Modeling of Market Risk by Elliptical Copulas Under the Condition of Coherence: Theory and Application

Ghizlane Kouaiba, Driss Mentagui

***Abstract*—The VaR (Value at Risk), apart from its widespread application in the banking sector to assess risks (operational, market, etc.), turns out to be an incoherence risk measure if it's based on non-elliptical distributions. Indeed, this part will be devoted to the study of the risk measure VaR in a space of elliptical distributions (elliptical copulas) for a given portfolio, and verify that this risk measure is coherent in the Artzner sense if we work in an elliptical space, or even spherical. Finally, the VaR copula calculation for the same portfolio made up of n risk factors will be compared to other VaR calculation methods (historical VaR, parametric VaR, Monte Carlo VaR and Expected-Shortfall).**

***Index Terms*—elliptical copula, VaR (Value-at-Risk), Monte Carlo, Kolmogorov-Smirnov.**

I. Introduction

In the most frequent cases, risk analysts (traders, actuaries, etc.) often have visibility on the marginal distribution functions of a vector of random variables rather than on their joint distribution function. Therefore, the basic concept of a copula is based on a mechanism for modeling the structure of dependence of a random variables set. In a bi-variate approach, for exemple, copulas can be used to define nonparametric measures of dependency for pairs of random variables. When general dependency patterns are relevant, such as those that go beyond correlation or linear association, the copulas can play an essential role in developing additional concepts and measures of dependency. Each individual random variable has an univariate marginal probability density function (and by construction, a marginal cumulative distribution function). The copula function "connects" all marginal distributions to create a multivariate distribution function.

II. Coherent risk measures

A. Definitions

In their seminal article, Artzner et al. (1999) [3] describe the desirable properties that an ideal coherent measure should verify: subadditivity, translation invariance, positive homogeneity and monotony.

Let Ω be a set of random variables, a function $\rho : \Omega \to R$ is said to be a coherent risk measure if it has the following properties:

Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$ for each X, Y $\in\Omega$.

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Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for all X $\in \Omega$ and $\lambda > 0$, $\lambda \in R$.

Monotonicity: if $X \leq Y$ then $\rho(X) \leq \rho(Y)$ for all X, Y$\in \Omega$

Translation invariance: $\rho(X+\alpha) = \rho(X) + \alpha$ for all X $\in \Omega$, $\rho \in R$.

A risk measure is said to be monetary if it is monotonic and invariant by translation.

In the case where ρ satisfies the first two conditions, we say that it is convex. That means for each $\beta \in [0, 1[$, and we will have:

$$\rho(\beta X + (1-\beta)Y) \leq \beta\rho(X) + (1-\beta)\rho(Y) \tag{1}$$

In addition, the lack of subadditivity implies that the portfolio diversification can lead to increased risk and avoid adding VaR from different risk sources. Thus, VaR is not a coherent measure in the work of Artzner et al. (1999), unlike conditional VaR, and regulators should be careful to insist on its use.

1) Counterexample to show that VaR (Value-at Risk) is not always a sub additive measure: Let X and Y be two random variables which follow the Pareto distribution [17] such that $X \sim par(1,1)$ and $Y \sim par(2,1)$:

$$\begin{aligned} X \sim par(1,1) \Rightarrow P(X \leq t_x) &= \frac{-1+t_x}{t_x}, t_x > 1 \\ \Rightarrow VaR(X,\alpha) = t_x &= \frac{1}{1-\alpha} \end{aligned} \tag{2}$$

$$\begin{aligned} Y \sim par(2,1) \Rightarrow P(Y \leq t_y) &= \frac{-2+t_y}{t_y}, t_y > 2 \\ \Rightarrow VaR(Y,\alpha) = t_y &= \frac{2}{1-\alpha} \end{aligned} \tag{3}$$

$$\begin{aligned} f_{\lambda,X,Y}(x,y) = \lambda(\lambda+1)(\theta_1\theta_2)^{(\lambda+1)}(\theta_2 x + \theta_1 y - \theta_1\theta_2)^{-(\lambda+2)} \\ x > \theta_1, y > \theta_2, \lambda > 0 \end{aligned} \tag{4}$$

In our case, we have: $\theta_1 = \lambda = 1; \theta_2 = 2$

$$\Rightarrow f_{X,Y}(x,y) = 8(2x+y-2)^{-3} \tag{5}$$

Then:

$$\begin{aligned} P[x+y \leq t] &= \int_1^{+\infty}\int_2^{t-x} 8(2x+y-2)^{-3}dydx \\ &= \int_1^{+\infty} -[4(2x+y-2)^{-2}]_2^{t-x}dx \\ &= \int_1^{+\infty}(-4(t+x-2)^{-2} + 4(2x)^{-2})dx \\ &= [4(t+x-2)^{-1}]_1^{+\infty} + [\frac{-4(2x)^{-1}}{2}]_1^{+\infty} \end{aligned} \tag{6}$$

That means:

$$P[x+y\le t] = -4(t-1)^{-1}+1=\alpha \tag{7}$$

To compare $P[x+y\le t]$ with $P[x+y\le t_x+t_y]$ we must study the sign of $P[x+y\le t_x+t_y]-\alpha$:

$$\begin{aligned} P[x+y\le t_x+t_y]-\alpha &= -4(\frac{1}{1-\alpha}+\frac{2}{1-\alpha}-1)^{-1} \\ &\quad +(1-\alpha) \\ P[x+y\le t_x+t_y]-\alpha &= (1-\alpha)(\frac{-4}{2+\alpha}+1)<0 \\ &\forall\alpha: 0\le\alpha\le 1 \end{aligned} \tag{8}$$

So :

$$t_x+t_y<t_{x+y} \tag{9}$$

That means :

$$VaR(X,\alpha)+VaR(Y,\alpha)<VaR(X+Y,\alpha) \tag{10}$$

Whatever $0\le\alpha\le 1$.

The interest of what follows in this chapter is to study an example of coherent risk measures, the VaR [11] despite its widespread application in the financial world, remains an incoherent measure in the sense of Artzner et al. (1999) (example of the Pareto distribution). In addition, when we are in a space of elliptical distributions, the VaR becomes a coherent risk measure.

III. Spherical and elliptical distributions

Spherical distributions [3] provide a family of symmetric distributions of uncorrelated random vectors with mean zero.

Let $X=(X_1,...,X_n)$ be a random vector, the distribution of X is said to be spherical if $\forall U\in R^{n*n}$ satisfies: $UU^t=U^tU=I_{nn}$ we will have:

$$UX =_d X \tag{11}$$

The characteristic function $\varphi(t)=E[exp(it^tX)]$ of such distribution admits a particular form:

Let $\phi: R^+\to R$ such that $\varphi(t)=\phi(t^tt)=\phi(t_1^2+..........+t_n^2)$

And we write:

$$X\sim S_n(\phi) \tag{12}$$

If X admits a density$f(X)=f(X_1,...,X_n)$ and that $f(X)=g(X^tX)=g(x_1^2+..........+x_n^2)$ such as: $g:R^+\to R^+$, so these spherical distributions are well interpreted as being distributions whose density is constant in the spheres.

These distributions admit an alternative stochastic representation:

$$X\sim S_n(\phi) \text{ if only if } X=_d RU \tag{13}$$

With U est is a uniform random variable on the unitary hypersphere $S_{n-1}=X\in R^n/X^tX=1$ and $R\ge 0$ is an independent variable of U (see Frang, Kotz and Ng (1987))

Example: in the case of a reduced multivariate distribution $R\sim\sqrt{\chi_n^2}$.

Let T: $R^n\to R^n, Y\to X=T(Y)=AY+\mu$, $A\in R^{n*n}$, $\mu\in R^n$. X admits an elliptical distribution [3], and we write $X\sim E_n(\mu,\Sigma,\phi)$, if only if $Y\sim S_n(\phi)$. The characteristic function of X is given by:

$$\begin{aligned} \varphi(t) &= E[exp(it^tX)] \\ &= E[exp(it^t(AY+\mu))] \\ &= E[exp(it^t\mu)]E[exp(i(A^tt)^tY)] \\ &= E[exp(it^t\mu)]\phi(t^t\Sigma t) \end{aligned} \tag{14}$$

Where: $\Sigma=AA^t$

IV. The Joint and Copula Distribution Functions

Let $(X_1,...,X_m)$ be a vector of random variables, the joint distribution function is defined by:

$$F(X_1,...,X_m)=Pr[X_i\le x_i, i=1,......,m] \tag{15}$$

To see the relation between the marginal distribution functions and the copulas, consider a continuous m-varied distribution function $F(Y_1,...,Y_m)$ with the univariate distributions: $F_1(Y_1),....,F_m(Y_m)$ and inverse functions (quantiles): $F_1^{-1}(a_1),........,F_m^{-1}(a_m)$. Such as $a_1,...,a_m$ are uniformly distributed variables, therefore:

$$\begin{aligned} F(y_1,...,y_1) &= F(F_1^{-1}(a_1),........,F_m^{-1}(a_m)) \\ &= Pr(A_1\le a_1,........A_m\le a_m) \\ &= C(a_1,........,a_m) \end{aligned} \tag{16}$$

C is the unique copula [1], [4], [13], [16] associated to the multivariate distribution function (see Sklar's theorem). However, if Y$\sim F$, and F is continuous:
(F$_1(Y_1)$...., , $F_m(Y_m))\sim C($a$_1,........,a_m)$.

A. Some examples of copulas

The simplest copula formula, for uniformly distributed variables u_1 ,$u_2,......,u_d$, is that of the product copula:

$$C(u_1,u_2,...,u_d)=u_1u_2\ldots u_d \tag{17}$$

The Gaussian copula [9] also takes the following form:

$$\begin{aligned} &C_\rho^{Ga}(\Phi^{-1}(u_d)\ldots,\Phi^{-1}(u_2),\Phi^{-1}(u_1))= \\ &\int_{-\infty}^{\Phi^{-1}(u_d)}\int_{-\infty}^{\Phi^{-1}(u_2)}\int_{-\infty}^{\Phi^{-1}(u_1)}\frac{1}{(2\pi)^{\frac{d}{2}}\sqrt{det(\Sigma)}} \\ &\qquad exp(\frac{-X'\Sigma^{-1}X}{2})dx_1dx_2\ldots dx_d \end{aligned} \tag{18}$$

With Φ is the reduced Gaussian distribution and Σ is the correlation matrix.

V. Elliptical distributions in risk-management:

Let $X\sim E_n(\mu,\Sigma,\phi)$ be a vector of n risks composing an investment portfolio P, such as $P=\{z=\sum\lambda_ix_i=\lambda X\in R\}$.

The subadditivity of the VaR is verified in an elliptical space, i.e :

$$(Z_1,Z_2)\in P\Rightarrow VaR_\alpha(Z_1+Z_2)\le VaR_\alpha(Z_1)+VaR_\alpha(Z_2) \tag{19}$$

1) Subadditivity:
   Proof: Let q_α be the quantile of this reduced distribution (mean=0, sd=1), with $0.5\le\alpha\le 1 (i.e: q_\alpha>0)$,

if we consider the distribution of the values of portfolios made up of financial securities (stocks, bonds, etc.), the VaR often corresponds to the quantile of the distribution of portfolios at the threshold $\alpha = 99\%$, and therefore:

$$\begin{aligned} VaR_\alpha(Z_1) &= \mu_1 + \sigma[Z_1]q_\alpha \\ VaR_\alpha(Z_2) &= \mu_2 + \sigma[Z_2]q_\alpha \end{aligned} \tag{20}$$

And

$$VaR_\alpha(Z_1 + Z_2) = \mu_1 + \mu_2 + \sigma[Z_2 + Z_1]q_\alpha \tag{21}$$

Let us show that $\sigma[Z_2 + Z_1] \leq \sigma[Z_2] + \sigma[Z_1]$:
According to the Minkowski inequality we have:

$$\begin{aligned} &\sigma(Z_1 + Z_2) \\ &= \sqrt{\iint_{R^2} (Z_1 + Z_2 - E(Z_1 + Z_2))^2 f_{Z_1,Z_2} dz_1 dz_2} \\ &= \sqrt{\iint_{R^2} (Z_1 + Z_2 - E(Z_1) - E(Z_2))^2 f_{Z_1,Z_2} dz_1 dz_2} \\ &= \sqrt{\iint_{R^2} ((Z_1 - E(Z_1) + Z_2 - E(Z_2))\sqrt{f_{Z_1,Z_2}})^2 dz_1 dz_2} \\ &\leq \sqrt{\iint_{R^2} ((Z_1 - E(Z_1))\sqrt{f_{Z_1,Z_2}})^2 dz_1 dz_2} \\ &+ \sqrt{\iint_{R^2} ((Z_2 - E(Z_2))\sqrt{f_{Z_1,Z_2}})^2 dz_1 z_2} \\ &= \sigma[Z_2] + \sigma[Z_1] \end{aligned} \tag{22}$$

Therefore:

$$\mu_1 + \mu_2 + \sigma[Z_2 + Z_1]q_\alpha \leq \mu_1 + \sigma[Z_1]q_\alpha + \mu_2 + \sigma[Z_2]q_\alpha \tag{23}$$

Finally:

$$VaR_\alpha(Z_1 + Z_2) \leq VaR_\alpha(Z_1) + VaR_\alpha(Z_2) \tag{24}$$

In principle, the reduced quantile of $Z_1 + Z_2$ must be deduced from the bivariate distribution of (Z_1, Z_2) and then compare $VaR_\alpha(Z_1 + Z_2)$ to $VaR_\alpha(Z_1) + VaR_\alpha(Z_2)$, therefore it's useful to call the elliptical bivariate copula to deduce the reduced quantile of $Z_1 + Z_2$ $(i.e : q'_\alpha)$ and subsequently confirm the subadditivity. In this case reduced quantile is calculated by using the following formula:

$$\alpha = \int_{-\infty}^{\phi^{-1}(u_2=1)} \int_{-\infty}^{t-y} c(x, y) dx dy \tag{25}$$

And who must verify $q'_\alpha = \frac{t}{\text{standard deviation } (x+y)} \leq q_\alpha$ (case of a Gaussian copula and the t-copula), because if standard deviation(x)=1 and standard deviation(y)=1, the standard deviation$(x + y) \neq 1$. The property of the subadditivity will be verified through the case study proposed in this paper.

2) Positive homogeneity:
Consider:

$$\begin{aligned} VaR_\alpha(Z_i) &= \mu_i + \sigma[Z_i]q_\alpha \\ &i = 1, 2 \end{aligned} \tag{26}$$

$$VaR_\alpha(\lambda Z_i) = \lambda VaR_\alpha(Z_i) = \lambda\mu_i + \lambda\sigma[Z_i]q_\alpha, \lambda > 0 \tag{27}$$

Because :

$$variance(\lambda Z_i) = \lambda^2 variance(Z_i) = (\lambda\sigma[Z_i])^2 \tag{28}$$

And

$$E(\lambda Z_i) = \lambda E(Z_i) = \lambda\mu_i \tag{29}$$

Which means:

$$\begin{aligned} \lambda VaR_\alpha(Z_i) = \lambda(\mu_i + \sigma[Z_i]q_\alpha) &= \lambda\mu_i + \lambda\sigma[Z_i]q_\alpha \\ &= VaR_\alpha(\lambda Z_i) \end{aligned} \tag{30}$$

3) Monotony:
If we suppose that: $Z_1 \leq Z_2$. And we know already that: $VaR_\alpha(X) = \{inf(X)/F(X) \geq \alpha\}$ and $F(X)$ is the cumulative distribution function of X, which increasing, we will have the following:

$$\begin{aligned} VaR_\alpha(Z_1) = \{inf(Z_1)/F(Z_1) \geq \alpha\} &\leq \\ VaR_\alpha(Z_2) = \{inf(Z_2)/F(Z_2) \geq \alpha\} & \end{aligned} \tag{31}$$

Which means:

$$VaR_\alpha(Z_1) \leq VaR_\alpha(Z_2) \tag{32}$$

4) Invariance by translation:

$$\begin{aligned} VaR_\alpha(Z_i + \lambda) &= (\mu_i + \lambda) + \sigma[Z_i + \lambda]q_\alpha \\ &= (\mu_i + \lambda) + \sigma[Z_i]q_\alpha \end{aligned} \tag{33}$$

Because:

$$variance(Z_i + \lambda) = variance(Z_i) = (\sigma[Z_i])^2 \tag{34}$$

And

$$E(Z_i + \lambda) = \mu_i + \lambda \tag{35}$$

Finally:

$$\begin{aligned} VaR_\alpha(Z_i + \lambda) &= (\mu_i + \lambda) + \sigma[Z_i]q_\alpha \\ &= \mu_i + \sigma[Z_i]q_\alpha + \lambda \\ &= VaR_\alpha(Z_i) + \lambda \end{aligned} \tag{36}$$

VI. CASE STUDY

Consider a portfolio that consists of the shares of two French companies (Faurécia (800 shares in €) which operates in the automotive sector, and Icade (900 shares in €) [5] which for its part operates in the real estate sector). The set of observations contains 261 observations since 08/10/2018.

First, it is necessary to adjust the loss distributions of these shares to adequate probability distributions. Once these marginal distributions are determined, it then comes to determine the appropriate elliptical copula [2] in order to calculate the VaR [15] of the portfolio at $\alpha = 99\%$ of the distribution of losses (the quantile of the sum of the using the bivariate copula concerning the calculation of $VaR_\alpha(Z_2 + Z_1)$. That said, we go through a test of distribution adjustment for each share, which is assumed to be continuous (the Kolmogorov-Smirnov test, the Cramer-Von-Mises test. . .).

It will also be appropriate to compare the $VaR_\alpha(Z_2 + Z_1)$ deduced from the bivariate copula to different calculations of the VaR (historical, parametric VaR, Monte Carlo, the Expected Shortfall .....).

A. VaR calculation:

1) Historical VaR: For a horizon t = N (261 days in our case), we can assess the portfolio with the historical risk factors. This means that we determine for each date N-1 potential daily variations, which are assimilated to N-1 potential losses (some losses are in fact gains). To be able to extract the quantile at α, it suffices for that to arrange the N potential losses and to take the absolute value of $\lceil N(1-\alpha)\rceil$i-th worst value.

2) Parametric VaR: This approach [8] assumes that variations in market risk factors follow a Gaussian distribution, on the one hand, and these factors have a linear risk profit, on the other hand. Therefore, for m risk factors, the variation of the portfolio is as following:

$$\Delta V = \sum_i n_i \Delta x_i \tag{37}$$

Such as:
ΔV: The variation of the portfolio.
n_i: The quantity of the i-th risk factor.
x_i: i-th risk factor with $\Delta x_i \sim N(\mu_i, \sigma_i)$.
$\mu = (\mu_1, \mu_2 \ldots\ldots, \mu_m)$, $n = (n_1, n_2 \ldots\ldots, n_m)$ and Σ is the variance-covariance matrix of Δx_i. Therefore:

$$\Delta V \sim N(n\mu', n\Sigma n') \tag{38}$$

And:

$$P(\Delta V \leq VaR_\alpha(p)) = \alpha \Rightarrow P(\frac{\Delta V - n\mu'}{\sqrt{n\Sigma n'}} \leq q_\alpha) = \alpha \tag{39}$$

Finally we have:

$$VaR_\alpha(p) = q_\alpha \sqrt{n\Sigma n'} + n\mu' \tag{40}$$

3) Copula VaR: This VaR calculation method will be based on the quantile q_α' , not deduced from the univariate reduced distribution but from the multivariate copula. In the case of a Gaussian copula or the t-copula.

we will have, as already mentioned:

$$\alpha = \int_{-\infty}^{\phi^{-1}(u_2=1)} \int_{-\infty}^{t-y} c(x,y)dxdy \tag{41}$$

And who must verify $q_\alpha' = \frac{t}{\text{standard deviation } (x+y)} \leq q_\alpha$ (case of a Gaussian copula and the t-copula).

For the case of the Gaussian copula [1]:
$x = \phi^{-1}(u_1)$, $y = \phi^{-1}(u_2)$

For the case of the t-copula [1]:
$x = t_\nu^{-1}(u_1)$, $y = t_\nu^{-1}(u_2)$

With:

t_ν is the student distribution at ν degree of freedom, and ϕ is the reduced Gaussian distribution.
Finally:

$$VaR_\alpha(p) = q_\alpha' \sqrt{n\Sigma n'} + n\mu' \tag{42}$$

In the case of a portfolio made up of two securities $X_1$and X_2:

$$VaR_\alpha(p) = q_\alpha' \sigma(n_1 X_1 + n_2 X_2) + (E(n_1 X_1) + E(n_2 X_2)) \tag{43}$$

This equality will be used later in the empirical study.

4) Monte Carlo VaR: This method is based on the same principle of parametric simulation except that, instead of estimating the distribution of returns based on past scenarios, Monte-Carlo simulation [7] uses the scenarios which are generated according to a model based on data history. It is assumed that the returns of the securities are governed by a parametric distribution with known parameters. The calculation steps are given as follows:

- We generate m independent realizations of the m shares yield process such that each process is a vector of length T.
- For each $j \in (1, \ldots.., n)$, we calculate the corresponding$VaR_i^{(j)}$ for the i-th security.
- Using the principle of the strong law of large numbers, the VaR_i of the i-th security is given as follows:

$$VaR_i = \frac{1}{n} \sum_j^n VaR_i^{(j)} \tag{44}$$

The VaR_p of the portfolio is given by:

$$\sqrt{(VaR_1, VaR_2, \ldots.., VaR_m) C (VaR_1, VaR_2, \ldots.., VaR_m)'} \tag{45}$$

Such as C is the correlation matrix of the m securities.

5) The Expected-Shortfall: The Expected-Shortfall (ES) [10], [12], [14] is a risk measure that gives visibility of the losses of a given portfolio, that can be observed beyond the VaR. In principle, it is the average of the losses that a portfolio can sustain beyond the VaR:

$$ES = E(X/X \geq VaR_\alpha)$$

$ES = E(-X/X \leq -VaR_\alpha)$ (in elliptical distribution case)

$$\begin{aligned} ES &= \frac{E(-X, X \leq -VaR_\alpha)}{P(X \leq -VaR_\alpha)} \\ &= \frac{E(-X, X \leq -VaR_\alpha)}{1-\alpha} \\ &= \frac{\int_{-\infty}^{-VaR_\alpha} -xf(x)dx}{1-\alpha} \end{aligned} \tag{46}$$

$f(x)$ is the distribution of the losses of the risk factor.

In the case of two securities, and using the copula, the calculation will be as follows:

$$ES(p) = E(q/q \geq q_\alpha') \sqrt{n\Sigma n'} + n\mu' \tag{47}$$

$$\begin{aligned} E(q/q \geq q_\alpha') &= E(-q/q \leq -q_\alpha')(\text{ elliptical distribution}) \\ &= \frac{E(-(x+y)/(x+y) \leq -t)}{standard\_deviation(x+y)}(\text{ case of 2 securities}) \\ &= \frac{\int_{-\infty}^{\phi^{-1}(u_2=1)} \int_{-\infty}^{-t-y} -(x+y)c(x,y)dxdy}{(1-\alpha)(standard\_deviation(x+y))} \end{aligned} \tag{48}$$

And t is already deduced from the relation (42) in the parametric-copula VaR part.

It should also be noted that for a symmetric distribution: $VaR_\alpha = -VaR_{1-\alpha}$.

B. Empirical part

Using the R software [6], the Kolmogorov-Smirnov adjustment test for the Gaussian distribution and the Student distribution gives as output the following values:

It is important to highlight that:

. (y2Faurécia) and (y21Icad) are respectively the daily variations of Faurécia and Icad shares.

. y2 = (y2Faurécia) * 800, y21 = (y21Icad)* 900

. y3 and y4 are respectively the reduced values of y2 and y21.

. d = y21 + y2.

According to the output of the software R, the values of p-values of the Kolmogorov-Smirnov are greater than 5%, and therefore, the two portfolios can be fitted to the reduced Gaussian distribution or to the student distribution at df =(260-1) degree of freedom, but to choose between the two distributions, we are based on the distribution whose the p-value is $>$ 5%: the output of the software R leads us to choose the Gaussian distribution for the two shares.

TABLE I
KOLMOGOROV-SMIRNOV TEST FOR Y2FAURÉIA AND Y21ICAD LOSSES

R software console code:				
Distribution test	Stutent		Gaussian	
Data	y2	y21	y2	y21
D statistic	0.50192	0.49042	0.034987	0.052034
p value	$<$ 2.2e-16	$<$ 2.2e-16	0.9067	0.4796
Alternative hypothesis	two-side	two-side	two-side	two-side

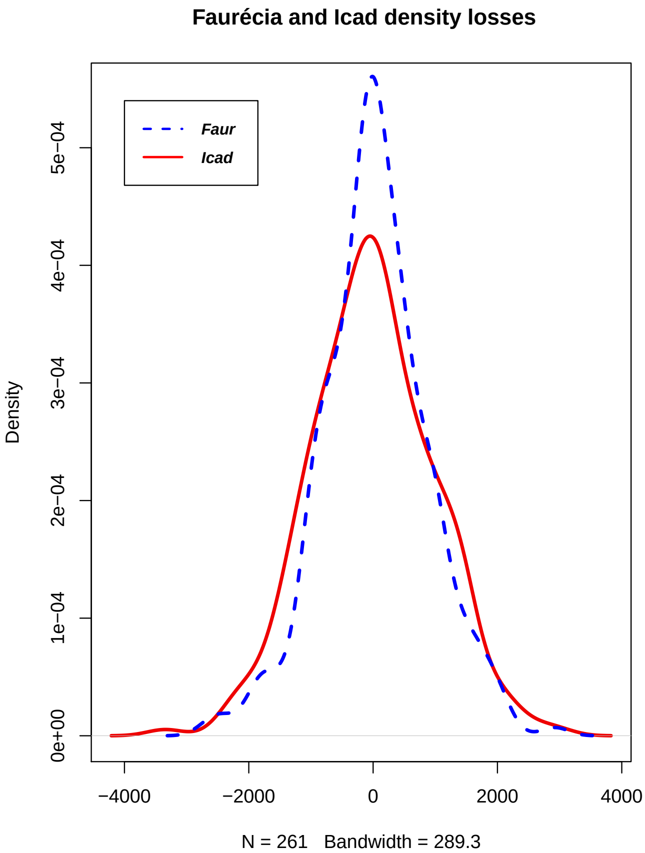

Fig. 1. Density of the two shares y2Faurécia and y21Icad losses

The next step will be the construction of the Gaussian copula of these two shares, that said, the calculation of the Spearman dependency parameter via the following code:

TABLE II
THE CALCULATION OF THE SPEARMAN DEPENDENCY PARAMETER

R software console code:
>rho=cor.test(y3+y4,data=data1,method='spearman',conf.level=0.9)
$estimate
>rho
rho
0.2534864

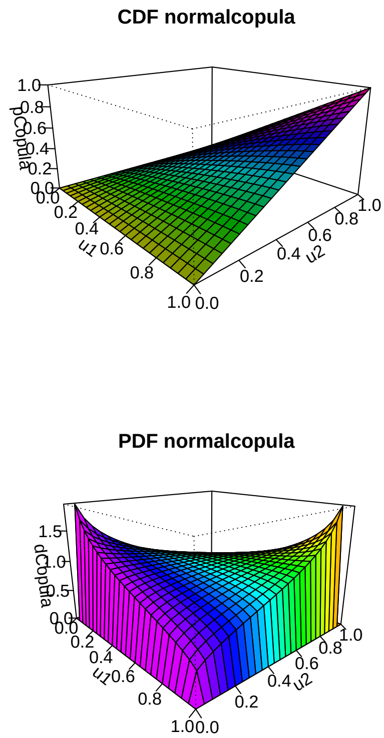

Fig. 2. The cumulative distribution and probability density function of the Gaussian copula with the Spearman dependence value = 0.2534864

Once the copula is determined, we are led to calculate the reduced quantile of y21 + y2 by using the Gaussian copula, which require finding the quantile t such as:

$$P[x+y \leq t] = \int_{-\infty}^{+\infty} \int_{-\infty}^{t-y} \frac{1}{2\pi\sqrt{(1-rho^2)}} \\ exp\left(\frac{-(s^2 - 2*rho*s*t+t^2)}{2(1-rho^2)}\right) dsdt \tag{49}$$

And see if using quantile $q_\alpha^{'}$ will respects the condition of subadditivity since it is the exact compute of the reduced quantile of d = y21 + y2, and then compare it with the quantile q_α using the univariate reduced Gaussian distribution since d also follows a Gaussian distribution ($q_\alpha^{'}$ is the quantile used in the demonstration of the subadditivity under condition of an elliptical copula).

TABLE III
CRAMER-VON MISES NORMALITY TEST FOR (D=Y21+Y2)

```
R software console code:

> cvm.test(d)

        Cramer-von Mises normality test
data: d
W = 0.072335, p-value = 0.2597
```

TABLE IV
REIMANN INTEGRAL ALGORITHM ON R SOFTWARE

```
R software console code:

Reiman <- function (h, ax, bx, by, fun3) {
t=-8.578
o=5
  while(o < 0.01 | o >  0.02) {
   o=0
   yi=bx+(h/2)
    while (yi<=by) {
    ay=t-yi
    xi=ax+(h/2)
     while (xi<=ay) {
     o=o+fun3(xi, yi) * (h^2)
     xi=xi+h }
    yi=yi+h }
   t=t+0.03 }
return (t) }
```

The quantile t resulting from the Gaussian copula formula is deduced from a developed algorithm of the Riemann integral principle in dimension d = 2, as following:
For t = -3.694 we retain $P[y3 + y4 \leq t] = 0.01013091 = 1 - \alpha$. However, we know that:

$$VaR_\alpha(p) = q_\alpha^{'}\sigma(n_1X_1 + n_2X_2) + (E(n_1X_1) + E(n_2X_2)) \tag{50}$$

If y3 and y4 are reduced the y3 + y4 are not necessarily reduced: Because in this case, the standard deviation of (y3 + y4) = $(1 + 1 + 2cov(y3, y4))^{1/2}$ and for an elliptical distribution the $q^{'}_{\alpha=99\%} = -q^{'}_{1-\alpha=1\%}$, so the $q^{'}_{\alpha=99\%} = \frac{-(-3.694)}{\text{standard deviation(y3+y4)}} = 2.327662 \approx -qnorm(0.01) = 2.326348$. Finally the $VaR_\alpha(p)$ using the copula is given by:
Copula VaR = 3407,028 € (see Table VI) and which is less than $VaR_\alpha(n_1X_1) + VaR_\alpha(n_2X_2)$ (the subadditivity is verified), given by "somme 2": somme2 = 4232,734 €.

TABLE V
$VaR_\alpha(n_1X_1) + VaR_\alpha(n_2X_2)$

```
R software console code:

>somme2=vary21+vary2
>somme2
    4232.734€
```

By comparing this risk value with another risk valuation method: historical, parametric, Monte Carlo and the Expected-Shortfall (ES) that we have already programmed on the R software, we obtain: .

TABLE VI
ALL VaR CALCULATING METHODS

R software console code:	
Historical VaR	3211.000 €
Parametric VaR	3371.209 €
Copula VaR	3407.028 €
Monte Carlo VaR	3757.244 €
Expected Shortfall	3914.921 €

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