Almost Periodic Solution for a Modified Leslie-Gower System with Single Feedback Control

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Abstract—**This paper concerns a modified Leslie-Gower system subject to single feedback control. By applying some preliminary lemmas, the permanence of the system is obtained. Based on this permanence result, the uniqueness of a globally attractive positive almost periodic solution of the system is established by Lyapunov function method. An example with computer simulation supports the feasibility of our theoretical findings.**

Index Terms—**Permanence, Global attractivity, Almost periodic solution, Modified Leslie-Gower system, Single feedback control.**

I. Introduction

For any continuous bounded function $\varphi : R \to R$, we denote

$$\varphi^l = \inf_{t\in R} \varphi(t)\,,\ \ \varphi^u = \sup_{t\in R} \varphi(t).$$

Recently, many scholars have studied the following modified Leslie-Gower predator-prey model which was firstly proposed and investigated by Aziz-Alaoui and Daher Okiye [4]:

$$\begin{cases} \dot{x}(t) =x(t)\Big[r_1 - b_1x(t) - \dfrac{a_1y(t)}{x(t)+k_1}\Big], \\ \dot{y}(t) =y(t)\Big[r_2 - \dfrac{a_2y(t)}{x(t)+k_2}\Big]. \end{cases} \tag{1}$$

Some outstanding results have been obtained: such as global attractivity and bifurcation analysis [4–18] for autonomous model with delay, interference, impulses, Lévy jumps, stage-structured, harvesting, refuge and so on; permanence, periodic solution and almost periodic solution [19–27] for nonautonomous model with different functional response, refuge and feedback controls. It is obvious that those works [22, 23, 27] which considering modified Leslie-Gower system with feedback controls are based on at least two feedback control variables. This arrangement implies that different species are influenced by different strategy. However, one strategy could affect on both species in the real world. For instance, spraying pesticide can keep down weeds and also have some bad side effects on corps or beneficial animals at the same time [28]. Chemotherapeutic drugs not only make cancer cells diminish quickly but also cause damage to normal cells and human immunity [29]. These phenomena reveal the theoretical and practical values of discussing single feedback control variable. Motivated by above reasons and some recent papers [30–39], we propose a modified Leslie-Gower system with single feedback control as follows:

$$\begin{cases} \dot{x}(t) =x(t)\Big[r_1(t) - b_1(t)x(t) - \dfrac{a_1(t)y(t)}{x(t)+k_1(t)} - f_1(t)u(t)\Big], \\ \dot{y}(t) =y(t)\Big[r_2(t) - \dfrac{a_2(t)y(t)}{x(t)+k_2(t)} - f_2(t)u(t)\Big], \\ \dot{u}(t) =-\beta(t)u(t) + e_1(t)x(t) + e_2(t)y(t), \end{cases} \tag{2}$$

where $x(t)$ and $y(t)$ stand for densities of prey and predator, respectively. $u(t)$ is the single feedback control variable. All the parameters are continuous bounded functions whose lower and upper bounds are positive. The initial condition for system (2) is:

$$x(0) > 0, y(0) > 0, u(0) > 0. \tag{3}$$

The remainder of this work is arranged as follows. The permanence of system (2) is considered in Section II. In Section III and IV, the global attractivity and uniqueness of a positive almost periodic solution of system (2) are discussed. Then, in Section V, our results are verified by one example with numerical simulation. Finally, we conclude in Section VI.

II. Permanence

In this section, we recall the following useful lemma at first.

Lemma 2.1 ([1]). *Suppose $c > 0$, $d > 0$. For $t \geq 0$ and $x(0) > 0$, the following statements hold*

(1) *if $\dot{x} \geq x(d - cx)$, then $\liminf\limits_{t\to+\infty} x(t) \geq \frac{d}{c}$;*

(2) *if $\dot{x} \leq x(d - cx)$, then $\limsup\limits_{t\to+\infty} x(t) \leq \frac{d}{c}$;*

(3) *if $\dot{x} \geq d - cx$, then $\liminf\limits_{t\to+\infty} x(t) \geq \frac{d}{c}$;*

(4) *if $\dot{x} \leq d - cx$, then $\limsup\limits_{t\to+\infty} x(t) \leq \frac{d}{c}$.*

Theorem 2.1. *Assume*

$$r_1^lk_1^l - a_1^uW_2 > f_1^uk_1^lW_3, \tag{Q_1}$$

and

$$r_2^l > f_2^uW_3, \tag{Q_2}$$

hold, where $W_1 = \dfrac{r_1^u}{b_1^l}$, $W_2 = \dfrac{r_2^u(W_1 + k_2^u)}{a_2^l}$, $W_3 = \dfrac{e_1^uW_1 + e_2^uW_2}{\beta^l}$, then system (2) with initial condition (3) is permanent.

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Proof. From (Q_1) and (Q_2), there exists $\varepsilon > 0$ satisfying

$$r_1^l k_1^l - \frac{a_1^u r_2^u (W_1 + \varepsilon + k_2^u)}{a_2^l} - (a_1^u - f_1^u k_1^l)\varepsilon > \frac{f_1^u k_1^l [(a_2^l e_1^u + e_2^u r_2^u)(W_1 + \varepsilon) + e_2^u r_2^u k_2^u + e_2^u a_2^l \varepsilon]}{a_2^l \beta^l}, \tag{4}$$

and

$$r_2^l - f_2^u \varepsilon > \frac{f_2^u [(a_2^l e_1^u + e_2^u r_2^u)(W_1 + \varepsilon) + e_2^u r_2^u k_2^u + e_2^u a_2^l \varepsilon]}{a_2^l \beta^l}. \tag{5}$$

We can deduce from the first equation of (2) that

$$\dot{x}(t) \le x(t)\Big(r_1^u - b_1^l x(t)\Big). \tag{6}$$

Using Lemma 2.1 and (6), one has

$$\limsup_{t\to+\infty} x(t) \le \frac{r_1^u}{b_1^l} \triangleq W_1. \tag{7}$$

In view of (7), we could choose $T_1 > 0$ such that

$$x(t) \le W_1 + \varepsilon \triangleq W_{1\varepsilon},\ t > T_1. \tag{8}$$

Substituting (8) into the second equation of (2) gives

$$\dot{y}(t) \le y(t)\Big(r_2^u - \frac{a_2^l y(t)}{W_{1\varepsilon} + k_2^u}\Big),\ t > T_1. \tag{9}$$

Applying Lemma 2.1 to (9) yields

$$\limsup_{t\to+\infty} y(t) \le \frac{r_2^u (W_{1\varepsilon} + k_2^u)}{a_2^l}. \tag{10}$$

Therefore, for above ε, one could choose $T_2 \ge T_1$ satisfying

$$y(t) \le \frac{r_2^u (W_{1\varepsilon} + k_2^u)}{a_2^l} + \varepsilon \triangleq W_{2\varepsilon},\ t > T_2. \tag{11}$$

Substituting (8) and (11) into the third equation of (2), we deduce

$$\dot{u}(t) \le -\beta^l u(t) + e_1^u W_{1\varepsilon} + e_2^u W_{2\varepsilon},\ t > T_2. \tag{12}$$

By Lemma 2.1, one gets

$$\limsup_{t\to+\infty} u(t) \le \frac{e_1^u W_{1\varepsilon} + e_2^u W_{2\varepsilon}}{\beta^l}. \tag{13}$$

So, there exists $T_3 \ge T_2$, such that

$$u(t) \le \frac{e_1^u W_{1\varepsilon} + e_2^u W_{2\varepsilon}}{\beta^l} + \varepsilon \triangleq W_{3\varepsilon},\ t > T_3. \tag{14}$$

We can obtain from (2), (11) and (14) that

$$\dot{x}(t) \ge x(t)\Big[r_1^l - b_1^u x(t) - \frac{a_1^u W_{2\varepsilon}}{k_1^l} - f_1^u W_{3\varepsilon}\Big],\ t > T_3. \tag{15}$$

Using (15), (4) and Lemma 2.1, we have

$$\liminf_{t\to+\infty} x(t) \ge \frac{r_1^l k_1^l - a_1^u W_{2\varepsilon} - f_1^u k_1^l W_{3\varepsilon}}{b_1^u k_1^l}.$$

This implies the existence of T_4 satisfying $T_4 > T_3$ and

$$x(t) \ge \frac{r_1^l k_1^l - a_1^u W_{2\varepsilon} - f_1^u k_1^l W_{3\varepsilon}}{b_1^u k_1^l} - \varepsilon \triangleq w_{1\varepsilon},\ t > T_4. \tag{16}$$

From (2), (14) and (16), one can get

$$\dot{y}(t) \ge y(t)\Big[r_2^l - \frac{a_2^u y(t)}{w_{1\varepsilon} + k_2^l} - f_2^u W_{3\varepsilon}\Big],\ t > T_4. \tag{17}$$

Using (5) and applying Lemma 2.1 to (17) lead to

$$\liminf_{t\to+\infty} y(t) \ge \frac{(r_2^l - f_2^u W_{3\varepsilon})(w_{1\varepsilon} + k_2^l)}{a_2^u}. \tag{18}$$

Similarly, there exists $T_5 > T_4$, such that

$$y(t) \ge \frac{(r_2^l - f_2^u W_{3\varepsilon})(w_{1\varepsilon} + k_2^l)}{a_2^u} - \varepsilon \triangleq w_{2\varepsilon},\ t > T_5. \tag{19}$$

By (2), (16) and (19), we derive

$$\dot{u}(t) \ge -\beta^u u(t) + e_1^l w_{1\varepsilon} + e_2^l w_{2\varepsilon},\ t > T_5. \tag{20}$$

Using Lemma 2.1 again, one has

$$\liminf_{t\to+\infty} u(t) \ge \frac{e_1^l w_{1\varepsilon} + e_2^l w_{2\varepsilon}}{\beta^u}. \tag{21}$$

Setting $\varepsilon \to 0$, we get

$$\begin{aligned}
\limsup_{t\to+\infty} y(t) &\le \frac{r_2^u (W_1 + k_2^u)}{a_2^l} \triangleq W_2,\\
\limsup_{t\to+\infty} u(t) &\le \frac{e_1^u W_1 + e_2^u W_2}{\beta^l} \triangleq W_3,\\
\liminf_{t\to+\infty} x(t) &\ge \frac{r_1^l k_1^l - a_1^u W_2 - f_1^u k_1^l W_3}{b_1^u k_1^l} \triangleq w_1,\\
\liminf_{t\to+\infty} y(t) &\ge \frac{(r_2^l - f_2^u W_3)(w_1 + k_2^l)}{a_2^u} \triangleq w_2,\\
\liminf_{t\to+\infty} u(t) &\ge \frac{e_1^l w_1 + e_2^l w_2}{\beta^u} \triangleq w_3.
\end{aligned} \tag{22}$$

Thus, Theorem 2.1 can be established by (7) and (22). □

III. Global Attractivity

The global attractivity of model (2) will be discussed in this part.

Theorem 3.1. *Assume (Q_1) and (Q_2), further suppose*

$$\Big[b_1(t) - \frac{W_2 a_1(t)}{(w_1 + k_1(t))^2} - \frac{W_2 a_2(t)}{(w_1 + k_2(t))^2} - e_1(t)\Big]^l > 0, \tag{Q_3}$$

$$\Big[\frac{a_2(t)}{W_1 + k_2(t)} - \frac{a_1(t)}{w_1 + k_1(t)} - e_2(t)\Big]^l > 0, \tag{Q_4}$$

and

$$\Big[\beta(t) - f_1(t) - f_2(t)\Big]^l > 0, \tag{Q_5}$$

hold, where w_i and W_i $(i = 1, 2)$ are given by Theorem 2.1, then model (2) is globally attractive.

Proof. Assume $(x(t), y(t), u(t))^T$ and $(x^*(t), y^*(t), u^*(t))^T$ are any two positive solutions of (2) with initial condition (3). It follows from Theorem 2.1, (Q_3), (Q_4) and (Q_5) that there exist $\varepsilon_1 > 0$ and $t_6 > t_5$ such that for $t > t_6$, we have

$$\begin{aligned}
&\Big[b_1(t) - \frac{(W_2 + \varepsilon)a_1(t)}{(w_1 - \varepsilon_1 + k_1(t))^2} - \frac{(W_2 + \varepsilon_1)a_2(t)}{(w_1 - \varepsilon_1 + k_2(t))^2}\\
&- e_1(t)\Big]^l > \varepsilon_1,\\
&\Big[\frac{a_2(t)}{W_1 + \varepsilon_1 + k_2(t)} - \frac{a_1(t)}{w_1 - \varepsilon_1 + k_1(t)} - e_2(t)\Big]^l > \varepsilon_1,\\
&\Big[\beta(t) - f_1(t) - f_2(t)\Big]^l > \varepsilon_1,
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
w_1 - \varepsilon_1 &\le x(t),\ x^*(t) \le W_1 + \varepsilon_1,\\
w_2 - \varepsilon_1 &\le y(t),\ y^*(t) \le W_2 + \varepsilon_1,\\
w_3 - \varepsilon_1 &\le u(t),\ u^*(t) \le W_3 + \varepsilon_1.
\end{aligned} \tag{24}$$

Set $\Theta_1(t)=|\ln x(t)-\ln x^*(t)|$, $\Theta_2(t)=|\ln y(t)-\ln y^*(t)|$ and $\Theta_3(t)=|u(t)-u^*(t)|$, then direct calculation leads to

$$\begin{aligned}&D^+\Theta_1(t)\\ \le&\Big[-b_1(t)+\frac{a_1(t)y(t)}{(x(t)+k_1(t))(x^*(t)+k_1(t))}\Big]|x(t)-x^*(t)|\\ &+\frac{a_1(t)|y(t)-y^*(t)|}{x^*(t)+k_1(t)}+f_1(t)|u(t)-u^*(t)|,\\ &D^+\Theta_2(t)\\ \le&f_2(t)|u(t)-u^*(t)|+\frac{a_2(t)y^*(t)|x(t)-x^*(t)|}{(x^*(t)+k_2(t))(x(t)+k_2(t))}\\ &-\frac{a_2(t)|y(t)-y^*(t)|}{x(t)+k_2(t)},\end{aligned}\tag{25}$$

and

$$\begin{aligned}D^+\Theta_3(t)\le&-\beta(t)|u(t)-u^*(t)|+e_1(t)|x(t)-x^*(t)|\\ &+e_2(t)|y(t)-y^*(t)|.\end{aligned}\tag{26}$$

Let $\Theta(t)=\Theta_1(t)+\Theta_2(t)+\Theta_3(t)$, one can get from (24)-(26) that

$$\begin{aligned}&D^+\Theta(t)\\ \le&\Big[-b_1(t)+\frac{a_1(t)y(t)}{(x(t)+k_1(t))(x^*(t)+k_1(t))}\\ &+\frac{a_2(t)y^*(t)}{(x^*(t)+k_2(t))(x(t)+k_2(t))}+e_1(t)\Big]|x(t)-x^*(t)|\\ &+\Big[\frac{a_1(t)}{x^*(t)+k_1(t)}-\frac{a_2(t)}{x(t)+k_2(t)}+e_2(t)\Big]|y(t)-y^*(t)|\\ &+\Big[f_1(t)+f_2(t)-\beta(t)\Big]|u(t)-u^*(t)|\\ \le&-\Big[b_1(t)-\frac{(W_2+\varepsilon_1)a_1(t)}{(w_1-\varepsilon_1+k_1(t))^2}-\frac{(W_2+\varepsilon_1)a_2(t)}{(w_1-\varepsilon_1+k_2(t))^2}\\ &-e_1(t)\Big]|x(t)-x_*(t)|-\Big[\frac{a_2(t)}{W_1+\varepsilon_1+k_2(t)}\\ &-\frac{a_1(t)}{w_1-\varepsilon_1+k_1(t)}-e_2(t)\Big]|y(t)-y^*(t)|\\ &-\Big[\beta(t)-f_1(t)-f_2(t)\Big]|u(t)-u^*(t)|,\ t>t_6.\end{aligned}\tag{27}$$

For $t>t_6$, combining (23) with (27) leads to

$$D^+\Theta(t)\le-\varepsilon_1\Big[|x(t)-x^*(t)|+|y(t)-y^*(t)|+|u(t)-u^*(t)|\Big],\tag{28}$$

which shows Θ is non-increasing on $[t_6,+\infty)$. Integrating (28) from t_6 to t, we obtain

$$\begin{aligned}&\Theta(t)+\varepsilon_1\Big[\int_{t_6}^t|x(s)-x^*(s)|\mathrm{d}s+\int_{t_6}^t|y(s)-y^*(s)|\mathrm{d}s\\ &+\int_{t_6}^t|u(s)-u^*(s)|\mathrm{d}s\Big]<\Theta(t_6)<+\infty,\ t>t_6.\end{aligned}$$

By the proof of [20, Theorem 3.1], one can similarly deduce

$$\begin{aligned}&\lim_{t\to+\infty}|x(t)-x^*(t)|=\lim_{t\to+\infty}|y(t)-y^*(t)|\\ &=\lim_{t\to+\infty}|u(t)-u^*(t)|=0.\end{aligned}$$

This ends the proof. □

IV. Almost periodic solution

Now we come to deal with the existence and uniqueness of positive almost periodic solution of model (2) when $b_1(t),\beta(t)$ and $a_i(t),f_i(t),e_i(t),k_i(t),r_i(t)$ $(i=1,2)$ are continuous bounded almost periodic functions whose lower and upper bounds are positive. One can refer to [2, 3] for some basic theory about almost periodic function.

Let (E) be the set of all solutions $(x(t),y(t),u(t))^T$ of model (2) with $w_1\le x(t)\le W_1$, $w_2\le y(t)\le W_2$, $w_3\le u(t)\le W_3$.

Lemma 4.1. $(E)\neq\emptyset$.

Proof. According to definition of almost periodic function, there exists a sequence $\{t_n\}$ satisfying $t_n\to\infty$ and

$$\begin{aligned}&b_1(t+t_n)\to b_1(t),\ \beta(t+t_n)\to\beta(t),a_i(t+t_n)\to a_i(t),\\ &f_i(t+t_n)\to f_i(t),\ e_i(t+t_n)\to e_i(t),k_i(t+t_n)\to k_i(t),\\ &r_i(t+t_n)\to r_i(t)\ (i=1,2),\end{aligned}$$

as $n\to\infty$ uniformly. Suppose $v(t)=(x(t),y(t),u(t))^T$ is a solution of model (2) with $w_1\le x(t)\le W_1,w_2\le y(t)\le W_2,w_3\le u(t)\le W_3$ for $t>t_6$. Then $v(t+t_n)$ are evidently equi-continuous and uniformly bounded on each bounded subset of R. In virtue of Ascoli's theorem, going if necessary to a subsequence, we suppose that $v(t+t_n)\to q(t)=(q_1(t),q_2(t),q_3(t))^T$ as $n\to\infty$ uniformly on each bounded subset of R. Select $t_7\in R$ satisfying $t_n+t_7>t_6$ for any n, so for $t>0$, one has

$$\begin{aligned}&x(t+t_n+t_7)-x(t_n+t_7)\\ =&\int_{t_7}^{t+t_7}x(s+t_n)\Big[r_1(s+t_n)-b_1(s+t_n)x(s+t_n)\\ &-\frac{a_1(s+t_n)y(s+t_n)}{x(s+t_n)+k_1(s+t_n)}-f_1(s+t_n)u(s+t_n)\Big]\mathrm{d}s,\\ &y(t+t_n+t_7)-y(t_n+t_7)\\ =&\int_{t_7}^{t+t_7}y(s+t_n)\Big[r_2(s+t_n)-\frac{a_2(s+t_n)y(s+t_n)}{x(s+t_n)+k_2(s+t_n)}\\ &-f_2(s+t_n)u(s+t_n)\Big]\mathrm{d}s,\end{aligned}$$

and

$$\begin{aligned}&u(t+t_n+t_7)-u(t_n+t_7)\\ =&\int_{t_7}^{t+t_7}\Big[-\beta(s+t_n)u(s+t_n)+e_1(s+t_n)x(s+t_n)\\ &+e_2(s+t_n)y(s+t_n)\Big]\mathrm{d}s.\end{aligned}$$

Letting $n\to\infty$, then Lebesgue's dominated convergence theorem shows that

$$\begin{aligned}q_1(t+t_7)-q_1(t_7)=&\int_{t_7}^{t+t_7}q_1(s)\Big[r_1(s)-b_1(s)q_1(s)\\ &-\frac{a_1(s)q_2(s)}{q_1(s)+k_1(s)}-f_1(s)q_3(s)\Big]\mathrm{d}s,\\ q_2(t+t_7)-q_2(t_7)=&\int_{t_7}^{t+t_7}q_2(s)\Big[r_2(s)-\frac{a_2(s)q_2(s)}{q_1(s)+k_2(s)}\\ &-f_2(s)q_3(s)\Big]\mathrm{d}s,\\ q_3(t+t_7)-q_3(t_7)=&\int_{t_7}^{t+t_7}\Big[-\beta(s)q_3(s)+e_1(s)q_1(s)\\ &+e_2(s)q_2(s)\Big]\mathrm{d}s.\end{aligned}$$

By the arbitrariness of t_7, $q(t)$ is a solution of (2). Obviously, $w_i\le q_i(t)\le W_i$ $(i=1,2,3)$ on R, so $q(t)\in(E)$. □

Theorem 4.1. *If all conditions in Theorem 3.1 hold, then model (2) admits a unique positive almost periodic solution.*
Proof. From Lemma 4.1, model (2) admits a bounded positive solution $\mu(t) = (\mu_1(t), \mu_2(t), \mu_3(t))^T$, $t > 0$. Hence, we can find a sequence $\{t_k^{'}\}$ such that $\{t_k^{'}\} \to \infty$ as $k \to \infty$ and $(\mu_1(t + t_k^{'}), \mu_2(t + t_k^{'}), \mu_3(t + t_k^{'}))^T$ satisfies

$$\begin{cases} \dot{x_1}(t) = x_1(t)\Big[r_1(t + t_k^{'}) - b_1(t + t_k^{'})x_1(t) \\ \qquad - \dfrac{a_1(t + t_k^{'})y(t)}{x(t) + k_1(t + t_k^{'})} - f_1(t + t_k^{'})u(t)\Big], \\ \dot{y}(t) = y(t)\Big[r_2(t + t_k^{'}) - \dfrac{a_2(t + t_k^{'})y(t)}{x(t) + k_2(t + t_k^{'})} \\ \qquad - f_2(t + t_k^{'})u(t)\Big], \\ \dot{u}(t) = -\beta(t + t_k^{'})u(t) + e_1(t + t_k^{'})x(t) \\ \qquad + e_2(t + t_k^{'})y(t). \end{cases}$$

It follows from Theorem 2.1 and the assumption of almost periodic coefficient that $\{\mu_i(t + t_k^{'})\}$ $(i = 1, 2, 3)$ are equicontinuous and uniformly bounded. Then, Ascoli's theorem shows the existence of a uniformly convergent subsequence $\{\mu_i(t + t_k)\} \subseteq \{\mu_i(t + t_k^{'})\}$ satisfying for any $\varepsilon_2 > 0$, we can choose $K(\varepsilon_2) > 0$ such that when $m, k \geq K(\varepsilon_2)$,

$$|\mu_i(t + t_m) - \mu_i(t + t_k)| < \varepsilon_2, \ \ i = 1, 2, 3,$$

which implies that $\mu_i(t)$ $(i = 1, 2, 3)$ are asymptotically almost periodic functions. So, there exist continuous functions $d_i(t{+}t_k)$ and almost periodic functions $c_i(t{+}t_k)$ $(i = 1, 2, 3)$ satisfying

$$\mu_i(t + t_k) = c_i(t + t_k) + d_i(t + t_k), \ i = 1, 2, 3,$$

and

$$\lim_{k\to+\infty} c_i(t + t_k) = c_i(t), \ \lim_{k\to+\infty} d_i(t + t_k) = 0, \ i = 1, 2, 3.$$

Hence, $\lim\limits_{k\to+\infty} \mu_i(t + t_k) = c_i(t)$ and $c_i(t)\,(i = 1, 2, 3)$ are almost periodic functions too.
Moreover,

$$\begin{aligned} \lim_{k\to+\infty} \dot{\mu_i}(t + t_k) &= \lim_{k\to+\infty} \lim_{h\to 0} \frac{\mu_i(t + t_k + h) - \mu_i(t + t_k)}{h} \\ &= \lim_{h\to 0} \lim_{k\to+\infty} \frac{\mu_i(t + t_k + h) - \mu_i(t + t_k)}{h} \\ &= \lim_{h\to 0} \frac{c_i(t + h) - c_i(t)}{h}, \ i = 1, 2, 3. \end{aligned}$$

So $\dot{c_i}(t)$ $(i = 1, 2, 3)$ is existence.
According to definition of almost periodic function, we can choose a sequence $\{t_n\}$ with $\{t_n\} \to \infty$ and

$$\begin{gathered} b_1(t + t_n) \to b_1(t), \ \beta(t + t_n) \to \beta(t), a_j(t + t_n) \to a_j(t), \\ f_j(t + t_n) \to f_j(t), \ e_i(t + t_n) \to e_i(t), k_j(t + t_n) \to k_j(t), \\ r_j(t + t_n) \to r_j(t) \ (j = 1, 2), \end{gathered}$$

uniformly on R as $n \to \infty$.

Evidently, $\mu_i(t{+}t_n) \to c_i(t)$ $(i = 1, 2, 3)$ as $n \to \infty$. Hence,

$$\begin{aligned} \dot{c_1}(t) &= \lim_{n\to+\infty} \dot{\mu_1}(t + t_n) \\ &= \lim_{n\to+\infty} \mu_1(t + t_n)\Big[r_1(t + t_n) - b_1(t + t_n)\mu_1(t + t_n) \\ &\quad - \frac{a_1(t + t_n)\mu_2(t + t_n)}{\mu_1(t + t_n) + k_1(t + t_n)} - f_1(t + t_n)\mu_3(t + t_n)\Big] \\ &= c_1(t)\Big[r_1(t) - b_1(t)c_1(t) - \frac{a_1(t)c_2(t)}{c_1(t) + k_1(t)} \\ &\quad - f_1(t)c_3(t)\Big], \end{aligned}$$

$$\begin{aligned} \dot{c_2}(t) &= \lim_{n\to+\infty} \dot{\mu_2}(t + t_n) \\ &= \lim_{n\to+\infty} \mu_2(t + t_n)\Big[r_2(t + t_n) \\ &\quad - \frac{a_2(t + t_n)\mu_2(t + t_n)}{\mu_1(t + t_n) + k_2(t + t_n)} - f_2(t + t_n)\mu_3(t + t_n)\Big] \\ &= c_2(t)\Big[r_2(t) - \frac{a_2(t)c_2(t)}{c_1(t) + k_2(t)} - f_2(t)c_3(t)\Big], \end{aligned}$$

and

$$\begin{aligned} \dot{c_3}(t) &= \lim_{n\to+\infty} \dot{\mu_3}(t + t_n) \\ &= \lim_{n\to+\infty} \Big[-\beta(t + t_n)\mu_3(t + t_n) \\ &\quad + e_1(t + t_n)\mu_1(t + t_n) + e_2(t + t_n)\mu_2(t + t_n)\Big] \\ &= -\beta(t)c_3(t) + e_1(t)c_1(t) + e_2(t)c_2(t). \end{aligned}$$

Therefore, $(c_1(t), c_2(t), c_3(t))^T$ is a positive almost periodic solution of model (2) and Theorem 3.1 further shows the uniqueness of this solution. □

V. Example and numeric simulation

In this part, we will give one example with numerical simulation to support our results.
Example 5.1. Consider this model:

$$\begin{cases} \dot{x}(t) = x(t)\Big(4.8 + \cos\sqrt{2}t - (10 - \sin\sqrt{5}t)x(t) \\ \qquad - \dfrac{(0.7 + 0.2\cos\sqrt{13}t)y(t)}{x(t) + 3.8 + \cos t} - 0.02u(t)\Big), \\ \dot{y}(t) = y(t)\Big(0.5 + 0.2\cos\sqrt{7}t - \dfrac{(1.5 + 0.2\cos\sqrt{3}t)y(t)}{x(t) + 1.6} \\ \qquad - 0.05u(t)\Big), \\ \dot{u}(t) = -(1.3 + 0.1\cos(\sqrt{3}t))u(t) + 0.3x(t) + 0.2y(t). \end{cases} \tag{29}$$

One could easily verify that conditions in Theorem 4.1 are all fulfilled. Hence, system (29) is permanent and admits a unique positive almost periodic solution which is globally attractive. These results are illustrated in Fig. 1.

VI. Conclusion

A modified Leslie-Gower predator-prey system with single feedback control is considered. By applying some preliminary lemmas and Lyapunov function method, we obtained the permanence and uniqueness of a globally attractive positive almost periodic solution for this model. These results show that single feedback control can greatly affect the dynamic behaviors of this system which is different from systems with two or more feedback control variables. On the other

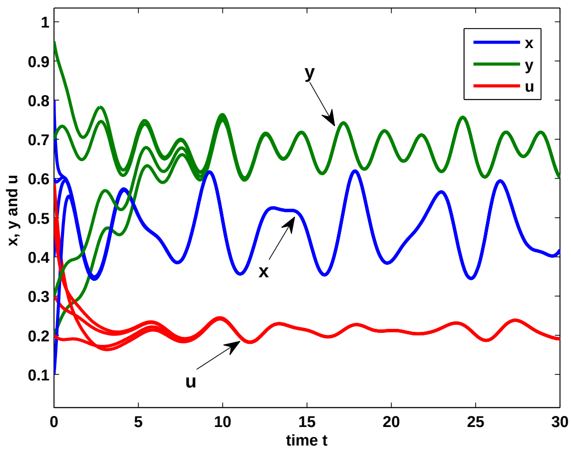

Fig. 1. Numeric simulations of system (29) with the initial conditions $(x(0), y(0), u(0))^T = (0.4, 0.2, 0.6)^T$, $(0.8, 0.7, 0.3)^T$, $(0.1, 0.95, 0.5)^T$ and $(0.6, 0.3, 0.2)^T$, respectively.

hand, we all know that time delay is an important influence factor for the dynamic behaviors of ecological model and we will study Leslie-Gower predator-prey system with time delay and single feedback control in the future.

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