Agents, Activity Levels and Utility Distributing Mechanism: Game-theoretical Viewpoint

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Abstract—In general, agents and its activity levels might be two essential factors under real-world situations. Thus, we propose a consistent solution to analyze utility distributing mechanism by focusing on the agents and its activity levels at the same time. Two existing concepts from traditional game theory are also applied to reinterpret this paper. First, by applying consistency which related to an extended reduction, two axiomatic results are offered to discuss the rationality of this solution. Second, based on excess mapping, two dynamic processes are constructed to illustrate that this solution can be obtained by agents who begin from an arbitrary efficient outcome and make succeeding modifications.

Index Terms—Utility distributing mechanism, reduction, axiomatic result, dynamic process.

I. INTRODUCTION

In most interactive systems (economic systems, management systems, operating systems, environmental systems, etc.), models, or programs, attention is usually paid to the optimal or equilibrium state of an allocation or processing concept, often called the solution. To highlight the advantages of a solution, simply proclaiming how superior it is does not necessarily lead to a majority of people accepting this solution. In axiomaticization, various mathematical theories are often used to analyze and propose the optimal or equilibrium states of these solutions. Axiomatization is a mathematical theory that first uses various mathematical theories to model systems, models, procedures, and certain fair, just, and well-recognized properties based on game theory. A relevant solution is then proposed, analyzed, and proven to be the only solution that satisfies certain fair, just, and well-recognized properties. It is important that these properties are indispensable. In this way, agreeing with these fair, just and well-recognized properties is equivalent to agreeing with the solution. On the other hand, if an allocation or processing concept is not agreed upon by the majority simultaneously, usually some form of communication, debate or negotiation will be used to revise the concept so that it gradually becomes one that can be agreed upon by the majority – such is the so-called dynamic process.

Under the theory of traditional cooperative games, a characteristic mapping is determined over whole the subsets of the collection of agents. This implies that the choices available for each agent are either to partake completely or not to partake at all. Under real-world situations, however, agents might take different activity levels to partake. Therefore, it is reasonable that the agents could be allowed to adopt different activity levels of participation in a coalition. A multi-choice game could be looked upon as a natural analogue of a traditional game in which each agent takes several activity levels. For example, an application appears in a large enterprise with many sections, where the income-making depends on its expressions. This forms a multi-choice condition in which the agents are the sections and the worth of a coalition where each section functions at a certain level is the corresponding income secured by the enterprise. As a result, the domain of the characteristic mapping is extended to allow multi-choice coalition.

In the axiomatic formulation for solutions in game theory, consistency is an important property. Consistency declares the independence of an outcome with respect to fixing several agents with its allotted payoffs. It claims that the proposal made for any issue should always assent with the proposal made in the subissue that arises if the payoffs of several agents are settled on. It has been defined in distinct situations depending upon how the payoffs of the agents that "leave the bargaining" are determined. This axiom has been prove under various issues by applying reductions. In addition to axiomatic analysis, dynamic processes could be depicted that lead the agents to that solution, starting from an efficient outcome.

This article focuses on the solution concept of the equal allocation of non-separable costs (EANSC, Ransmeier [12]). Based on the notion of the EANSC, all agents firstly obtain its marginal contributions from the grand coalition, and further allot the rest of utilities equally. Under traditional games, Moulin [10] introduced a notion of reduction and related consistency to show that the EANSC could provide a fair rule for distributing utility. By determining overall outcomes for a given agents on multi-choice games, Liao [5] proposed an extended EANSC by applying the maximal marginal contributions of agents. Inspired by the notion of replication due to Nouveland et al. [11], Liao [6] introduce the duplicate EANSC to compute overall values for a given agents. Related researches also could be refered to Cheng et al. [1], Hwang and Liao [3], Liao et al. [8], and so on. In real-world situations, however, agents and its activity levels might
be two essential factors at the same time. These mentioned above raise one motivation under multi-choice consideration:

- whether the EANSC could extended by considering agents and its activity levels at the same time.

This article is aimed at answering above motivation. Some existing results of traditional game theory would be extended in this article. The main results are as follows.

1) By considering the agents and the activity levels on multi-choice games simultaneously, a generalization of the traditional EANSC, the multi-choice equal allocation of non-separable costs (MCEANSC), is defined in Section 2.

2) By extending the reduction proposed by Moulin [10] to multi-choice games, we provide two axiomatic results to present the rationality of the MCEANSC in Section 3. We show that

- the MCEANSC is the unique allocation matching the properties of bilateral consistency and standard for two-person games;
- the MCEANSC is the unique allocation matching the properties of efficiency, bilateral consistency, symmetry and zero-independence.

3) A solution could be analyzed by axiomatic justification. Alternatively, dynamic processes could be depicted that lead the agents to that solution, starting from an efficient outcome. The basis of a dynamic analysis was laid by Stearns [14]. Different from dynamic result due to Maschler and Owen [9], the excess mapping is introduced to analyze dynamic processes leading to the MCEANSC in Section 4, starting from an efficient outcome. Some more applications, comparisons, connections and statements are also discussed throughout this article.

II. PRELIMINARIES

Let $U$ be the universe of agents. For $i \in U$, one could set $G_i = \{0, 1, \cdots, g_i\}$ as the activity level collection of agent $i$ and $G_i^+ = G_i \setminus \{0\}$, where $g_i \in \mathbb{N}$ and level 0 implies not partaking. For $A \subseteq U$, $A \neq \emptyset$, let $G^A = \bigcap_{i \in A} G_i$ be the product set of the activity level collections for agents in $A$. A multi-choice coalition is a vector $\zeta \in G^A$. The i-th coordinate $\zeta_i$ of $\zeta$ is the activity level of agent $i$ in $\zeta$. A multi-choice coalition could be depicted as a set of economic agents, i.e., agents, who deliver fractions of its representation to a group strategy maker, the multi-choice coalition. The term multi-choice coalition also appears if the importance for examining the qualification of a agent in a coalition is pondered. Denote the zero vector in $\mathbb{R}^A$ to be $0_A$. The multi-choice coalition 0 corresponds to the empty agent-coalition. Let $\zeta \in G^A$ and $K \subseteq A$. $|K|$ is the amount of agents in $K$, $N(\zeta) = \{i \in A | \zeta_i \neq 0\}$ and $\zeta_K \in \mathbb{R}^K$ is the restriction of $\zeta$ to $K$.

A multi-choice transferable-utility (TU) game is $(A, g, u)$, where $A \neq \emptyset$ is finite set of agents and $u : G^A \rightarrow \mathbb{R}$ is a characteristic mapping which assigns to each $\zeta = (\zeta_i)_{i \in A} \in G^A$ the value that the agents can get when each agent $i$ partakes at level $\zeta_i$ with $u(0_A) = 0$. The mapping $u$ assigns to each multi-choice coalition $\zeta = (\zeta_i)_{i \in A} \in G^A$ a value, explaining what such a coalition can accomplish in cooperation.

Denote the class of all multi-choice TU games by $\Delta$. Given $(A, g, u) \in \Delta$, let $P^A = \{(i, k_i) | i \in A, k_i \in G_i^+\}$. A solution on $\Delta$ is a mapping $\kappa$ assorting to each $(A, g, u) \in \Delta$ a vector

$$\kappa(A, g, u) = (\kappa_{i,k_i}(A, g, u))_{(i,k_i) \in P^A} \in \mathbb{R}^{P^A}.$$  

Here $\kappa_{i,k_i}(A, g, u)$ is the value of the agent $i$ if it participates in a coalition with membership $k_i$ in $u$. For convenience, one could define $\kappa_{i,0}(A, g, u) = 0$ for each $i \in A$.

Next, a multi-choice generalization of the equal allocation of non-separable costs is provided as follows.

**Definition 1:** The multi-choice equal allocation of non-separable costs (MCEANSC), $\overline{\theta}$, is the mapping on $\Delta$ which associates to each $(A, g, u) \in \Delta$, each agent $i \in A$ and each $k_i \in G_i$, the value

$$\overline{\theta}_{i,k_i}(A, g, u) = \theta_{i,k_i}(A, g, u) + \frac{1}{|A|} \cdot \left[u(g) - \sum_{j \in A} \theta_{j,g_j}(A, g, u)\right],$$

where $\theta_{i,k_i}(A, g, u) = u(g_{A_k(\zeta_i)}, k_i) - u(g_{A_{k_i}(\zeta)}, 0)$ is the level-marginal contribution of the agent $i$ and its activity level $k_i$. Under $\overline{\theta}$, all agents get its level-marginal contributions respectively, and further allot the rest of utility equally.

Subsequently, one would like to demonstrate that the MCEANSC could provide “optimal or balanced allocating mechanisms” among all agents, in the sense that this agency can obtain payoff from each combination of operational levels of all agents under multi-choice performances.

III. REDUCTION AND AXIOMATIC RESULTS

This section would show that there exists a specific reduction that could be applied to characterize the MCEANSC.

Let $\kappa$ be a solution. $\kappa$ matches efficiency (EFF) if for all $(A, g, u) \in \Delta$, $\sum_{i \in A} \kappa_{i,g_i}(A, g, u) = u(g)$. $\kappa$ matches standard for two-person games (STPG) if for all $(A, g, u) \in \Delta$ and $|A| \leq 2$, $\kappa(A, g, u) = \overline{\theta}(A, g, u)$. $\kappa$ matches symmetry (SYM) if for all $(A, g, u) \in \Delta$ with $u(\zeta, k_i, g_j) - u(\zeta, 0, g_j) = u(\zeta, g_i, k_j) - u(\zeta, g_i, 0)$ for some $(i, k_i), (j, k_j) \in P^A$ and for all $\zeta \in G^A(\{j\})$, $\kappa_{i,k_i}(A, g, u) = \kappa_{j,k_j}(A, g, u) \cdot \kappa$ matches zero-independence (ZI) if for all $(A, g, u), (A, g, v) \in \Delta$ with $u(\zeta, v(\zeta) \in E(\zeta, c_{ij}) \subseteq \mathbb{R}^A$, and for all $c \in \mathbb{R}^A$, $\kappa_{i,k_i}(A, g, u) = \kappa_{i,g_i}(A, g, v) + c$.

Axioms EFF is famous and diffusely acceptable. EFF claims that all agents allot the usability completely if all agents partake at full steam. Axiom STPG is a generalized analogue of Hart and Mas-Colell’s [2] two-person standardness property, as generated for the Shapley value [13]. STPG claims that each agent obtains the payoff based on $\overline{\theta}$ in two-person situations. Axiom SYM claims that the payoffs of two agents should be the same if the marginal contributions among them are equal. Axiom ZI could be explained as a mighty weak analogue of additivity. By Definition 2.1, it is easy to examine that the MCEANSC matches EFF, STPG, SYM and ZI.

Given a subdivision of a group of agents, and an outcome vector assigned by a solution under some game. Moulin [10] defined the reduction as that in which each alliance in this subdivision could attain outcomes to its elements only if they are consonant with the initial outcomes to “all” the elements.

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outside of this subdivision. In the following a multi-choice analogue of the Moulin’s reduction is defined. Let \((A, g, u) \in \Delta, S \subseteq A\) and \(\kappa\) be a solution. The reduction \((S, g_S, u^\kappa_S)\) is defined as that for all \(\zeta \in G^S\),

\[
    u^\kappa_S(\zeta) = \begin{cases}
        0 & \text{if } \zeta = 0_S, \\
        u(\zeta, g_A \setminus S) - \sum_{i \in A \setminus S} \kappa_{i,g_i}(A, g, u) & \text{otherwise}.
    \end{cases}
\]

The bilateral consistency prerequisite could be depicted briefly as follows. For arbitrary group of two agents under a multi-choice game, one could introduces a “reduced mechanism” among them by considering the measures remaining after the rest of the agents are given the outcomes stipulated via a solution \(\kappa\) on \(\Delta\). \(\kappa\) is bilateral consistent if it emerges the same outcomes as under the initial situation when it is applied to arbitrary reduction. Officially, a solution \(\kappa\) satisfies bilateral consistency (BCON) if for every \((A, g, u) \in \Delta\) with \(|A| \geq 3\), for every \(S \subseteq A\) with \(|S| = 2\) and for every \((i, k) \in P^S\), \(\kappa_{i,k}(S, g_S, u^\kappa_S)\) is defined as that for all \(\zeta \in G^S\),

**Lemma 1:** The MCEANSC \(\theta\) matches BCON.

**Proof:** Let \((A, g, u) \in \Delta\) with \(|A| \geq 3\) and \(S \subseteq A\) with \(|S| = 2\). Assume that \(S = \{i, j\}\). By definition of \(\theta\), for every \((p, k) \in P^S\),

\[
    \theta_{p,k}(S, g_S, u^\kappa_S) = u^\kappa_S(k, g_j) - u^\kappa_S(0, g_j) + u^\kappa_S(g_A \setminus \{i\}, k) - u^\kappa_S(g_A \setminus \{i\}, 0).
\]

By definitions of \(u^\kappa_S\) and \(\theta\) and equations (1), (2),

\[
    \theta_{i,k}(S, g_S, u^\kappa_S) = \frac{1}{|S|} \left[ u^\kappa_S(k, g_j) - u^\kappa_S(0, g_j) + u^\kappa_S(g_A \setminus \{i\}, k) - u^\kappa_S(g_A \setminus \{i\}, 0) \right].
\]

Hence, the MCEANSC matches BCON.

Next, the MCEANSC would be characterized by means of related properties of efficiency, bilateral consistency, symmetry and zero-independence.

**Theorem 1:** Solution \(\kappa\) matches BCON and STPG on \(\Delta\). By BCON and STPG of \(\kappa\), it is easy to conclude that \(\kappa\) matches EFF. Let \((A, g, u) \in \Delta\). By STPG of \(\kappa\), \(\kappa(A, g, u) = \theta(A, g, u)\) if \(|A| = 2\). Similar to traditional games by putting a “dummy” agent to one-person situation, the proof is done if \(|A| = 1\). The condition \(|A| > 2\). Let \(i \in A\) and \(S = \{i, j\}\) for some \(j \in A \setminus \{i\}\), then for all \(k_i \subseteq G_i, k_j \subseteq G_j\),

\[
    \begin{align*}
    \kappa_{i,k_i}(A, g, u) - \kappa_{j,k_j}(A, g, u) &= \kappa_{i,k_i}(S, g_S, u^\kappa_S) - \kappa_{j,k_j}(S, g_S, u^\kappa_S) \\
    &= \theta_{i,k_i}(S, g_S, u^\kappa_S) - \theta_{j,k_j}(S, g_S, u^\kappa_S) \\
    &= u^\kappa_S(k_i, g_j) - u^\kappa_S(0, g_j) - u^\kappa_S(g_A \setminus \{i\}, k) + u^\kappa_S(g_A \setminus \{i\}, 0) \\
    &= u^\kappa_S(g_j, k_i) - u^\kappa_S(g, 0) - u^\kappa_S(g_A \setminus \{i\}, k_j) + u^\kappa_S(g_A \setminus \{i\}, 0).
    \end{align*}
\]

Similarly, \(\theta\) instead of \(\kappa\) in equation (3), one could derive that

\[
    \begin{align*}
    \theta_{i,k_i}(A, g, u) - \theta_{j,k_j}(A, g, u) &= u(g_A \setminus \{i\}, k_i) - u(g_A \setminus \{i\}, 0) \\
    &= u(g_A \setminus \{j\}, k_j) - u(g_A \setminus \{j\}, 0).
    \end{align*}
\]

By equations (3), (4),

\[
    \kappa_{i,k_i}(A, g, u) - \kappa_{j,k_j}(A, g, u) = \theta_{i,k_i}(A, g, u) - \theta_{j,k_j}(A, g, u).
\]

This implies that \(\kappa_{i,k_i}(A, g, u) - \theta_{i,k_i}(A, g, u) = w\) for every \((i, k_i)\). It remains to verify that \(w = 0\). By EFF of \(\kappa\) and \(\theta\),

\[
    0 = \sum_{i \in A} \left[ \kappa_{i,g_i}(A, g, u) - \theta_{i,g_i}(A, g, u) \right] = |A| \cdot w.
\]

Thus, \(w = 0\).
\[ v(g) = \kappa_{i,g}(A,g,v) + \kappa_{j,g}(A,g,v) = 2 \cdot \kappa_{i,g}(A,g,v). \]

By definition of \( v \) and equation (5),
\[ \kappa_{i,g}(A,g,v) = \frac{1}{2} \cdot \left[ u(g) - \theta_{i,g}(A,g,u) - \theta_{j,g}(A,g,u) \right]. \]

By ZI of \( \kappa \),
\[ \kappa_{i,k}(A,g,u) = \frac{1}{2} \cdot \left[ u(g) - \theta_{i,g}(A,g,u) - \theta_{j,g}(A,g,u) \right] + \theta_{i,k}(A,g,u) \]
\[ = \bar{\theta}_{i,k}(A,g,u). \]

Similarly, \( \kappa_{i,k}(A,g,u) = \bar{\theta}_{i,k}(A,g,u) \) for all \( k \in G_i \).
Hence, \( \kappa \) matches STPG.

**Theorem 2:** Solution \( \kappa \) matches EFF, BCON, SYM and ZI if and only if \( \kappa = \bar{\theta} \).

**Proof:** \( \bar{\theta} \) matches EFF, SYM and ZI by Definition 1.
The rest of proofs follow from Lemmas 1, 2 and Theorem 1.

The following examples could examine that each of the axioms applied in Theorems 1 and 2 is logically independent of the rest of axioms.

**Example 1:** Define a solution \( \kappa^1 \) to be as for every \((A,g,u) \in \Delta \) and for every \((i,k) \in P^A\),
\[ \kappa^1_{i,k}(A,g,u) = \begin{cases} \bar{\theta}_{i,k}(A,g,u) & \text{if } |A| \leq 2, \\ 0 & \text{otherwise}. \end{cases} \]
Clearly, \( \kappa^1 \) matches STPG, but it violates BCON.

**Example 2:** Define a solution \( \kappa^2 \) to be as for every \((A,g,u) \in \Delta \) and for every \((i,k) \in P^A\),
\[ \kappa^2_{i,k}(A,g,u) = \theta_{i,k}(A,g,u). \]
Clearly, \( \kappa^2 \) matches BCON, SYM and ZI, but it violates STPG and EFF.

**Example 3:** Define a solution \( \kappa^3 \) to be as for every \((A,g,u) \in \Delta \) and for every \((i,k) \in P^A\),
\[ \kappa^3_{i,k}(A,g,u) = \frac{u(g) - \theta_{i,g}(A,g,u)}{|A|} \]
Clearly, \( \kappa^3 \) matches EFF, BCON and SYM, but it violates ZI.

**Example 4:** Define a solution \( \kappa^4 \) to be as for every \((A,g,u) \in \Delta \) and for every \((i,k) \in P^A\),
\[ \kappa^4_{i,k}(A,g,u) = \left[ u(g) - u(g_{A \setminus \{i\}},0) \right] + \frac{1}{|A|} \cdot \left[ u(g) - \sum_{k \in A} \left[ u(g) - u(g_{A \setminus \{k\}},0) \right] \right]. \]
Clearly, \( \kappa^4 \) matches EFF, BCON and ZI, but it violates SYM.

**Example 5:** Define a solution \( \kappa^5 \) to be as for every \((A,g,u) \in \Delta \) and for every \((i,k) \in P^A\),
\[ \kappa^5_{i,k}(A,g,u) = \sum_{S \subseteq A \setminus S} \sum_{i \in S} \left[ u(g_{A \setminus \{i\} \cup \{k\}},0) - u(g_{A \setminus \{i\}}) \right] \]
Clearly, \( \kappa^5 \) matches EFF, SYM and ZI, but it violates BCON.

**IV. Excess Mapping and Dynamic Results**

Different from existing outcomes due to Maschler and Owen [9], this section introduces the excess mapping to present two dynamic results that leads the agents to the MCENASC, starting from an efficient outcome.

The set of efficient outcomes under a game \((A,g,u)\) is defined to be \( X(A,g,u) = \{\kappa(A,g,u) | \kappa \) is a efficient solution on \( \Delta \}. \)
Let \((A,g,u) \in \Delta \) and \( \kappa(A,g,u) \in X(A,g,u). \)

The excess of a level vector \( \zeta \in G^A \) at \( \kappa \) is \( EC(\zeta,u,\kappa) = u(\zeta) - \sum_{i \in A} \kappa_{i,g}(A,g,u). \)

The exess can be regarded to be the discontent of level vector \( \zeta \) if all agents obtain its outcomes from \( \kappa \) in \((A,g,u)\).

Let \((A,g,u) \in \Delta \) with \( A \geq 3 \), \( \kappa(A,g,u) \in X(A,g,u) \) and \( t > 0 \). Define the correction mapping \( r : X(A,g,u) \rightarrow X(A,g,u) \) to be as for all \((i,k) \in P^A\),
\[ r_{i,k}(\kappa(A,g,u)) = \kappa_{i,k}(A,g,u) + t \sum_{j \in A \setminus \{i\} \cup \{k\}} \left( EC(g_{A \setminus \{j\}},0),u,\kappa \right) - EC(g_{A \setminus \{j\}},0),u,\kappa), \]
where \( t \) is a regular positive calculation, which makes known the assumption that agent \( i \) does not require for full modification (if \( t = 1 \)) but only (frequently) a part of it. The calculation \( t \) behaves how much the excess is revised. When it partsake in a game, some mutations or discontent may be arisen from different conditions. The correction mappings are based on the idea that, each agent cuts down the discontent relating to its own and others’ non-participation, and apply these modifications to regulate the initial outcome.

The following result displays that the correction mapping is well-defined, i.e., \( r(\kappa(A,g,u)) \in X(A,g,u) \) if \( \kappa(A,g,u) \in X(A,g,u). \) This result plays an important role to generate the necessary convergence result.

**Lemma 3:** Let \((A,g,u) \in \Delta \) with \(|A| \geq 3 \) and \( \kappa(A,g,u) \in X(A,g,u). \) Then for all \( i \in A, \)
\[ \sum_{j \in A \setminus \{i\}} \left( EC(g_{A \setminus \{j\}},0),u,\kappa \right) - EC(g_{A \setminus \{j\}},0),u,\kappa) \]
and
\[ \sum_{i \in A} \left( EC(g_{A \setminus \{i\}},0),u,\kappa \right) - EC(g_{A \setminus \{i\}},0),u,\kappa) \]
\[ = 0. \]

**Proof:** Let \((A,g,u) \in \Delta, i,j \in A \) and \( \kappa(A,g,u) \in X(A,g,u). \)
\[ \sum_{j \in A \setminus \{i\}} \left( EC(g_{A \setminus \{j\}},0),u,\kappa \right) - EC(g_{A \setminus \{j\}},0),u,\kappa) \]
\[ = \sum_{j \in A \setminus \{i\}} (u(g_{A \setminus \{j\}},0) - \sum_{k \in A \setminus \{j\}} \kappa_{k,g}(A,g,u)) \]
\[ = \sum_{j \in A \setminus \{i\}} (u(g_{A \setminus \{j\}},0) - u(g_{A \setminus \{j\}})) \]
\[ - \kappa_{i,g}(A,g,u) + \kappa_{j,g}(A,g,u). \]
Clearly, \( \kappa^5 \) matches EFF, SYM and ZI, but it violates BCON.
By equations (6) and (7),
\[
\begin{align*}
\sum_{j \in \mathcal{A}_i(1)} & \left( EC((g_{A \setminus \{j\}}, 0), u, x) - EC((g_{A \setminus \{j\}}, 0), u, x) \right) \\
= & \sum_{j \in \mathcal{A}_i(1)} \left( \bar{\theta}_{i,g_i}(A, g, u) - \bar{\theta}_{j,g_j}(A, g, u) \right) \\
= & (|A| - 1) \left[ \bar{\theta}_{i,g_i}(A, g, u) - \bar{\theta}_{i,g_i}(A, g, u) \right] \\
& - \sum_{j \in \mathcal{A}_i(1)} \bar{\theta}_{j,g_j}(A, g, u) + \sum_{j \in \mathcal{A}_i(1)} \kappa_{j,g_j}(A, g, u) \\
= & |A| \left[ \bar{\theta}_{i,g_i}(A, g, u) - \kappa_{i,g_i}(A, g, u) \right] - u(g) + u(g) \\
(by \ EFF \ of \ \bar{\theta} \ and \ \kappa) \\
= & |A| \left[ \kappa_{i,g_i}(A, g, u) - \kappa_{i,g_i}(A, g, u) \right].
\end{align*}
\]
Moreover,
\[
\begin{align*}
\sum_{i \in \mathcal{A}} \sum_{j \in \mathcal{A}_i(1)} & \left( EC((g_{A \setminus \{j\}}, 0), u, \kappa) - EC((g_{A \setminus \{j\}}, 0), u, \kappa) \right) \\
= & \sum_{i \in \mathcal{A}} |A| \cdot \left[ \bar{\theta}_{i,g_i}(A, g, u) - \kappa_{i,g_i}(A, g, u) \right] \\
= & |A| \cdot \left[ u(g) - u(g) \right] \\
(by \ EFF \ of \ \bar{\theta} \ and \ \kappa) \\
= & 0.
\end{align*}
\]

Let \((A, g, u) \in \Delta \) and \(\kappa(A, g, u) \in X(A, g, u)\). We define the dynamic sequences \(\kappa^q(A, g, u)_{q=1}^{\infty}\) for every \(q \in \mathbb{N}\) as follows.
\[
\kappa^q(A, g, u) = \kappa(A, g, u), \ldots, \kappa^q(A, g, u) = r(\kappa^{q-1}(A, g, u)).
\]

**Theorem 3:** Let \((A, g, u) \in \Delta\). If \(0 < t < \frac{2}{|\mathcal{A}|}\), then \(\kappa^q(A, g, u)_{q=1}^{\infty}\) converges to \(\bar{\theta}_{i,g_i}(A, g, u)\) for each \((A, g, u) \in X(A, g, u)\) and for all \(i \in \mathcal{A}\).

**Proof:** Let \((A, g, u) \in \Delta\). Then \(\kappa(A, g, u) \in X(A, g, u)\). By Lemma 3, for some \(h\),
\[
r_{i,g_i}(\kappa(A, g, u)) - \kappa_{i,g_i}(A, g, u) = t \cdot \sum_{j \in \mathcal{A}_i(1)} \left( EC((g_{A \setminus \{j\}}, 0), u, \kappa) - EC((g_{A \setminus \{j\}}, 0), u, \kappa) \right) \\
= t \cdot |A| \cdot \left[ \bar{\theta}_{i,g_i}(A, g, u) - \kappa_{i,g_i}(A, g, u) \right].
\]
Hence,
\[
\begin{align*}
\bar{\theta}_{i,g_i}(A, g, u) & = r_{i,g_i}(\kappa(A, g, u)) \\
= & \bar{\theta}_{i,g_i}(A, g, u) - \kappa_{i,g_i}(A, g, u) + \kappa_{i,g_i}(A, g, u) \\
& - t \cdot |A| \cdot \left[ \bar{\theta}_{i,g_i}(A, g, u) - \kappa_{i,g_i}(A, g, u) \right] \\
= & \left( 1 - t \cdot |A| \right) \cdot \bar{\theta}_{i,g_i}(A, g, u) - \kappa_{i,g_i}(A, g, u). 
\end{align*}
\]
For all \(q \in \mathbb{N}\),
\[
\begin{align*}
\bar{\theta}_{i,g_i}(A, g, u) & = r_{i,g_i}(\kappa^q(A, g, u)) \\
= & \left( 1 - t \cdot |A| \right)^q \cdot \bar{\theta}_{i,g_i}(A, g, u) - \kappa_{i,g_i}(A, g, u). 
\end{align*}
\]
If \(0 < t < \frac{2}{|\mathcal{A}|}\), then \(-1 < \left( 1 - t \cdot |A| \right) < 1\) and \(\{r^q_{i,g_i}(A, g, u)\}_{q=1}^{\infty}\) converges to \(\bar{\theta}_{i,g_i}(A, g, u)\).

Efficiency of a solution is essential in the techniques of dynamic analysis. In Theorem 3, some values among the MCEANSC of a game could not be reached by applying efficiency and related dynamic analysis. Similar to Liao [4], a different definition of efficiency is offered on as follows.

Let \((A, g, u) \in \Delta\). A solution \(\kappa\) on \(\Delta\) matches **plurality-efficiency (PEFF)** in \((A, g, u)\) if for all \((i, k_i) \in P^A\),
\[
\kappa_{i,k_i}(A, g, u) + \sum_{j \in \mathcal{A}_i(1)} \kappa_{j,g_j}(A, g, u) = u\left(g_{A \setminus \{i\}}, k_i\right).
\]
It also matches EFF in \((A, g, u)\) if there exists \((A, g, u)\) such that a solution matches PEFF in \((A, g, u)\). An interpretation of plurality-efficiency due to Liao [4] is stated as follows.

Under a traditional game \((A, u^{TU}\)) the foremost supposition is that the grand alliance \(A\) forms, and then that \(u^{TU}(A)\) is the utility that has to be allocated. Hence, a solution \(\kappa_i^{TU}(A)\) is a mapping \(\kappa^{TU}\) appointing to \((A, u^{TU})\) an outcome \(\kappa(A, u^{TU}) = \left( \kappa_i(A, u^{TU}) \right)_{i \in \mathcal{A}} \in \mathbb{R}^A\) where \(\kappa_i(A, u^{TU})\) is the value assigned to agent \(i\), and \(TU\)-efficiency claims that \(\sum_{i \in \mathcal{A}} \kappa_i(A, u^{TU}) = u^{TU}(A)\), all the incomes (maybe losses) are to be allocated among the agents. In a multi-choice game \((A, g, u)\), the foremost supposition is still that the grand coalition \(A\) takes shape, and then there exist various cooperative aspects of \(A\). This implies that for each \(\zeta \in \mathcal{C}^A\) with \(\zeta \neq 0\) for every \(i \in A\), it is probable that \(u(\zeta)\) is the utility that has to be allocated. In order to attain the maximal and beneficial result of “identity”, each individual agent expects that all other agents are assumed to partake at its maximum level of energy if it partakes at level \(\zeta\), which is also of significant condition.

**Theorem 4:** Let \((A, g, u) \in \Delta\) such that the solution \(\bar{\theta}\) matches PEFF in \((A, g, u)\). If \(0 < t < \frac{2}{|\mathcal{A}|}\), then for each solution \(\kappa\) which matches PEFF in \((A, g, u)\), \(\kappa^q(A, g, u)_{q=1}^{\infty}\) converges to \(\bar{\theta}(A, g, u)\).

**Proof:** The proof of this theorem is direct analogue of the proof of Theorem 3, therefore it would be omitted.

V. DISCUSSION

The merit of the method in this article are that the MCEANSC of a multi-choice game continuously exists and to determine an outcome for a given agent partaking at a given level that different from the general type with multi-choice games, which determining a type of entire outcome for a given agent by gathering the contribution of this agent among total levels. In order to explain how the MCEANSC can be applied and to cause its implication clear, an application due to Liao [8] is quoted as follows. (Liao [8])

Let \((A, g, u) \in \Delta\) and \(A\) be a set of investors. Suppose that the capital of each \(i \in A\) is \(c_i\). In this model the capital of a agent can be non-positive; in fact, some agents may be in need of capital (in this case an investment of a negative capital is a financing process). For all \(\zeta \in \mathcal{C}^A\), \(\zeta\) could be treated as a multi-choice coalition. A multi-choice coalition \(\zeta\) is seen as an organization meant to achieve some goals, which are common to its members. The endowment of a multi-choice coalition \(\zeta\) with the capital it needs for its activities is done by the members and the degree of membership of agent \(i \in A\) to multi-choice coalition \(\zeta\) is measured by the level of capital \(c_i\) agent \(i\) invests in the multi-choice coalition \(\zeta\). Observe that this way of measuring the degree of membership is different from the more usual one in which the degree of membership is measured by the share of coalitional capital a agent owns. It better reflects
the risks agents are ready to take over when investing in
a specific organization and also their personal interest in
realizing the goals the organization is meant to achieve: if a
agent with a capital of $100 and another agent with a capital
of $10000 invest the same amount of $100 in organization
ζ, it means that the first agent is much more interested in ζ
and, consequently, more personally involved and assuming
a higher risk than the second agent for the realization of
the goals of ζ. In that follows we interpret the membership
degree of a agent to a multi-choice coalition as a measure
the goals of the agent assumes by transferring a part of its
capital to the coalition considered as a collective decision
maker.

Another application for resource-distributing under a
sports organization is also stated as follows. Let \( A = \{1, 2, \cdots, a\} \) be a collection of all elements of the operational
committee of a sports organization. In the operational
committee, all elements are picked by recommendation or
voting from sections of the sports organization. All elements
have the authority to raise, consult, originate, and veto
all projects for resource allocating. All elements dedicate
distinct levels of observation and involvement to different
projects depending on its professional expertise and the
common observation they represent. The level of affection
is also closely connected with the coalitional decision
constituted for the interests of different divisions. For the
foregoing reasons, decisions applied by each element of the
operational committee present distinct levels of involvement
and particular metes of multiplicity. The mapping \( u \) could
be regarded as an affect mapping which appoints to each level
vector \( \zeta = (\zeta_i)_{i \in A} \in G^A \) the affect that the elements can contribute if each element \( i \) partakes at operational decision
\( \zeta_i \in G_i \). Modeled under this notion, the decision-making
processes of the operational committee of a sports organi-
zation \((A, g, u)\) could be regarded as a multi-choice TU
game, with \( u \) being a characteristic mapping and \( G_i \) being
the collection of all operational decisions of the element \( i \).
To evaluate the effect of each element in the operational
committee, applying the power indexes this article proposed,
one could first assess the affect each element under each level
over previous resource-distributing project meetings based
on various performances, which is the the level-marginal
contribution defined in Definition 1. The rest of shared affect
should also be equally distributed, which is the MCEANSC
declared in Definition 1.

Subsequently, one would explore the realistic implications
of the properties presented in Section 3. In this way, we can
also explore whether the MCEANSC can be regarded as an
appropriate allocation and processing principle in real-world
situations.

- **Efficiency (EFF)** represents the situation where re-
sources are completely allocated, which in real-world
situations usually means "resources must be used com-
pletely and properly".
- **Symmetry (SYM)** represents the situation where, if two
people make the same amount of marginal contribution,
they should eventually receive the same pay, which in
real-world situations usually means "equal pay for equal
work".
- **Zero-independence (ZI)** represents the situation where
any conditions that occur during the game must be
reflected in the final allocation, which in real-world con-
ditions usually implies that production and allocation
must be synchronized and proportional.
- **Standard for two-person games (STPG)** represent a
self-sufficient situation if there is only one agent in the
game, but if there are two agents in the game, each of
them first receive what they could have occurred alone,
and they partake all the rest of losses and profits at the
tail of the game. In reality, many concepts of allocation
and processing usually depend on the characteristics of
individual behavior and the states of interaction
between two people. In real-world situations, STPG
usually represents "self-sufficiency in the case of one
person, and helping both yourself and each other in the
case of two people". As stated in the nature of
STPG, the state of interaction between two people has a
decisive influence on the overall situation of allocation.
No allocation concept will match everyone.
- **What bilateral consistency (BCON)** suggests in real-
world situations is that if any two people are dissatisfied,
they are allowed to restart the process and perform
another allocation with the best conditions possible, and
if the result of the allocation turns out to be the same
as the original result, then the allocation concept has a
stable and consistent criterion.

With the above statement, together with the relevant results
presented for the MCEANSC in Section 3, we can clearly
summarize what can be considered an appropriate allocation
and processing principle for the MCEANSC in real-world
situations.

VI. CONCLUDING REMARKS

1) Cheng et al. [1], Hwang and Liao [3], Liao [5], [6],
and Liao et al. [8] proposed several extensions of the
Banzhaf-Owen index, the EANSC, the PEANSC and
related results by respectively considering different no-
tions on multi-choice games. By both considering the
agents and the activity levels, this article proposed an
extension of the EANSC and related results on multi-
choice games. One might compare these published
results with the results of this article. Several major
differences are as follows:

- Cheng et al. [1] defined the multi-choice normal-
ized Banzhaf-Owen index to determine a type of
entire outcome for a given agent by applying the
maximal-utilities related to the sizes of coalitions.
Differing from the results due to Cheng et al.
[1], this article proposed the MCEANSC, several
axioms and the reduction by considering the agents
and its activity levels at the same time. The other
main disparity is the fact that this article proposed
the dynamic result of the MCEANSC. The dy-
namic result does not introduce in Cheng et al.
[1]. The techniques of axiomatic results in this article
are similar to the axiomatic results of Moulin [10].
- Hwang and Liao [3] defined three extensions of
the EANSC to determine an outcome of a given
level for a given agent by allocating the rest of
utility among all levels. Differing from the results
due to Hwang and Liao [3], this article proposed
the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Hwang and Liao [3]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].

- Liao [5] defined the maximal EANSC to determine a type of entire outcome for a given agent by applying the maximal marginal contributions of agents among all levels. Differing from the results due to Liao [5], this article proposed the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Liao [5]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].

- Liao [6] proposed the duplicate EANSC to determine a type of entire outcome for a given agent by applying the replicated behavior of agents among all levels. Differing from the results due to Liao [6], this article proposed the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Liao [6]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].

- Liao et al. [8] defined the multi-choice pseudo equal allocation of non-separable costs to determine a value of a given level for a given agent by extending the PEANSC. Differing from the results due to Liao [8], this article proposed the MCEANSC, several axioms and the reduction by considering the agents and its activity levels at the same time. The other main disparity is the fact that this article proposed the dynamic result of the MCEANSC. The dynamic result does not introduce in Liao [8]. The techniques of axiomatic results in this article are similar to the axiomatic results of Moulin [10].

- Inspired by Maschler and Owen [9], Liao [4] adopted the plurality-efficiency to offer the dynamic result of a solution. Inspired by Liao [4], this article proposed two dynamic results of the MCEANSC. In view of the correcting mappings due to Maschler and Owen [9] and Liao [4], the “reduction” is a key factor. However, the correcting mapping of this paper is generated from “excess mapping”.

2) This paper offer several axiomatic and two dynamic results of the MCEANSC respectively. Due to bilateral consistency property, this article present two axiomatic results which are analogues of the results of Hart and Mas-Colell [2] and Moulin [10]. Due to efficiency property, some outcomes of the MCEANSC of a game could not be generated from dynamic results. One would try to investigate axiomatic results by discarding consistency and investigate dynamic results by discarding efficiency.

3) This article has combined proofs with mathematical theories, statements with practical examples, and cross arguments between them to derive an allocation and processing principle that can be applied in real-world situations. Some might wonder whether the concept of allocation in other game theories can also be applied in real-world situations. This article leaves it to the researches to explore this in future researches.

REFERENCES