right)}t^{\alpha-1}+\frac{D_{0+}^{\alpha-2}u\left(0\right)}{\Gamma\left(\alpha-1\right)}t^{\alpha-2}.$$

Using $\mathrm{T}_1\left(u\right)=0$, $\mathrm{T}_1\left(t^{\alpha-1}\right)=\gamma_2$, $\mathrm{T}_1\left(t^{\alpha-2}\right)=\gamma_1$, we have

$$\begin{aligned}R\left(u,\lambda\right)&=I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}N_\lambda u\right)\right)-\frac{\mathrm{T}_1\left(I_{0+}^{\alpha}\phi_q\left(I_{0+}^{\beta}N_\lambda u\right)\right)}{\gamma_1^2\Gamma\left(\alpha\right)+\gamma_2^2\Gamma\left(\alpha-1\right)}\\&\quad\times\left(\gamma_1\Gamma\left(\alpha\right)t^{\alpha-2}+\gamma_2\Gamma\left(\alpha-1\right)t^{\alpha-1}\right)\\&=u\left(t\right)-\frac{D_{0+}^{\alpha-1}u\left(0\right)}{\Gamma\left(\alpha\right)}t^{\alpha-1}-\frac{D_{0+}^{\alpha-2}u\left(0\right)}{\Gamma\left(\alpha-1\right)}t^{\alpha-2}\\&\quad-\frac{\mathrm{T}_1\left(u\left(t\right)-\frac{D_{0+}^{\alpha-1}u(0)}{\Gamma(\alpha)}t^{\alpha-1}-\frac{D_{0+}^{\alpha-2}u(0)}{\Gamma(\alpha-1)}t^{\alpha-2}\right)}{\gamma_1^2\Gamma\left(\alpha\right)+\gamma_2^2\Gamma\left(\alpha-1\right)}\\&\quad\times\left(\gamma_1\Gamma\left(\alpha\right)t^{\alpha-2}+\gamma_2\Gamma\left(\alpha-1\right)t^{\alpha-1}\right)\\&=u\left(t\right)-\frac{D_{0+}^{\alpha-1}u\left(0\right)}{\Gamma\left(\alpha\right)}t^{\alpha-1}-\frac{D_{0+}^{\alpha-2}u\left(0\right)}{\Gamma\left(\alpha-1\right)}t^{\alpha-2}\\&\quad+\frac{\frac{D_{0+}^{\alpha-1}u(0)}{\Gamma(\alpha)}\gamma_2+\frac{D_{0+}^{\alpha-2}u(0)}{\Gamma(\alpha-1)}\gamma_1}{\gamma_1^2\Gamma\left(\alpha\right)+\gamma_2^2\Gamma\left(\alpha-1\right)}\\&\quad\times\left(\gamma_1\Gamma\left(\alpha\right)t^{\alpha-2}+\gamma_2\Gamma\left(\alpha-1\right)t^{\alpha-1}\right)\\&=u\left(t\right)+\frac{\gamma_2D_{0+}^{\alpha-2}u\left(0\right)-\gamma_1D_{0+}^{\alpha-1}u\left(0\right)}{\gamma_1^2\Gamma\left(\alpha\right)+\gamma_2^2\Gamma\left(\alpha-1\right)}\cdot\gamma_1t^{\alpha-1}\\&\quad-\frac{\gamma_2D_{0+}^{\alpha-2}u\left(0\right)-\gamma_1D_{0+}^{\alpha-1}u\left(0\right)}{\gamma_1^2\Gamma\left(\alpha\right)+\gamma_2^2\Gamma\left(\alpha-1\right)}\cdot\gamma_2t^{\alpha-2}\\&=u\left(t\right)-Pu=\left(I-P\right)u.\end{aligned}$$

Thus, the condition (c) of Definition 2.2 is fulfilled. For $u\in\overline{\Omega}$, one has

$$M\left[Pu+R\left(u,\lambda\right)\right]=N_\lambda u-QN_\lambda u=\left(I-Q\right)N_\lambda u.$$

That is, the condition (d) of Definition 2.2 is fulfilled. Hence, N_λ is M-compact in $\overline{\Omega}$. □

Lemma 3.5. Suppose (H_1) and (H_2) hold, then

$$\Omega_1=\{u\in\mathrm{dom}M\,|Mu=N_\lambda u\ ,\ \lambda\in(0,1)\}$$

is bounded in X.

Proof. For $u\in\mathrm{dom}M$, Lemma 2.3 implies that $u(t)=I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}$. Using $u(0)=0$, one has $c_3=0$. That is,

$$u(t)=I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2}.$$

Thus,

$$\begin{aligned}&D_{0+}^{\alpha-1}u(t)=I_{0+}^{1}D_{0+}^{\alpha}u(t)+c_1\Gamma\left(\alpha\right),\\&D_{0+}^{\alpha-2}u(t)=I_{0+}^{2}D_{0+}^{\alpha}u(t)+c_1\Gamma\left(\alpha\right)t+c_2\Gamma\left(\alpha-1\right),\\&c_1=\frac{1}{\Gamma\left(\alpha\right)}\left(D_{0+}^{\alpha-1}u(t)-\int_0^{t}D_{0+}^{\alpha}u(s)ds\right),\\&c_2=\frac{1}{\Gamma\left(\alpha-1\right)}\left[D_{0+}^{\alpha-2}u(t)-\int_0^{t}\left(t-s\right)D_{0+}^{\alpha}u(s)ds\right.\\&\qquad\left.-\left(D_{0+}^{\alpha-1}u(t)-\int_0^{t}D_{0+}^{\alpha}u(s)ds\right)t\right].\end{aligned}$$

For $u\in\Omega_1$, one has $QN_\lambda u=0$. Hypothetical condition (H_1) means that there exists $t_0\in[0,1]$ such that $\left|D_{0+}^{\alpha-1}u(t_0)\right|\leq M_0$, $\left|D_{0+}^{\alpha-2}u(t_0)\right|\leq M_0$. By using the

relations

$$D_{0+}^{\alpha-1}u(t)=D_{0+}^{\alpha-1}u(t_0)+\int_{t_0}^{t}D_{0+}^{\alpha}u(s)ds,$$
$$D_{0+}^{\alpha-2}u(t)=D_{0+}^{\alpha-2}u(t_0)+\int_{t_0}^{t}D_{0+}^{\alpha-1}u(s)ds,$$

we can get

$$\begin{aligned}&\left\|D_{0+}^{\alpha-1}u\right\|_{\infty}\le M_0+\left\|D_{0+}^{\alpha}u\right\|_{\infty},\\&\left\|D_{0+}^{\alpha-2}u\right\|_{\infty}\le 2M_0+\left\|D_{0+}^{\alpha}u\right\|_{\infty},\\&|c_1|\le\frac{1}{\Gamma(\alpha)}\left(M_0+2\left\|D_{0+}^{\alpha}u\right\|_{\infty}\right),\\&|c_2|\le\frac{1}{\Gamma(\alpha-1)}\left(3M_0+\frac{7}{2}\left\|D_{0+}^{\alpha}u\right\|_{\infty}\right).\end{aligned}$$

Thus,

$$\|u\|_{\infty}\le A_1\left\|D_{0+}^{\alpha}u\right\|_{\infty}+B_1,$$

where $A_1=\frac{1}{\Gamma(\alpha+1)}+\frac{2}{\Gamma(\alpha)}+\frac{7}{2\Gamma(\alpha-1)}$, $B_1=\left(\frac{1}{\Gamma(\alpha)}+\frac{3}{\Gamma(\alpha-1)}\right)M_0$.
The hypothetical condition (H_2), combined with $Mu=N_\lambda u$ and $D_{0+}^{\alpha}u(0)=0$, means

$$\begin{aligned}&\left|\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right|\\&\le\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\left|f\left(s,u(s),D_{0+}^{\alpha-2}u(s),D_{0+}^{\alpha-1}u(s),D_{0+}^{\alpha}u(s)\right)\right|ds\\&\le\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}[a(s)+b(s)|u(s)|^{p-1}+c(s)\left|D_{0+}^{\alpha-2}u(s)\right|^{p-1}\\&\quad+d(s)\left|D_{0+}^{\alpha-1}u(s)\right|^{p-1}+e(s)\left|D_{0+}^{\alpha}u(s)\right|^{p-1}]ds\\&\le\frac{1}{\Gamma(\beta+1)}\left(\|a\|_{\infty}+\|b\|_{\infty}\|u\|_{\infty}^{p-1}+\|c\|_{\infty}\left\|D_{0+}^{\alpha-2}u\right\|_{\infty}^{p-1}\right.\\&\quad\left.+\|d\|_{\infty}\left\|D_{0+}^{\alpha-1}u\right\|_{\infty}^{p-1}+\|e\|_{\infty}\left\|D_{0+}^{\alpha}u\right\|_{\infty}^{p-1}\right).\end{aligned}$$

This, together with $\left|\phi_p\left(D_{0+}^{\alpha}u(t)\right)\right|=\left|D_{0+}^{\alpha}u(t)\right|^{p-1}$ and Lemma 2.6, means

$$\begin{aligned}&\|D_{0+}^{\alpha}u\|_{\infty}^{p-1}\\&\le\frac{1}{\Gamma(\beta+1)}[\|a\|_{\infty}+\|b\|_{\infty}\left(A_1\|D_{0+}^{\alpha}u\|_{\infty}+B_1\right)^{p-1}\\&\quad+\|c\|_{\infty}\left(2M_0+\|D_{0+}^{\alpha}u\|_{\infty}\right)^{p-1}\\&\quad+\|d\|_{\infty}\left(M_0+\|D_{0+}^{\alpha}u\|_{\infty}\right)^{p-1}+\|e\|_{\infty}\|D_{0+}^{\alpha}u\|_{\infty}^{p-1}]\\&\le B_2+\frac{A_1^{p-1}L\|b\|_{\infty}+L\|c\|_{\infty}+L\|d\|_{\infty}+\|e\|_{\infty}}{\Gamma(\beta+1)}\cdot\|D_{0+}^{\alpha}u\|_{\infty}^{p-1},\end{aligned}$$

where $L=\max\left\{1,2^{p-2}\right\}$, $B_2=\frac{1}{\Gamma(\beta+1)}\cdot\left[\|a\|_{\infty}+L\|b\|_{\infty}B_1^{p-1}+L\|c\|_{\infty}(2M_0)^{p-1}+L\|d\|_{\infty}(M_0)^{p-1}\right]$.
The inequality (2) implies that there exists $M_2>0$ such that

$$\begin{aligned}&\left\|D_{0+}^{\alpha}u\right\|_{\infty}\le M_2,\left\|D_{0+}^{\alpha-1}u\right\|_{\infty}\le M_0+M_2:=M_3,\\&\left\|D_{0+}^{\alpha-2}u\right\|_{\infty}\le 2M_0+M_2:=M_4,\\&\|u\|_{\infty}\le A_1M_2+B_1:=M_5.\end{aligned}$$

Thus, one has

$$\begin{aligned}\|u\|_X&=\max\left\{\|u\|_{\infty},\left\|D_{0+}^{\alpha-2}u\right\|_{\infty},\left\|D_{0+}^{\alpha-1}u\right\|_{\infty},\left\|D_{0+}^{\alpha}u\right\|_{\infty}\right\}\\&\le\max\{M_5,M_4,M_3,M_2\}:=M_6.\end{aligned}$$

Hence, Ω_1 is bounded. □

Lemma 3.6. If the hypothetical condition (H_3) holds, then

$$\Omega_2=\{u\in\mathrm{Ker}M\,|QNu=0\}$$

is bounded in X.

Proof. For $u\in\Omega_2$, one has $u(t)=c\left(\gamma_2t^{\alpha-2}-\gamma_1t^{\alpha-1}\right)$, $c\in\mathbb{R}$, $t\in[0,1]$ and $F(Nu)=0$. The hypothetical condition (H_3) implies $|c|\le M_1$. Thus,

$$\begin{aligned}&\|u\|_X\\&\le M_1\left\|\gamma_2t^{\alpha-2}-\gamma_1t^{\alpha-1}\right\|_X\\&\le\max\left\{M_1\left(|\gamma_1|+|\gamma_2|\right),M_1\left(|\gamma_1|\Gamma(\alpha)+|\gamma_2|\Gamma(\alpha-1)\right)\right\}\\&:=M_7.\end{aligned}$$

Hence, Ω_2 is bounded. □

The main results of this paper are as follows.

Theorem 3.1 . Let $f\in C([0,1]\times\mathbb{R}^4,\mathbb{R})$. Suppose that the hypothetical conditions (A_0)-(A_2) and (H_1)-(H_3) hold. Then FFBVPs (1) has at least a solution.

Proof. Let $\Omega\supset\overline{\Omega_1}\cup\overline{\Omega_2}\cup\{u|u\in X,\|u\|_X\le\max\{M_6,M_7\}+1\}$ be an open and bounded set of X. Lemmas 3.5, 3.6 imply that $Mu\ne N_\lambda u$, $u\in\mathrm{dom}M\cap\partial\Omega$ and $QNu\ne0$, $u\in\mathrm{Ker}M\cap\partial\Omega$.

Let $H(u,\delta)=\rho\delta u+(1-\delta)JQNu$, $\delta\in[0,1]$, $u\in\mathrm{Ker}M\cap\overline{\Omega}$, where $J:\mathrm{Im}Q\to\mathrm{Ker}M$ is a homeomorphism with $Jc=c(r_2t^{\alpha-2}-r_1t^{\alpha-1})$, $\rho=1$ or $\rho=-1$, if (3) or (4) hold, respectively. For $u\in\mathrm{Ker}M\cap\partial\Omega$, one has $u=c(r_2t^{\alpha-2}-r_1t^{\alpha-1})\ne0$. If $\delta=1$, then $H(u,1)=\rho c(r_2t^{\alpha-2}-r_1t^{\alpha-1})\ne0$. If $\delta=0$, by applying Lemma 3.6, then $H(u,0)=QN(c(r_2t^{\alpha-2}-r_1t^{\alpha-1}))(r_2t^{\alpha-2}-r_1t^{\alpha-1})\ne0$. For $0<\delta<1$, $u(t)=c(r_2t^{\alpha-2}-r_1t^{\alpha-1})$. Lemma 3.6 implies $\|u\|_X\le M_1\left\|\gamma_2t^{\alpha-2}-\gamma_1t^{\alpha-1}\right\|_X$. So, $|c|>M_1$. If

$$\begin{aligned}&H\left(c\left(\gamma_2t^{\alpha-2}-\gamma_1t^{\alpha-1}\right),\delta\right)=\rho\delta c\left(\gamma_2t^{\alpha-2}-\gamma_1t^{\alpha-1}\right)\\&+(1-\delta)QN\left(c\left(\gamma_2t^{\alpha-2}-\gamma_1t^{\alpha-1}\right)\right)\left(\gamma_2t^{\alpha-2}-\gamma_1t^{\alpha-1}\right)=0,\end{aligned}$$

using (3), one has

$$c^2=-\frac{1-\delta}{\rho\delta}\cdot cQN\left(c\left(\gamma_2t^{\alpha-2}-\gamma_1t^{\alpha-1}\right)\right)<0,$$

which is a contradiction. Hence, $H(u,\delta)\ne0$, $u\in\mathrm{Ker}M\cap\partial\Omega$, $\delta\in[0,1]$. Based on the homotopy invariance of degree, we have

$$\begin{aligned}\deg(JQN,\Omega\cap\mathrm{Ker}M,0)&=\deg(H(\cdot,0),\Omega\cap\mathrm{Ker}M,0)\\&=\deg(H(\cdot,1),\Omega\cap\mathrm{Ker}M,0)\\&=\deg(H(\rho I,\Omega\cap\mathrm{Ker}M,0)\ne0.\end{aligned}$$

Lemma 2.1 implies that $Mu=Nu$ has at least a solution in $\mathrm{dom}M\cap\overline{\Omega}$. That is, FFBVPs (1) has at least a solution in X. □

For another result of FFBVPs (1), assume the inequality $\left|D_{0+}^{\alpha-2}u(t)\right|+\left|D_{0+}^{\alpha-2}u(t)\right|>M_0$ in the condition (H_1) is replaced by $\left|D_{0+}^{\alpha-1}u(t)\right|>M_0{}'$ or $\left|D_{0+}^{\alpha-2}u(t)\right|>M_0{}''$, which may lead to slight changes in the proof of Lemma 3.5. In particular, we remember that Theorem 3.1 satisfies the relation $\gamma_1^2+\gamma_2^2\ne0$.

Theorem 3.2. Let $f\in C([0,1]\times\mathbb{R}^4,\mathbb{R})$. Suppose that $\gamma_1\ne0$ and the conditions (A_0)-(A_2) and (H_3)-(H_5) hold. Then FFBVPs (1) has at least a solution.

Proof. For $u \in \Omega_1,\ QNu = 0$. This hypothetical condition (H_4) implies that there exists $t_1 \in [0,1]$ such that $\left|D_{0+}^{\alpha-1}u\left(t_1\right)\right| \leq M_0'$. By Lemma 3.4, $R(u,\ \lambda) = (I-P)u$. So, $u(t) = Pu(t) + (I-P)u(t) = Pu(t) + R(u,\ \lambda)$. Then

$$\left|D_{0+}^{\alpha-1}Pu\left(t_1\right)\right| \leq M_0' + \|R(u,\ \lambda)\|_X.$$

According to the definition of P, one has

$$\left|\frac{\gamma_2 D_{0+}^{\alpha-2}u\left(0\right) - \gamma_1 D_{0+}^{\alpha-1}u\left(0\right)}{\gamma_1^2\Gamma\left(\alpha\right) + \gamma_2^2\Gamma\left(\alpha-1\right)}\right| \leq \frac{1}{\gamma_1\Gamma\left(\alpha\right)}(M_0' + \|R(u,\lambda)\|_X).$$

Therefore,

$$\begin{aligned}\|u\|_X &\leq \|Pu\|_X + \|R(u,\ \lambda)\|_X \\ &\leq \frac{\max\left\{\left(|r_1|+|r_2|\right),\ \left(|r_1|\Gamma\left(\alpha\right)+|r_2|\Gamma\left(\alpha-1\right)\right)\right\}}{\gamma_1\Gamma\left(\alpha\right)} \\ &\quad \times \left(M_0{}' + \|R(u,\ \lambda)\|_X\right) + \|R(u,\ \lambda)\|_X \\ &\leq C_1 + C_2\|R(u,\ \lambda)\|_X,\end{aligned}$$

where $C_1 = M'_0 M_\alpha/\gamma_1\Gamma\left(\alpha\right),\ C_2 = M_\alpha/\gamma_1\Gamma\left(\alpha\right)+1,\ M_\alpha = \max\left\{\left(|r_1|+|r_2|\right),\left(|r_1|\Gamma\left(\alpha\right)+|r_2|\Gamma\left(\alpha-1\right)\right)\right\}$.
Considering

$$\begin{aligned}\|R(u,\ \lambda)\|_X &\leq \left[1 + \frac{\|\mathrm{T}_1\|_\infty\left(|r_1|\Gamma\left(\alpha\right)+|r_2|\Gamma\left(\alpha-1\right)\right)}{\gamma_1^2\Gamma\left(\alpha\right) + \gamma_2^2\Gamma\left(\alpha-1\right)}\right] \\ &\quad \times \left\|I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}\left(I-Q\right)N_\lambda u\right)\right)\right\|_X \\ &\leq \left[1 + \frac{\|\mathrm{T}_1\|_\infty\left(|r_1|\Gamma\left(\alpha\right)+|r_2|\Gamma\left(\alpha-1\right)\right)}{\gamma_1^2\Gamma\left(\alpha\right) + \gamma_2^2\Gamma\left(\alpha-1\right)}\right] \\ &\quad \times \left(\frac{1}{\Gamma\left(\beta+1\right)}\right)^{q-1}\|N_\lambda u\|_\infty^{q-1},\end{aligned}$$

and then using the hypothetical condition (H_5) and Lemma 2.6, one has

$$\begin{aligned}\|u\|_X &\leq C_1 + C_2\left[1 + \frac{\|\mathrm{T}_1\|_\infty\left(|r_1|\Gamma\left(\alpha\right)+|r_2|\Gamma\left(\alpha-1\right)\right)}{\gamma_1^2\Gamma\left(\alpha\right) + \gamma_2^2\Gamma\left(\alpha-1\right)}\right] \\ &\quad \times \left(\frac{1}{\Gamma\left(\beta+1\right)}\right)^{q-1} \\ &\quad \times \Big(\|a\|_\infty + \|b\|_\infty\|u\|_\infty^{p-1} + \|c\|_\infty\left\|D_{0+}^{\alpha-2}u\right\|_\infty^{p-1} \\ &\quad + \|d\|_\infty\left\|D_{0+}^{\alpha-1}u\right\|_\infty^{p-1} + \|e\|_\infty\|D_{0+}^{\alpha}u\|_\infty^{p-1}\Big)^{q-1} \\ &\leq C_1 + C_2\left[1 + \frac{\|\mathrm{T}_1\|_\infty\left(|r_1|\Gamma\left(\alpha\right)+|r_2|\Gamma\left(\alpha-1\right)\right)}{\gamma_1^2\Gamma\left(\alpha\right) + \gamma_2^2\Gamma\left(\alpha-1\right)}\right] \\ &\quad \times \left(\frac{1}{\Gamma\left(\beta+1\right)}\right)^{q-1} \\ &\quad \times \left[\|a\|_\infty + \left(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty\right)\|u\|_X^{p-1}\right]^{q-1} \\ &\leq C_1 + C_2L_q\left[1 + \frac{\|\mathrm{T}_1\|_\infty\left(|r_1|\Gamma\left(\alpha\right)+|r_2|\Gamma\left(\alpha-1\right)\right)}{\gamma_1^2\Gamma\left(\alpha\right) + \gamma_2^2\Gamma\left(\alpha-1\right)}\right] \\ &\quad \times \left(\frac{1}{\Gamma\left(\beta+1\right)}\right)^{q-1} \\ &\quad \times \left[\|a\|_\infty^{q-1} + \left(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty\right)^{q-1}\|u\|_X\right].\end{aligned}$$

The inequality (5) implies that Ω_1 is bounded. The rest of the proof, similar to Theorem 3.1, which is ignored here.

Remark 3.1. When the inequality $\left|D_{0+}^{\alpha-1}u\left(t\right)\right| > M_0{}'$ of (H_4) is replaced by $\left|D_{0+}^{\alpha-2}u\left(t\right)\right| > M_0{}''$, the proof of the existence of the solution of FFBVPs (1) is similar to Theorem 3.2. It is not explained in detail here.

The above conclusions are discussed under resonance conditions. For non-resonance conditions, we give the following conclusion.

Theorem 3.3. Let $f \in C([0,1]\times\mathbb{R}^4,\mathbb{R})$. If $\mathrm{T}_1,\ \mathrm{T}_2 : C[0,1] \to \mathbb{R}$ are continuous linear functionals satisfying the condition $\mathrm{T}_1\left(t^{\alpha-1}\right)\mathrm{T}_2\left(t^{\alpha-2}\right) \neq \mathrm{T}_1\left(t^{\alpha-2}\right)\mathrm{T}_2\left(t^{\alpha-1}\right)$, then FFBVPs (1) has a unique solution if and only if the following operator $A : C[0,1] \to C[0,1]$ has a unique fixed point, where

$$\begin{aligned}&(Au)(t) \\ &= I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t))) \\ &+ \frac{D_1}{D}t^{\alpha-1} + \frac{D_2}{D}t^{\alpha-2},\end{aligned}$$

$$\begin{aligned}D &= \begin{vmatrix}\mathrm{T}_1\left(t^{\alpha-1}\right) & \mathrm{T}_1\left(t^{\alpha-2}\right) \\ \mathrm{T}_2\left(t^{\alpha-1}\right) & \mathrm{T}_2\left(t^{\alpha-2}\right)\end{vmatrix} \\ &= \mathrm{T}_1\left(t^{\alpha-1}\right)\mathrm{T}_2\left(t^{\alpha-2}\right) - \mathrm{T}_1\left(t^{\alpha-2}\right)\mathrm{T}_2\left(t^{\alpha-1}\right) \neq 0,\end{aligned}$$

$$D_1 = \begin{vmatrix}-\mathrm{T}_1(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)))) \\ -\mathrm{T}_2(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)))) \\ \mathrm{T}_1\left(t^{\alpha-2}\right) \\ \mathrm{T}_2\left(t^{\alpha-2}\right)\end{vmatrix},$$

$$D_2 = \begin{vmatrix}\mathrm{T}_1\left(t^{\alpha-1}\right) \\ \mathrm{T}_2\left(t^{\alpha-1}\right) \\ -\mathrm{T}_1(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)))) \\ -\mathrm{T}_2(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t))))\end{vmatrix}.$$

Proof. If u is a solution to $Au = u$, we have

$$D_{0+}^{\beta}\phi_p\left(D_{0+}^{\alpha}u\left(t\right)\right) = f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)).$$

By Lemma 2.3, we get

$$\begin{aligned}&\phi_p\left(D_{0+}^{\alpha}u\left(t\right)\right) \\ &= I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)) + c_0t^{\beta-1}.\end{aligned}$$

It follows from $D_{0+}^{\alpha}u(0) = 0$ that $c_0 = 0$. So, we obtain

$$\phi_p\left(D_{0+}^{\alpha}u\left(t\right)\right) = I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t)).$$

Then

$$D_{0+}^{\alpha}u\left(t\right) = \phi_q(I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t))).$$

By Lemma 2.3, one has

$$\begin{aligned}&u\left(t\right) \\ &= I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t))) \\ &+ c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3}.\end{aligned}$$

It follows from $u(0) = 0$ that $c_3 = 0$. Thus,

$$\begin{aligned}&u\left(t\right) \\ &= I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f(t,u(t),D_{0+}^{\alpha-2}u(t),D_{0+}^{\alpha-1}u(t),D_{0+}^{\alpha}u(t))) \\ &+ c_1t^{\alpha-1} + c_2t^{\alpha-2}.\end{aligned}$$

From $\mathrm{T}_1(u) = \mathrm{T}_2(u) = 0$, we have

$$\begin{cases}\mathrm{T}_1(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f) + c_1t^{\alpha-1} + c_2t^{\alpha-2}) = 0, \\ \mathrm{T}_2(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f) + c_1t^{\alpha-1} + c_2t^{\alpha-2}) = 0.\end{cases}$$

Considering the T_1, T_2 are continuous linear functionals, we get

$$\begin{cases} \mathrm{T}_1(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f)) + c_1\mathrm{T}_1(t^{\alpha-1}) + c_2\mathrm{T}_1(t^{\alpha-2}) = 0, \\ \mathrm{T}_2(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f)) + c_1\mathrm{T}_2(t^{\alpha-1}) + c_2\mathrm{T}_2(t^{\alpha-2}) = 0. \end{cases}$$

Then,

$$\begin{cases} c_1\mathrm{T}_1(t^{\alpha-1}) + c_2\mathrm{T}_1(t^{\alpha-2}) = -\mathrm{T}_1(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f)), \\ c_1\mathrm{T}_2(t^{\alpha-1}) + c_2\mathrm{T}_2(t^{\alpha-2}) = -\mathrm{T}_2(I_{0+}^{\alpha}\phi_q(I_{0+}^{\beta}f)). \end{cases}$$

According to Cramer's Rule, we obtain

$$c_1 = \frac{D_1}{D},\ c_2 = \frac{D_2}{D}.$$

Thus, u is a solution to FFBVPs (1). If u is a solution of FFBVPs (1), then $(Au)(t) = u(t)$. Therefore, FFBVPs (1) has one unique solution if and only if the operator equation $Au = u$ has a unique solution. □

An example is given below to verify the rationality of Theorem 3.2.

Example 3.1. Consider the following FFBVPs:

$$\begin{cases} D_{0+}^{\frac{1}{2}}\phi_{\frac{3}{2}}\left(D_{0+}^{\frac{5}{2}}u(t)\right) = \\ \quad f\left(t, u(t), D_{0+}^{\frac{1}{2}}u(t), D_{0+}^{\frac{3}{2}}u(t), D_{0+}^{\frac{5}{2}}u(t)\right), t\in(0,1), \\ u(0) = D_{0+}^{\frac{5}{2}}u(0) = 0, \\ \mathrm{T}_1(u) = D_{0+}^{\frac{3}{2}}u(1) + D_{0+}^{\frac{1}{2}}u(1) = 0, \\ \mathrm{T}_2(u) = 3D_{0+}^{\frac{3}{2}}u(1) + 2\int_0^1 D_{0+}^{\frac{1}{2}}u(t)\,dt = 0, \end{cases} \tag{11}$$

where $\alpha = \frac{5}{2}, \beta = \frac{1}{2}, p = \frac{3}{2}$, and

$$\begin{aligned} &f\left(t, u(t), D_{0+}^{\frac{1}{2}}u(t), D_{0+}^{\frac{3}{2}}u(t), D_{0+}^{\frac{5}{2}}u(t)\right) \\ &= \frac{1}{36} + \frac{1}{36}\times\left[\sin\left(\sqrt{|u(t)|}\right) + \sin\left(\sqrt{\left|D_{0+}^{\frac{1}{2}}u(t)\right|}\right)\right. \\ &\quad \left. + \left(\sqrt{\left|D_{0+}^{\frac{3}{2}}u(t)\right|}\right) + \sin\left(\sqrt{\left|D_{0+}^{\frac{5}{2}}u(t)\right|}\right)\right]. \end{aligned}$$

FFBVPs (11) is at resonance with

$$\begin{aligned} &\mathrm{Ker}M = \left\{c\left(\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} - \frac{2t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right), c\in\mathbb{R}\right\}, \\ &D_{0+}^{\frac{3}{2}}u(t) = c, D_{0+}^{\frac{1}{2}}u(t) = c(t-2), \\ &\mathrm{T}_1\left(t^{\frac{1}{2}}\right) = \gamma_1 = \frac{\sqrt{\pi}}{2} \neq 0 \\ &\mathrm{T}_1\left(t^{\frac{3}{2}}\right) = \gamma_2 = \frac{3\sqrt{\pi}}{2}, \\ &k = 2, \|\mathrm{T}_1\| = 2. \end{aligned}$$

Obviously, these assumptions (A_0)-(A_1) satisfy. Take $\|a\|_\infty = \|b\|_\infty = \|c\|_\infty = \|d\|_\infty = \|e\|_\infty = \frac{1}{36}$ and $q = 3$, then

$$\begin{aligned} C_2 &= \frac{\max\left\{\left(|\gamma_1|+|\gamma_2|\right), \left(|\gamma_1|\Gamma(\alpha)+|\gamma_2|\Gamma(\alpha-1)\right)\right\}}{\gamma_1\Gamma(\alpha)} + 1 \\ &= \frac{\max\left\{2\sqrt{\pi}, \frac{9}{8}\pi\right\}}{\frac{3}{8}\pi} + 1 \\ &= \frac{16}{3\sqrt{\pi}} + 1, \end{aligned}$$

$$\begin{aligned} &\frac{C_2\max\left\{1, 2^{q-2}\right\}}{\left(\Gamma(\beta+1)\right)^{q-1}}\times\left(\|b\|_\infty + \|c\|_\infty + \|d\|_\infty + \|e\|_\infty\right)^{q-1} \\ &\times\left[1 + \frac{\|\mathrm{T}_1\|_\infty\left(|\gamma_1|\Gamma(\alpha)+|\gamma_2|\Gamma(\alpha-1)\right)}{\gamma_1^2\Gamma(\alpha)+\gamma_2^2\Gamma(\alpha-1)}\right] \\ &= \frac{8\left(\frac{16}{3\sqrt{\pi}}+1\right)\left(1+\frac{4}{\sqrt{\pi}}\right)}{81\pi} < 1, \end{aligned}$$

and

$$\begin{aligned} &|f(t,u,v,w,z)| \\ &\le \frac{1}{36} + \frac{1}{36}\sqrt{|u|} + \frac{1}{36}\sqrt{|v|} + \frac{1}{36}\sqrt{|w|} + \frac{1}{36}\sqrt{|z|} \\ &= \frac{1}{36} + \frac{1}{36}|u|^{p-1} + \frac{1}{36}|v|^{p-1} + \frac{1}{36}|w|^{p-1} + \frac{1}{36}|z|^{p-1}. \end{aligned}$$

Clearly, condition (H_5) holds. If $\left|D_{0+}^{\frac{3}{2}}u(t)\right| > M_0' = 4$, then

$$\begin{aligned} &f(t,u,v,w,z) \\ &= \frac{1}{36}\left(1 + \sin\sqrt{|u|} + \sin\sqrt{|v|} + \sqrt{|w|} + \sin\sqrt{|z|}\right) > 0, \end{aligned}$$

and

$$\begin{aligned} F(Nu) &= \left(\mathrm{T}_2 - k\mathrm{T}_1\right)\left(I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}Nu\right)\right)\right) \\ &= 3D_{0+}^{\frac{3}{2}}\left(I_{0+}^{\frac{5}{2}}\left(\phi_q\left(I_{0+}^{\beta}Nu\right)\right)\right)(1) \\ &\quad + 2\int_0^1 D_{0+}^{\frac{1}{2}}\left(I_{0+}^{\frac{5}{2}}\left(\phi_q\left(I_{0+}^{\beta}Nu\right)\right)\right)(s)ds \\ &\quad - 2\left[D_{0+}^{\frac{3}{2}}\left(I_{0+}^{\frac{5}{2}}\left(\phi_q\left(I_{0+}^{\beta}Nu\right)\right)\right)(1)\right. \\ &\quad \left. + D_{0+}^{\frac{1}{2}}\left(I_{0+}^{\frac{5}{2}}\left(\phi_q\left(I_{0+}^{\beta}Nu\right)\right)\right)(1)\right] \\ &= \int_0^1 \phi_q\left(I_{0+}^{\beta}Nu\right)ds - 2\int_0^1 (1-s)\phi_q\left(I_{0+}^{\beta}Nu\right)ds \\ &\quad + 2\int_0^1\int_0^t (t-s)\,\phi_q\left(I_{0+}^{\beta}Nu\right)dsdt \\ &= \int_0^1 s^2\phi_q\left(I_{0+}^{\beta}Nu\right)ds \neq 0. \end{aligned}$$

This implies condition (H_4) holds. Similarly, choose $M_1 > 0$, such that for

$$u(t) = c\left(\frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} - \frac{2t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right), \left|D_{0+}^{\frac{3}{2}}u(t)\right| = |c| > M_0' > M_1,$$

$$F(Nu) = \left(\mathrm{T}_2 - k\mathrm{T}_1\right)\left(I_{0+}^{\alpha}\left(\phi_q\left(I_{0+}^{\beta}f(t,u(t)), c(t-2), c, 0\right)\right)\right) \neq 0.$$

provided $|c| > M_1$. So, condition (H_3) holds. Hence, FFBVPs (11) satisfies all conditions of Theorem 3.2. That is, there is at least one solution to FFBVPs (11).

IV. Conclusion

This paper studies the solvability of fractional p-Laplacian equations with functional boundary value conditions at non-resonance and resonance respectively. By applying the existence theorems and the extension of continuation theorems, three results on this problem are obtained (see Theorem 3.1, 3.2, 3.3). Finally, the article uses examples to verify the rationality of the results. Our article aims to further extend the results of [12,13] to non-linear cases, and to some extent, to generalize and enrich the existing results.

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