CTE Method and Nonlocal Symmetries for a High-order Classical Boussinesq-Burgers Equation

Jinming Zuo

***Abstract*—In this work, the consistent tanh expansion (CTE) method is developed for a high-order classical Boussinesq-Burgers (HCBB) equation. Via the CTE method, we obtain many exact significant solutions including soliton-resonant solutions, soliton-periodic wave interactions and soliton-rational wave interactions. The CTE related nonlocal symmetries are also proposed. The nonlocal symmetries can be localized to find finite Bäcklund transformations by prolonging the model to an enlarged one.**

***Index Terms*—high-order classical Boussinesq-Burgers (HCBB) equation, Bäcklund transformation (BT), Interaction solutions, Lie point symmetry.**

I. Introduction

It is well known that there are many approaches to find the exact solutions for a given partial differential equation in the nonlinear science, such as Hirota's bilinear method [1], Bäcklund transformation (BT) [2], Darboux transformation (DT) [3], Painlevé analysis [4], (G'/G)-expansion method [5] and so on. Recently, Lou and his group [6-8] propose the consistent Riccati expansion (CRE) and consistent tanh expansion (CTE) method through the nonlocal symmetry to find interaction solutions of NLEEs. On account of this, there are a lot of paper here to study this problem [9-12].

In this work, we consider the following high-order classical Boussinesq-Burgers equation [13]

$$u_t + \frac{3}{2}(1-\beta)(uu_x)_x - \frac{3}{2}(uv)_x - 3u^2u_x - \frac{1}{4}u_{xxx} = 0, \tag{1a}$$

$$v_t - 3\beta(1-\frac{1}{2}\beta)(2u_xu_{xx} + uu_{xxx}) - \frac{3}{2}vv_x - \frac{3}{2}(1-\beta)(uv_x)_x - 3(u^2v)_x - \frac{1}{4}v_{xxx} = 0. \tag{1b}$$

where $u = u(x,t)$ is the height deviating from the equilibrium position of water, $v = v(x,t)$ is the field of horizontal velocity, β is a constant representing different dispersive power. In the case of $\beta = 0$, the HCBB equation (1) reduces to a high-order classical Boussinesq system. And through the transformation $(u, v, x, t, \beta) \longrightarrow (-u, -v, -x, -t, 1)$, it reduces to a high-order Boussinesq-Burgers equation [14-18].

In [13], Geng and Wu constructed finite-band solutions of the HCBB equation (1) based on the Lax pairs of the stationary evolution equations. In this work, we should use the CTE method to seek the interaction solutions between solitons and potential STO waves. The CTE related nonlocal symmetries are also proposed. The nonlocal symmetries can be localized to find finite BT by prolonging the model to an enlarged one.

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II. CTE Method and Explicit Solutions

Based on the CTE method [6-12], the generalized tanh function expansion for the HCBB equation (1) can be read as

$$u = u_1 \tanh(w) + u_0 + \frac{1}{2}w_x, \tag{2a}$$

$$v = v_2 \tanh^2(w) + v_1 \tanh(w) + v_0. \tag{2b}$$

where u_1, u_0, v_2, v_1, v_0 and w are some arbitrary function of x, t. In the expansion (2a), we have written u_0 as $u_0 + \frac{1}{2}w_x$ for convenience later. So, we have the following two cases.

Case 1 We have

$$\begin{aligned} u_1 &= \tfrac{1}{2}w_x, \\ v_2 &= \tfrac{\beta w_x^2}{2} - w_x^2, \\ v_1 &= -\tfrac{\beta w_{xx}}{2} + w_{xx}, \\ v_0 &= (2-\beta)\Big(\tfrac{w_x^2 + w_{xx}}{2} + u_{0x}\Big). \end{aligned} \tag{3}$$

while u_0 and w are determined by the following two equations

$$u_{0t} - (\frac{1}{4}u_{0xx} + \frac{3}{2}u_0u_{0x} + u_0^3)_x = 0, \tag{4a}$$

$$w_t - (\frac{1}{4}w_{xx} + \frac{3}{4}w_x^2 + \frac{3}{2}u_0w_x)_x - w_x^3 - 3u_0w_x(u_0 + w_x) = 0. \tag{4b}$$

Case 2 We have

$$\begin{aligned} u_1 &= -\tfrac{1}{2}w_x, \\ v_2 &= -\tfrac{\beta w_x^2}{2}, \\ v_1 &= \tfrac{\beta w_{xx}}{2}, \\ v_0 &= \tfrac{\beta w_x^2}{2} - \tfrac{\beta w_{xx}}{2} - \beta u_{0x}. \end{aligned} \tag{5}$$

while u_0 and w are determined by the following two equations

$$u_{0t} - (\frac{1}{4}u_{0xx} - \frac{3}{2}u_0u_{0x} + u_0^3)_x = 0, \tag{6a}$$

$$w_t - (\frac{1}{4}w_{xx} - \frac{3}{4}w_x^2 - \frac{3}{2}u_0w_x)_x - w_x^3 - 3u_0w_x(u_0 + w_x) = 0. \tag{6b}$$

In order to make the solutions more clear, we have two theorems as follows

Theorem 1 If u_0 and w are the solutions of Eq. (4), then

$$u = \frac{1}{2}w_x[\tanh(w) + 1] + u_0, \tag{7a}$$

$$v = \frac{2-\beta}{2}\left[w_x^2 \text{sech}^2(w) + w_{xx}\tanh(w) + w_{xx} + 2u_{0x}\right]. \tag{7b}$$

is the solution of the HCBB eqution (1).

Theorem 2 If u_0 and w are the solutions of Eq. (6), then

$$u = -\frac{1}{2}w_x[\tanh(w) - 1] + u_0, \tag{8a}$$

$$v = \frac{\beta}{2}\left[w_x^2 \text{sech}^2(w) + w_{xx}\tanh(w) - w_{xx} - 2u_{0x}\right]. \tag{8b}$$

is the solution of the HCBB equation (1).

According to the above two theorems, by solving the u_0 and w equation (4) or (6), we can get various interaction solutions among different types of nonlinear excitations. In fact, the u_0 equations (4a) and (6a) are just two kinds of well known linearizable Sharma-Tasso-Olver (STO) equation [19-21]. The w equations (4b) and (6b) can also be linearized because they are two potential forms of the variable coefficient STO (PSTO) equations.

In this work, we only restrict the trivial STO solution

$$u_0 = c. \tag{9}$$

where c is an arbitrary constant. In this case the w equations (4b) and (6b) are each simplified to the following two constant coefficient PSTO equations

$$\begin{aligned} &w_t - (\tfrac{1}{4}w_{xx} + \tfrac{3}{4}w_x^2 + \tfrac{3}{2}cw_x + 3c^2 w)_x \\ &-w_x^3 - 3cw_x^2 = 0. \end{aligned} \tag{10}$$

and

$$\begin{aligned} &w_t - (\tfrac{1}{4}w_{xx} - \tfrac{3}{4}w_x^2 - \tfrac{3}{2}cw_x + 3c^2 w)_x \\ &-w_x^3 - 3cw_x^2 = 0. \end{aligned} \tag{11}$$

A. Single soliton solutions

Eqs. (10) and (11) have the following trivial solution

$$\begin{aligned} &w = kx + \omega t, \\ &\omega = k(k^2 + 3ck + 3c^2), \end{aligned} \tag{12}$$

where k is an arbitrary constant, which leads to the following single soliton solutions of the HCBB equation (1)

$$\begin{aligned} u = &\tfrac{1}{2}k \tanh\left[kx + k(k^2 + 3ck + 3c^2)t\right] \\ &+c + \tfrac{1}{2}k, \end{aligned} \tag{13a}$$

$$v = \tfrac{2-\beta}{2}k^2 \mathrm{sech}^2\left[kx + k(k^2 + 3ck + 3c^2)t\right]. \tag{13b}$$

and

$$\begin{aligned} u = &-\tfrac{1}{2}k \tanh\left[kx + k(k^2 + 3ck + 3c^2)t\right] \\ &+c + \tfrac{1}{2}k, \end{aligned} \tag{14a}$$

$$v = \tfrac{\beta}{2}k^2 \mathrm{sech}^2\left[kx + k(k^2 + 3ck + 3c^2)t\right]. \tag{14b}$$

Taking $k = \frac{1}{2}, c = 1, \beta = 1$ in (13), we can show the single soliton solutions of the HCBB equation (1) in Fig 1.

B. Interaction solutions

In order to obtain the interaction solutions of the HCBB equation (1), we consider w in the form

$$w = kx + \omega t + g, \tag{15}$$

where $g = g(x,t)$, on account of which, Eqs. (10) and (11) lead to the following PSTO waves

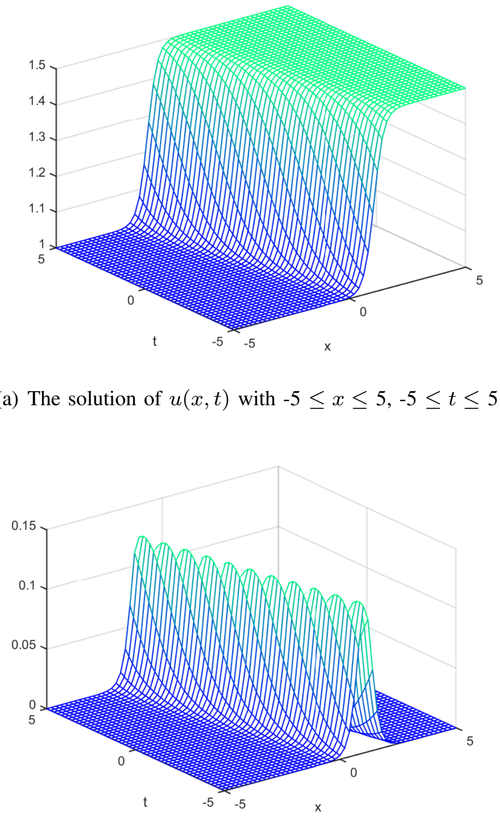
$$\begin{aligned} &g_t - \tfrac{1}{4}(g_{xx} + 6c_1 g_x + 12c_1^2 g + 3g_x^2)_x - g_x^3 \\ &-3c_1 g_x^2 + \omega_0 = 0, \end{aligned} \tag{16}$$

and

$$\begin{aligned} &g_t - \tfrac{1}{4}(g_{xx} - 6c_1 g_x + 12c_1^2 g - 3g_x^2)_x - g_x^3 \\ &-3c_1 g_x^2 + \omega_0 = 0, \end{aligned} \tag{17}$$

where c_1 and ω_0 are related to k, c and ω by

$$\begin{aligned} &c_1 = c + k, \\ &\omega_0 = \omega - k(k^2 + 3ck + 3c^2). \end{aligned} \tag{18}$$



(a) The solution of $u(x,t)$ with -5 $\leq x \leq$ 5, -5 $\leq t \leq$ 5

(b) The solution of $v(x,t)$ with -5 $\leq x \leq$ 5, -5 $\leq t \leq$ 5

Fig. 1. Single soliton solutions of the HCBB equation.

Substituting Eq. (15) along with Eq. (16) into Eq. (7), we get the interaction solution for the HCBB equation (1)

$$u = \tfrac{1}{2}(k + g_x)\left[\tanh(kx + \omega t + g) + 1\right] + c, \tag{19a}$$

$$\begin{aligned} v = &\tfrac{2-\beta}{2}\left[(k + g_x)^2 \mathrm{sech}^2(kx + \omega t + g)\right. \\ &\left.+g_{xx}\tanh(kx + \omega t + g) + g_{xx}\right]. \end{aligned} \tag{19b}$$

Similarly, substituting Eq. (15) along with Eq. (17) into Eq. (8), we also get the interaction solution for the HCBB equation (1)

$$u = -\tfrac{1}{2}(k + g_x)\left[\tanh(kx + \omega t + g) - 1\right] + c, \tag{20a}$$

$$\begin{aligned} v = &\tfrac{\beta}{2}\left[(k + g_x)^2 \mathrm{sech}^2(kx + \omega t + g)\right. \\ &\left.+g_{xx}\tanh(kx + \omega t + g) - g_{xx}\right]. \end{aligned} \tag{20b}$$

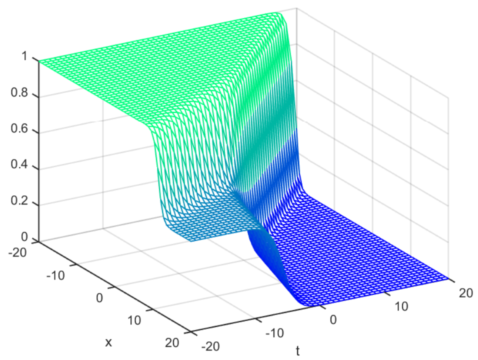
It is well known that the PSTO equation has many types of known exact solutions. Thus, we can use those known solutions to construct the interaction solutions between a soliton and those PSTO waves.

1) Multiple resonant soliton solutions

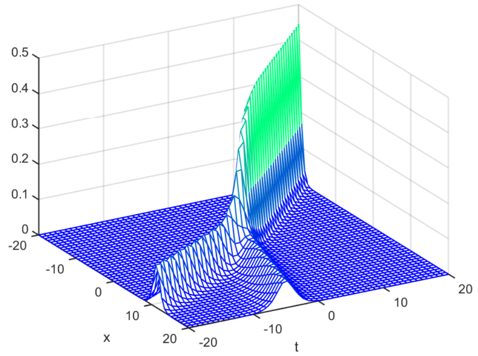
Eq. (16) possesses the following multiple wave solutions

$$g = \tfrac{1}{2}\ln\Big[\sum_{i=1}^{n} l_i e^{(k_i x + \omega_i t)}\Big] \tag{21}$$

with $\omega_i = -2\omega_0 + k_i(\frac{1}{4}k_i^2 + \frac{3}{2}c_1 k_i + 3c_1^2)$, and k_i, l_i are arbitrary constants. Substituting (21) into the expression (19), we can obtain $(n+1)$ resonant soliton solutions of the HCBB equation (1).



(a) The solution of $u(x,t)$ with -20 $\leq x \leq$ 20, -20 $\leq t \leq$ 20



(b) The solution of $v(x,t)$ with -20 $\leq x \leq$ 20, -20 $\leq t \leq$ 20

Fig. 2. Multiple resonant soliton solutions of the HCBB equation.

Similarly, Eq. (17) possesses the following multiple wave solutions

$$g = -\frac{1}{2}\ln\Big[\sum_{i=1}^{n} l_i e^{(k_i x+\omega_i t)}\Big] \tag{22}$$

with $\omega_i = 2\omega_0 + k_i(\frac{1}{4}k_i^2 - \frac{3}{2}c_1 k_i + 3c_1^2)$, and k_i, l_i are arbitrary constants. Substituting (22) into the expression (20), we can also obtain $(n+1)$ resonant soliton solutions of the HCBB equation (1).

Taking $n = 2, k = \frac{1}{2}, c = 1, \omega = 1, l_1 = 1, k_1 = 2, l_2 = 2, k_2 = 3, \beta = 1$ in (22), we can show multiple resonant soliton solutions of the HCBB equation (1) in Fig 2.

2) Multiple interactions with periodic waves

It is not difficult to find soliton interactions with sine-cosine periodic waves. Such as, the PSTO (16) possesses the following exact solutions

$$g = \frac{1}{2}\ln\sum_{i=1}^{n}\Big\{d_i\cos\big[l_i(x+a_i t)\big]e^{(k_i x+b_i t)}\Big\} \tag{23}$$

where $a_i = 3(c_1 + \frac{k_i}{2})^2 - \frac{1}{4}l_i^2, b_i = -2\omega_0 + 3k_i c_1^2 - \frac{3}{2}c_1(l_i^2 - k_i^2) + \frac{1}{4}k_i^3 - \frac{3}{4}k_i l_i^2$, and $d_i, k_i, l_i (i = 1, 2, \cdots, n)$ are arbitrary constants.

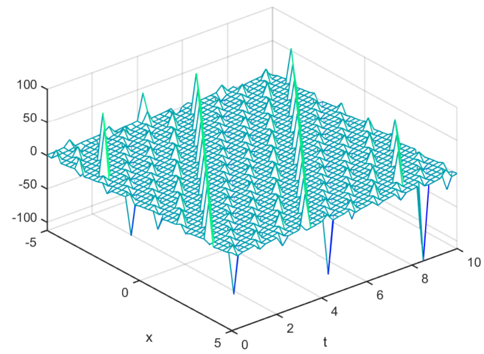
Similarly, the PSTO (17) possesses the following exact solutions

$$g = -\frac{1}{2}\ln\sum_{i=1}^{n}\Big\{d_i\cos\big[l_i(x+a_i t)\big]e^{(k_i x+b_i t)}\Big\} \tag{24}$$

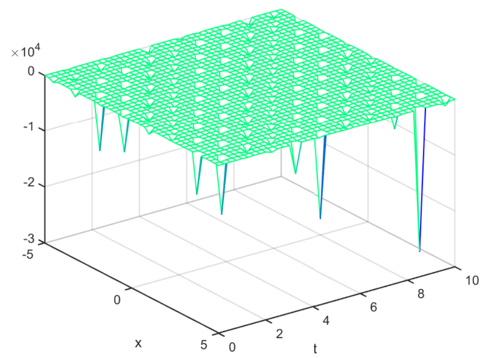
where $a_i = 3(c_1 - \frac{k_i}{2})^2 - \frac{1}{4}l_i^2, b_i = 2\omega_0 + 3k_i c_1^2 + \frac{3}{2}c_1(l_i^2 - k_i^2) + \frac{1}{4}k_i^3 - \frac{3}{4}k_i l_i^2$, and $d_i, k_i, l_i (i = 1, 2, \cdots, n)$ are arbitrary constants.

In other words, the expression (23) with (19), and the expression (24) with (20) also exhibit the interaction solutions of multiple solitons and multiple periodic waves.

Taking $n = 2, k = 1, c = 2, \omega = 1, l_1 = 2, k_1 = 1, d_1 = 1, l_2 = 3, k_2 = 2, d_2 = 2, \beta = 1$ in (24), we can show multiple interactions with periodic waves of the HCBB equation (1) in Fig 3.



(a) The solution of $u(x,t)$ with -5 $\leq x \leq$ 5, 0 $\leq t \leq$ 10



(b) The solution of $v(x,t)$ with -5 $\leq x \leq$ 5, 0 $\leq t \leq$ 10

Fig. 3. Multiple interactions with periodic waves of the HCBB equation.

3) Multiple interactions with rational waves

In order to obtain more solutions of Eqs. (16) and (17), we consider ω in the following result

$$\omega = k(k^2 + 3ck + 3c^2). \tag{25}$$

Thus, Eq. (16) is simplified to

$$\begin{aligned} &g_t - \frac{1}{4}(g_{xx} + 6c_1 g_x + 12c_1^2 g + 3g_x^2)_x \\ &-g_x^3 - 3c_1 g_x^2 = 0. \end{aligned} \tag{26}$$

It is not difficult to verify that Eq. (26) possesses the following solution

$$\begin{aligned} g = \frac{1}{2}\ln\big[&a_1 x^2 + a_2 x + 6a_1 c_1^2 xt + 9a_1 c_1^4 t^2 \\ &+3c_1(a_2 c_1 + a_1)t\big], \end{aligned} \tag{27}$$

which means the solution (27) along with (19) becomes an interaction solution between a soliton and a rational wave.

Similarly, according to (25), Eq. (17) is simplified to

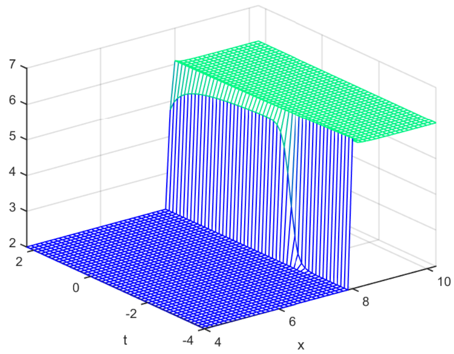
$$\begin{aligned} &g_t - \frac{1}{4}(g_{xx} - 6c_1 g_x + 12c_1^2 g - 3g_x^2)_x \\ &-g_x^3 - 3c_1 g_x^2 = 0, \end{aligned} \tag{28}$$

Eq. (28) possesses the following solution

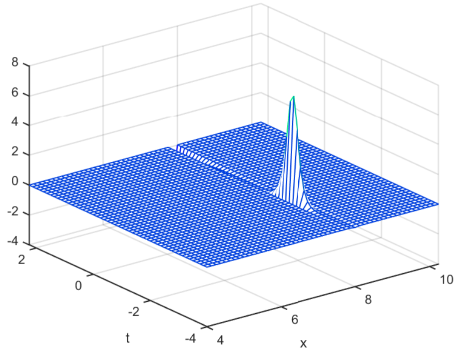
$$\begin{aligned} g = -\tfrac{1}{2}\ln\big[&a_1x^2 + a_2x + 6a_1c_1^2xt + 9a_1c_1^4t^2 \\ &+3c_1(a_2c_1 - a_1)t\big], \end{aligned} \tag{29}$$

which also means the solution (29) along with (20) becomes an interaction solution between a soliton and a rational wave.

Taking $k = 4, c = 2, a_1 = 1, a_2 = 2, \beta = 1$ in (27), we can show multiple interactions with rational waves of the HCBB equation (1) in Fig 4.



(a) The solution of $u(x,t)$ with $4 \le x \le 10$, $-4 \le t \le 2$



(b) The solution of $v(x,t)$ with $4 \le x \le 10$, $-4 \le t \le 2$

Fig. 4. Multiple interactions with rational waves of the HCBB equation.

III. Nonlocal Symmetries Related to CTE method

To find nonlocal symmetries related to CTE method , we write down non-auto Bäcklund (BT) theorems for the HCBB equation (1).

Theorem 3 If $\{u_0, w\}$ is a solution of Eq. (4), then

$$\begin{aligned} u &= w_x + u_0, \\ v &= (2-\beta)(w_{xx} + u_{0x}). \end{aligned} \tag{30}$$

is a solution of the HCBB equation (1).

Theorem 4 If $\{u_0, w\}$ is a solution of Eq. (6), then

$$\begin{aligned} u &= w_x + u_0, \\ v &= -\beta(w_{xx} + u_{0x}). \end{aligned} \tag{31}$$

is a solution of the HCBB equation (1).

Now it is ready to study the nonlocal symmetries of the HCBB equation (1). A symmetry

$$\sigma = \begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix}$$

of the HCBB equation is defined as a solution of its linearized system

$$\begin{aligned} \sigma_t^u + \Big[&\tfrac{3}{2}(1-\beta)(u\sigma^u)_x - 3u^2\sigma^u - \tfrac{3}{2}v\sigma^u \\ &-\tfrac{3}{2}u\sigma^v - \tfrac{1}{4}\sigma^u_{xx}\Big]_x = 0, \end{aligned} \tag{32a}$$

$$\begin{aligned} \sigma_t^v - \Big\{&3\beta(1-\tfrac{1}{2}\beta)\big[(u\sigma^u)_{xx} - u_x\sigma^u_x\big] \\ &+\tfrac{3}{2}(1-\beta)(u\sigma^v_x + v_x\sigma^u) + 3u^2\sigma^v + 6uv\sigma^u \\ &+\tfrac{3}{2}v\sigma^v + \tfrac{1}{4}\sigma^v_{xx}\Big\}_x = 0. \end{aligned} \tag{32b}$$

which means the HCBB equation (1) is form invariant under the transformation

$$\begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} \longrightarrow \begin{pmatrix} u \\ v \end{pmatrix} + \epsilon \begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} \tag{33}$$

with the infinitesimal parameter ϵ. Thus, the HCBB equation (1) has the following nonlocal symmetry theorems.

Theorem 5 If $\{u, v\}$ is related to $\{u_0, w\}$ by (30), and $\{u_0, w\}$ is a solution of (4), then

$$\begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} = \begin{pmatrix} w_x e^{-2w} \\ (2-\beta)(w_{xx} - 2w_x^2)e^{-2w} \end{pmatrix} \tag{34}$$

is a nonlocal symmetry of the HCBB equation (1).

Theorem 6 If $\{u, v\}$ is related to $\{u_0, w\}$ by (31), and $\{u_0, w\}$ is a solution of (6), then

$$\begin{pmatrix} \sigma^u \\ \sigma^v \end{pmatrix} = \begin{pmatrix} w_x e^{2w} \\ -\beta(w_{xx} + 2w_x^2)e^{2w} \end{pmatrix} \tag{35}$$

is a nonlocal symmetry of the HCBB equation (1).

The nonlocal symmetries can be localized by introducing an enlarged system. Thus, the nonlocal symmetry given in Theorem 5 has the following localization theorem for the enlarged system

$$\begin{aligned} &u_t + \tfrac{3}{2}(1-\beta)(uu_x)_x - \tfrac{3}{2}(uv)_x \\ &\quad -3u^2u_x - \tfrac{1}{4}u_{xxx} = 0, \\ &v_t - 3\beta(1-\tfrac{1}{2}\beta)(2u_xu_{xx} + uu_{xxx}) \\ &\quad -\tfrac{3}{2}(1-\beta)(uv_x)_x - \tfrac{3}{2}vv_x - 3(u^2v)_x \\ &\quad -\tfrac{1}{4}v_{xxx} = 0, \\ &u = w_x + u_0, \\ &v = (2-\beta)(w_{xx} + u_{0x}), \\ &w_1 = w_x, \\ &w_2 = -w_{1x}, \\ &u_{0t} - (\tfrac{1}{4}u_{0xx} + \tfrac{3}{2}u_0u_{0x} + u_0^3)_x = 0, \\ &w_t - (\tfrac{1}{4}w_{xx} + \tfrac{3}{4}w_x^2 + \tfrac{3}{2}u_0w_x)_x \\ &\quad -w_x^3 - 3u_0w_x(u_0 + w_x) = 0. \end{aligned} \tag{36}$$

Similarly, the nonlocal symmetry given in Theorem 6 has the following localization theorem for the enlarged system

$$\begin{aligned} &u_t + \tfrac{3}{2}(1-\beta)(uu_x)_x - \tfrac{3}{2}(uv)_x \\ &\quad -3u^2u_x - \tfrac{1}{4}u_{xxx} = 0, \\ &v_t - 3\beta(1-\tfrac{1}{2}\beta)(2u_xu_{xx} + uu_{xxx}) \\ &\quad -\tfrac{3}{2}(1-\beta)(uv_x)_x - \tfrac{3}{2}vv_x - 3(u^2v)_x \\ &\quad -\tfrac{1}{4}v_{xxx} = 0, \\ &u = w_x + u_0, \\ &v = -\beta(w_{xx} + u_{0x}), \\ &w_1 = w_x, \\ &w_2 = w_{1x}, \\ &u_{0t} - (\tfrac{1}{4}u_{0xx} - \tfrac{3}{2}u_0u_{0x} + u_0^3)_x = 0, \\ &w_t - (\tfrac{1}{4}w_{xx} - \tfrac{3}{4}w_x^2 \\ &\quad -\tfrac{3}{2}u_0w_x)_x - w_x^3 - 3u_0w_x(u_0 + w_x) = 0. \end{aligned} \tag{37}$$

Theorem 7 For Eq. (36), the HCBB equation (1) possesses a Lie point symmetry

$$\begin{aligned} &\sigma^u = w_1 \mathrm{e}^{-2w}, \\ &\sigma^v = -(2-\beta)(w_2 + 2w_1^2)\mathrm{e}^{-2w}, \\ &\sigma^w = -\tfrac{1}{2}\mathrm{e}^{-2w}, \\ &\sigma^{u_0} = 0, \\ &\sigma^{w_1} = w_1 \mathrm{e}^{-2w}, \\ &\sigma^{w_2} = (w_2 + 2w_1^2)\mathrm{e}^{-2w}. \end{aligned} \tag{38}$$

which is a localization of the nonlocal symmetry for the original HCBB equation (1).

Theorem 8 For Eq. (37), the HCBB equation (1) possesses a Lie point symmetry

$$\begin{aligned} &\sigma^u = w_1 \mathrm{e}^{2w}, \\ &\sigma^v = -\beta(w_2 + 2w_1^2)\mathrm{e}^{2w}, \\ &\sigma^w = \tfrac{1}{2}\mathrm{e}^{2w}, \\ &\sigma^{u_0} = 0, \\ &\sigma^{w_1} = w_1 \mathrm{e}^{2w}, \sigma^{w_2} = (w_2 + 2w_1^2)\mathrm{e}^{2w}. \end{aligned} \tag{39}$$

which is a localization of the nonlocal symmetry for the original HCBB equation (1). When a nonlocal symmetry is localized, it can be used to find its finite transformations and the related symmetry reductions. Thus, we have the following finite transformation theorems.

Theorem 9 if $\{u, v, w, u_0, w_1, w_2\}$ is a solution of the prolonged HCBB equation (36), so $\{u', v', w', u_0', w_1', w_2'\}$ is with

$$\begin{aligned} &u' = u + \tfrac{\epsilon w_1}{-\epsilon + \mathrm{e}^{2w}}, \\ &v' = v - (2-\beta)\left[\tfrac{\epsilon w_2}{-\epsilon + \mathrm{e}^{2w}} + \tfrac{2\epsilon w_1^2 \mathrm{e}^{2w}}{(-\epsilon + \mathrm{e}^{2w})^2}\right], \\ &w' = \tfrac{1}{2}\ln(-\epsilon + \mathrm{e}^{2w}), \\ &u_0' = u_0, \\ &w_1' = \tfrac{w_1 \mathrm{e}^{2w}}{-\epsilon + \mathrm{e}^{2w}}, \\ &w_2' = \tfrac{w_2 \mathrm{e}^{2w}}{-\epsilon + \mathrm{e}^{2w}} + \tfrac{2\epsilon w_1^2 \mathrm{e}^{2w}}{(-\epsilon + \mathrm{e}^{2w})^2}. \end{aligned} \tag{40}$$

Theorem 10 if $\{u, v, w, u_0, w_1, w_2\}$ is a solution of the prolonged HCBB equation (37), so $\{u', v', w', u_0', w_1', w_2'\}$ is with

$$\begin{aligned} &u' = u + \tfrac{\epsilon w_1}{-\epsilon + \mathrm{e}^{-2w}}, \\ &v' = v - \beta\left[\tfrac{\epsilon w_2}{-\epsilon + \mathrm{e}^{-2w}} + \tfrac{2\epsilon w_1^2 \mathrm{e}^{-2w}}{(-\epsilon + \mathrm{e}^{-2w})^2}\right], \\ &w' = \tfrac{1}{2}\ln(-\epsilon + \mathrm{e}^{-2w}), \\ &u_0' = u_0, \\ &w_1' = \tfrac{w_1 \mathrm{e}^{-2w}}{-\epsilon + \mathrm{e}^{-2w}}, \\ &w_2' = \tfrac{w_2 \mathrm{e}^{-2w}}{-\epsilon + \mathrm{e}^{-2w}} + \tfrac{2\epsilon w_1^2 \mathrm{e}^{-2w}}{(-\epsilon + \mathrm{e}^{-2w})^2}. \end{aligned} \tag{41}$$

From the finite BT transformation Theorem 9 and Theorem 10, we can obtain new solutions of the HCBB equation (1) from any seed solutions.

IV. Conclusions

In conclusion, the solitons and any other types of potential STO waves interaction solutions of the HCBB equation (1) are studied with the help of the CTE method. In particular, the multiple soliton-resonant solutions, soliton-periodic wave interactions and soliton-rational wave interactions are explicitly presented. The CTE related nonlocal symmetries are also proposed. The nonlocal symmetries can be localized to find finite BT by prolonging the model to an enlarged one.

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