Excellent Higher Order Iterative Scheme for Solving Non-linear Equations

Waikhom Henarita Chanu and Sunil Panday

Abstract—**In this paper, we propose a higher order extension of well-known fourth order Ostrowski's iterative method for solving algebraic and transcendental equations. The newly proposed scheme would involve the evaluation of the function and the first derivative of the function, similar to Ostrowski's method. The expansion of the Taylor series is used to achieve theoretical convergence of the newly proposed tenth order technique. Several numerical experiments support the underlying theory on the convergence order of the proposed method. The performance of the newly proposed method has been compared with the existing well known competitors on some classic academic problems. Numerical tests reveal that the new method is comparable to existing methods and produces better results within less CPU time.**

Index Terms—**Simple root, Nonlinear equation, Iterative methods, Error.**

I. INTRODUCTION

CONSTRUCTION of efficient higher order iterative method for simple zero of the nonlinear univariate function is one of the most challenging problems in numerical analysis. The analytical solution either does not exist or computationally cumbersome to find the simple roots of the following equations:

$$\phi(x) = 0 \tag{1}$$

where $\phi : \mathbb{D} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable univariate function defined on an open interval $\mathbb{D}$ of $\mathbb{R}$. Therefore, iterative approaches can be used to solve equation (1) numerically. An iterative method with an initial guess (x_0) in the neighbourhood of the root can manage to provide a better estimate of the root after performing several iterations. The iterative method is defined by

$$x_{n+1} = P(\phi)(x_n), \qquad for\ n = 0, 1, 2, 3, ... \tag{2}$$

where $P(\phi)$ is called the iteration function.

An iterative method consisting of one step is called a one-point iterative method and a method with multiple steps is called multi-point iterative method. The problem that occurs, when developing high-order one-point iteration methods is the evaluation of the higher derivative of the function. Computation of the second or higher-order derivative is extremely difficult for many nonlinear equations. Therefore, many researchers in the field of numerical analysis have paid attention to construct new multi-point iteration techniques to derive approximate solutions for nonlinear equations [1]–[3]. The Newton's method (NM), is the most popular root-finding routine for univariate functions. The Newton's method [4], [5] is given by

$$x_{n+1} = x_n - \frac{\phi(x_n)}{\phi'(x_n)}, \qquad n = 0, 1, 2, ... \tag{3}$$

The Newton's method is quadratically convergent in some neighbourhood of simple roots. One of the renowned and efficient multi-point iterative technique for solving equation (1) is Ostrowski's method [4], [5]. This method has a fourth-order of convergence and is expressed as

$$\begin{aligned} y_n &= x_n - \frac{\phi(x_n)}{\phi'(x_n)} \\ x_{n+1} &= x_n - \frac{\phi(x_n) - \phi(y_n)}{\phi(x_n) - 2\phi(y_n)} \frac{\phi(x_n)}{\phi'(x_n)}. \end{aligned} \tag{4}$$

In 2012, M. Matinfar et al. [6] had developed a tenth-order multi-point iterative method (MAM), which is given as follows:

$$\begin{aligned} y_n &= x_n - \frac{\phi(x_n)}{\phi'(x_n)} \\ z_n &= y_n - \left[\frac{\phi(y_n)}{x_n^2} + \frac{\phi(y_n)}{x_n}\right] \frac{\phi(x_n)}{\phi'(x_n)} \\ x_{n+1} &= z_n - \frac{\left\{\frac{1}{2}\alpha\theta_n^2 + (\alpha+1)\theta_n + 2 + \frac{1}{2}\alpha\right\}\phi(z_n)}{\psi_\phi(x_n, y_n, z_n)}, \end{aligned} \tag{5}$$

where

$$\theta_n = \frac{1}{z_n - \psi_\phi(x_n, y_n, z_n)}$$

and

$$\begin{aligned} \psi_\phi(x_n, y_n, z_n) = &- \frac{x_x + 2y_n - 3z_n}{(x_n - z_n)(y_n - z_n)}\phi(z_n) \\ &+ \frac{(x_n - z_n)^2}{(x_n - y_n)^2(y_n - z_n)}\phi(y_n) \\ &+ \frac{y_n - z_n}{x_n - y_n}\phi'(x_n) \\ &+ \frac{(y_n - z_n)(2y_n - 3x_n + z_n)}{(x_n - z_n)(x_n - y_n)^2}\phi(x_n). \end{aligned}$$

In 2013, M. A. Hafiz et al. [7] had developed another new tenth-order method (MHM), which is given as follows

$$\begin{aligned} y_n &= x_n - \frac{\phi(x_n)}{\phi'(x_n)} \\ z_n &= y_n - \frac{\phi(y_n)}{\phi'(y_n)} - \frac{[\phi(y_n)]^2 P_1(y_n)}{2[\phi'(y_n)]^3} \\ x_{n+1} &= z_n - \frac{\phi(z_n)}{\phi[z_n, y_n] - (z_n - y_n)\phi[z_n, y_n, y_n]}. \end{aligned} \tag{6}$$

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In 2014, Y. Y. Al-Husayni et al. [8] had developed the following tenth-order multi-point iterative method (YAM):

$$\begin{aligned} y_n =& x_n + \frac{\phi(x_n)}{\phi[G_n, x_n]} \\ z_n =& y_n - \left(\frac{\phi(x_n)\phi(G_n)}{\phi(y_n)-\phi(x_n)}\right)\left(\frac{1}{\phi[G_n, x_n]} - \frac{1}{\phi[G_n, y_n]}\right) \\ x_{n+1} =& z_n - \frac{\phi(z_n)}{2\phi[y_n, x_n] - \phi[G_n, x_n]} \\ & - w(t_n) \times \frac{\phi\left(z_n - \frac{\phi(z_n)}{2\phi[y_n,x_n]-\phi[G_n,x_n]}\right)}{2\phi[y_n, x_n] - \phi[G_n, x_n]}, \end{aligned} \tag{7}$$

where $\phi[y, x] = \frac{\phi(y)-\phi(x)}{y-x}$ and $G_n = x_n + \phi(x_n)^3$.

In 2019, D. Sharma et al. [9] developed a family of an iterative method having tenth order of convergence (DSM), which is rewritten as follows:

$$\begin{aligned} y_n &= x_n - \frac{\phi(x_n)}{s_x} \\ z_n &= x_n - \theta\frac{\phi(x_n)+\phi(y_n)}{s_x} - (1-\theta)\frac{\phi(x_2)^2}{s_x\big(\phi(x_n)-\phi(y_n)\big)} \\ w_n &= z_n - \frac{\phi(z_n)}{\phi[z_n, y_n] + \Psi_{z_n}(z_n - y_n)} \\ x_{n+1} &= w_n - \frac{\phi(w_n)}{\phi[w_n, z_n] + \Psi_{w_n}(w_n - z_n)}, \end{aligned} \tag{8}$$

where

$$\Psi_{z_n} = \phi[z_n, x_n, x_n] \approx \frac{\phi[z_n, x_n] - s_x}{z_n - x_n},$$

$$\Psi_{w_n} = \phi[w_n, x_n, x_n] \approx \frac{\phi[w_n, x_n] - s_x}{w_n - x_n},$$

$$\phi[w_n, z_n] = \frac{\phi(w_n) - \phi(z_n)}{w_n - z_n}$$

and

$$s_x = \frac{\phi(x_n + \phi(x_n)) - \phi(x_n - \phi(x_n))}{2\phi(x_n)}.$$

In 2019, K. Nouri et. al. [10] had also developed a new iterative method (KHM), having tenth order of convergence defined as follows:

$$\begin{aligned} y_n &= x_n - \frac{\phi(x_n)}{\phi'(x_n)} \\ z_n &= y_n - \frac{\phi(y_n)}{\phi'(y_n)} \\ w_n &= y_n + \frac{\phi(y_n)}{\phi'(y_n)} \\ x_{n+1} &= y_n - \frac{(y_n - z_n)\phi(y_n)^2(\phi(z_n)+\phi(w_n))}{\phi(y_n)^2(A) - 4\phi(y_n)\phi(z_n)^2 - 6\phi(z_n)^3}, \end{aligned} \tag{9}$$

where $A = \phi(w_n) - \phi(z_n)$.

Furthermore, Barrada et al. had introduced a new family of Halley's method having third order of convergence [11] and a family of Chebyshev's method [12]. Sharma et al. had introduced an optimal fourth order method [13]. Soliaiman et al. had developed two new efficient sixth order methods [14]. In this study, we propose a tenth order iterative technique for finding simple roots of a nonlinear equation. We have demonstrated a new iterative technique free from second and higher order derivative evaluation of the function. We have derived newly proposed method using Ostrowski's method and the expansion of Taylor series. The second-order derivative is replaced with an estimate including the function and its first derivative evaluations to reduce the total number of function evaluations at each iteration. We have checked that the numerical test supports the theoretical result. We have tested the performance of the proposed method on numerous numerical examples and found that newly proposed method outperformed than some existing well known methods available in the literature. The rest of the paper is organised as follow. In Section II, the construction of the method is described and convergence analysis is also performed to establish the order of convergence. In Section III, the newly proposed method is tested on some numerical examples and comparisons of the results of new method with other well-known methods of the same order are summarized in tables. Finally, the conclusion of the study is given in Section IV.

II. Construction of New Method

Suppose that a function $\phi(x) = 0$ is sufficiently differentiable univariate function defined on a given open interval $\mathbb{D}$ of $\mathbb{R}$. Let x_0 be the approximate to a simple root α of an equation $\phi(x) = 0$. We expand $\phi(x)$ using Taylor's series expansion about x_0 as follows:

$$\phi(x) = \phi(x_0) + (x - x_0)\phi'(x_0) + (x - x_0)^2\phi''(x_0). \tag{10}$$

Substituting $\phi(x) = 0$ in equation (10), to obtain

$$x = x_0 - \frac{\phi(x_0)}{\phi'(x_0)} - \frac{(x - x_0)^2\phi''(x_0)}{\phi'(x_0)}. \tag{11}$$

Now, from equation (4), we get

$$x - x_0 = \frac{\phi(y_0) - \phi(x_0)}{2\phi(y_0) - \phi(x_0)}\frac{\phi(x_0)}{\phi'(x_0)}, \tag{12}$$

where $y_0 = x_0 - \phi(x_0)/\phi'(x_0)$.

Substituting the value of $x - x_0$ from equation (12) in the right hand side of equation (11), we get

$$x = x_0 - \frac{\phi(x_0)}{\phi'(x_0)} + \left[\frac{\phi(x_0)[\phi(y_0) - \phi(x_0)]}{\phi'(x_0)[2\phi(y_0) - \phi(x_0)]}\right]^2\frac{\phi''(x_0)}{\phi'(x_0)}. \tag{13}$$

Rewriting equation (13) in iterative form, we get

$$x_{n+1} = x_n - \frac{\phi(x_n)}{\phi'(x_n)} + \left[\frac{\phi(x_n)[\phi(y_n) - \phi(x_n)]}{\phi'(x_n)[2\phi(y_n) - \phi(x_n)]}\right]^2\frac{\phi''(x_n)}{\phi'(x_n)}, \tag{14}$$

where $y_n = x_n - \phi(x_n)/\phi'(x_n)$.

This is the third order of convergence. To get the higher order of convergence, we use Ostrowski's methods as first two steps, and equation (14) as last step, and therefore we get the following new method:

$$\begin{aligned} y_n &= x_n - \frac{\phi(x_n)}{\phi'(x_n)} \\ z_n &= x_n - \frac{\phi(x_n)\big(\phi(y_n) - \phi(x_n)\big)}{\phi'(x_n)\big(2\phi(y_n) - \phi(x_n)\big)} \\ x_{n+1} &= z_n - \left(\frac{\phi(z_n)}{\phi'(z_n)} + \left(\frac{\phi(z_n)\big(\phi(z_n) - \phi(x_n)\big)}{\phi(z_n)\big(2\phi(z_n) - \phi(x_n)\big)}\right)^2 \right. \\ & \quad \left. \times\frac{\phi''(z_n)}{2\phi'(z_n)}\right). \end{aligned} \tag{15}$$

The method given in (15) requires the evaluation of second order derivative. Higher order derivative evaluation make the method more complicated, in order to reduce this difficulties, we have approximated the second-order derivative, using a combination of the known data steps.
Here, we have considered the function $Q(t) = a+b(t-z_n)+c(t-z_n)^2+d(t-z_n)^3$ where a, b, c, and d are unknown to be found. Using the following conditions of interpolation [14]

$$\phi(x_n) = Q(x_n),\ \phi(z_n) = Q(z_n),\ \phi'(x_n) = \phi'(x_n),\\ \phi'(z_n) = Q'(z_n),\ \phi''(z_n) = Q''(z_n).$$

Thus, we have a system of linear equation. The following approximation is obtained after solving the system of equation [14]:

$$\phi''(z_n) = \frac{2}{x_n - z_n}\left(3\frac{\phi(x_n)-\phi(z_n)}{x_n - z_n} - 2\phi'(z_n) - \phi'(x_n)\right). \tag{16}$$

Substituting the value of $\phi''(z)$ from equation (16) in equation (15), we obtain a new iterative technique as follows:

$$\begin{aligned} y_n &= x_n - \frac{\phi(x_n)}{\phi'(x_n)} \\ z_n &= x_n - \frac{\phi(x_n)\big(\phi(y_n)-\phi(x_n)\big)}{\phi'(x_n)\big(2\phi(y_n)-\phi(x_n)\big)} \\ x_{n+1} &= z_n - \left(\frac{\phi(z_n)}{\phi'(z_n)} + \left(\frac{\phi(z_n)\big(\phi(z_n)-\phi(x_n)\big)}{\phi(z_n)\big(2\phi(z_n)-\phi(x_n)\big)}\right)^2 \right. \\ &\quad \times \frac{1}{2\phi'(z_n)}\frac{2}{(x_n - z_n)}\left(3\frac{\phi(x_n)-\phi(z_n)}{x_n - z_n}\right. \\ &\quad \left.\left. -2\phi'(z_n) - \phi'(x_n)\right)\right). \end{aligned} \tag{17}$$

The order of convergence of the preceding method is analyzed in the following Theorem 1.

Theorem 1: Let $\phi(x)$ be a real-valued and sufficiently differentiable function in an open interval $\mathbb{D} \subset \mathbb{R}$. If $\phi(x)$ has a simple root $\alpha \in \mathbb{D}$ and x_0 is sufficiently close to α, then the method given in equation (17) has tenth order of convergence.

Proof: Let $e_n = x_n - \alpha$ be the n^{th} iteration error. Expanding $\phi(x_n)$ and $\phi'(x_n)$ using Taylor expansion about α, we get

$$\phi(x_n) = \phi'(\alpha)\{e_n + \sum_{j=2}^{12} D_j e_n^j + O[e_n^{13}]\}, \tag{18}$$

where $D_j = \phi^{(j)}(\alpha)/\phi'(\alpha)$, for $j = 2, 3, 4, ...$
And

$$\phi'(x_n) = \phi'(\alpha)\{1 + e_{j=2}^{12} jD_j e_n^{j-1} + O[e_n^{13}]\}. \tag{19}$$

From equations (18) and (19), we can write

$$\frac{\phi(x_n)}{\phi'(x_n)} = e_n - D_2e_n^2 + \sum_{i=3}^{12} A_i e_n^i + O[e_n^{13}], \tag{20}$$

where $A_i(D_2, D_3, D_4...)$ is the constant containing $D_j^{;s}$ i.e $A_3 = 2(-D_2^2 + D_3)$, $A_4 = (4D_3^3 - 7D_2D_3 + 3D_4)$,

$$A_6 = 16D_2^5 - 5D_2^3D_3 + 28D_2^2D_4 - 17D_3D_4D_2(3D_3^2 \\ - 13D_3) + 5D_6,$$

etc. Then using equation (20), we get

$$y_n - \alpha = D_2e_n^2 + \sum_{i=3}^{12} A_i e_n^i + O[e_n^{13}]. \tag{21}$$

Again, expanding $\phi(y_n)$ about α by using Taylor expansion, we get

$$\phi(y_n) = \phi'(\alpha)\big(D_2e_n^2 + \sum_{i=3}^{12} A_i e_n^i + O[e_n^{13}]\big). \tag{22}$$

Using equations (18), (19), (20) and (22) in the second step of (17), we get

$$\begin{aligned} z_n - \alpha &= (D_2^3 - D_2D_3)e_n^4 - 2(2D_2^4 - 4D_2^2D_3 + D_3^2 \\ &\quad + D_2D_4)e_n^5 + \sum_{k=6}^{12} B_k e_n^k + O[e_n^{13}], \end{aligned} \tag{23}$$

where $B_k(D_2, D_3, D_4, ...)$ is the of constant $D_j^{;s}$

$$i,e \quad B_6 = 10D2^5 - 30D_2^3D_3 + 12D_2^2D_4 - 7D_3D_4 \\ + 3D_2(6D_3^2 - D_5),$$

,

$$B_7 = 2(10D_2^6 - 40D_2^4D_3 - D_3^3 + 20D_2^3D_4 \\ + 3D_4^2 + 8D_2^2(5D_3^2 - D_5)5D_3D_5 \\ + 2D_2(D_6 - 13D_3D_4)),$$

etc. Expanding $\phi(z_n)$ and $\phi'(z_n)$ using Taylor expansion about α, we get

$$\begin{aligned} \phi(z_n) &= \phi'(\alpha)\big((D_2^3 - D_2D_3)e_n^4 - 2(2D_2^4 - 4D2^2D_3 \\ &\quad + D_3^2 + D_2D4)e_n^5 + \sum_{k=6}^{12} B_k e_n^k + O[e_n^{13}]\big) \end{aligned} \tag{24}$$

and

$$\begin{aligned} \phi'(z_n) &= \phi'(\alpha)\big(1 + 2D_2^2(D_2^2 - D_3)e_n^4 - 4D_2(2D_2^4 \\ &\quad - 4D_2^2D_3 + D_3^2 \\ &\quad + D_2D_4)e_n^5 + \sum_{l=6}^{12} C_l e_n^l + O[e_n^{13}]\big), \end{aligned} \tag{25}$$

where $C_l(D_2, D_3, D_4, ...)$ are constant.
Using equation (24) and equation (25) in the third step of (17), i.e x_{n+1}, we get

$$x_{n+1} - \alpha = (D_2^3 - D_2D_3)^2 D_4 e_n^{10} + O[e_n^{11}], \tag{26}$$

which is emplied as

$$e_{n+1} = (D_2^3 - D_2D_3)^2 D_4 e_n^{10} + O[e_n^{11}]. \tag{27}$$

Hence, the method given by equation (17) has tenth order of convergence and $(D_2^3 - D_2D_3)^2D_4$ is asymptotic error constant. ■

III. Numerical Applications of The Proposed Method

In this section, we illustrate the efficiency of newly proposed iterative method by applying the method to various nonlinear equations. Some examples of nonlinear functions with their initial guesses and the roots of the corresponding functions are given below:

Example 1: $\phi_1(x) = xe^{x^2} - cos(-x)$, $x_0 = 0.5$, $\alpha = 0.58840177650099628$.
Example 2: $\phi_2(x) = e^{(-x^2+x+3)} - cos(x-1) + x$, $x_0 = -1$, $\alpha = -1.1594672726420347$.
Example 3: $\phi_3(x) = x^{\frac{3}{2}} - 3x + 2$, $x_0 = 5$, $\alpha = 7.4777060274997321$.
Example 4: $\phi_4(x) = sin^{-1}(x^2-1) + \frac{x^2}{2} - 1$, $x_0 = 1$, $\alpha = 1.1528937224350386$.
Example 5: $\phi_5(x) = sin(2cosx) - 1 - x^2 + e^{sin(x^3)}$, $x_0 = -0.6$, $\alpha = -0.78489598766121254$.
Example 6: $\phi_6(x) = xe^{x^2} - sin^2x + 3cosx + 5$, $x_0 = -2$, $\alpha = -1.207647827130919$.

We test the newly proposed method on the following non-smooth function found in [15]–[17].
Example 7:

$$\phi_7(x) = \begin{cases} x(x-1) & if\ x < 0 \\ -2x(x+1) & if\ x \geq 0 \end{cases}, \quad x_0 = 1, \quad \alpha = 0.$$

Example 8:

$$\phi_8(x) = \begin{cases} 3(x^2-x) & if\ x < 0 \\ -6(x^3+x) & if\ x \geq 0 \end{cases}, \quad x_0 = 0.5, \quad \alpha = 0.$$

Aso, we test the newly proposed method on the following real life problems found in [18], [19].

Example 9: "Let's study the Planck's radiation law problem which used to calculate the energy density within an isothermal blackbody that is given as follows [18]:

$$\psi(\lambda) = \frac{8\pi ch\lambda^{-5}}{e^{\frac{ch}{\lambda kT}} - 1}, \tag{28}$$

where

λ = is the wavelength of the radiation,

T= absolute temperature of the blackbody,

k= Boltzmann's constant,

h= Planck's constant, and

c= speed of light,

We have to determine wavelength λ which corresponds to maximum energy density $\psi(\lambda)$. Differentiating equation (28) w.r.t λ, we get

$$\psi'(\lambda) = \left(\frac{8\pi ch\lambda^{-6}}{e^{\frac{ch}{\lambda kT}} - 1}\right)\left(\frac{(ch/\lambda kT)e^{\frac{ch}{\lambda kT}}}{e^{\frac{ch}{\lambda kT}} - 1} - 5\right). \tag{29}$$

The maximum of $\psi(\lambda)$ is obtained when

$$\left(\frac{(ch/\lambda kT)e^{\frac{ch}{\lambda kT}}}{e^{\frac{ch}{\lambda kT}} - 1} - 5\right) = 0$$

i.e

$$\left(\frac{(ch/\lambda kT)e^{\frac{ch}{\lambda kT}}}{e^{\frac{ch}{\lambda kT}} - 1}\right) = 5.$$

Substituting $x = ch/\lambda kT$, the above equation reduces to

$$1 - e^{-x} = \frac{x}{5}. \tag{30}$$

Then, we defined a function

$$\phi_9(x) = 1 - e^{-x} - \frac{x}{5} = 0. \tag{31}$$

Thus, the solutions of $\phi_9(x) = 0$ produce the maximum wavelength of radiatin λ by means of the following formula

$$\lambda \approx \frac{ch}{\alpha kT},$$

where α is the solution of $\phi_9(x) = 0$. Since $\phi_9(1) = -0.432121$ and $\phi_9(5) = 0.00673795$, clearly we see that a zero of $\phi_9(x)$ appeares in the intervel [1, 5]. The approximate root of the equation (31) is given by $x \approx 4.9651142317442763$. using $x_0 = 3$" [18].

Example 10: "In the study of multi-factor effect the trajectory of an electron in the air gap between two parallel plates is given by [19]

$$\begin{aligned} x(t) = x_0 + \big(v_0 + e\frac{E_0}{mw}sin(wt_0+\theta)(t-t_0) \\ + e\frac{E_0}{mw}(cos(wt_0-\theta) + sin(wt_0+\theta))\big), \end{aligned} \tag{32}$$

where e and m are the charge and the mass of the electron at rest, x_0 and v_0 are the position and velocity of the electron at time t_0 and $E_0 sin(wt_0 - \theta)$ is the RF (radio fequency) electric field between the plates.
Choosing the particulars parameters in the expression in order to deal with a simpler expression, we get the following nonlinear equation [19]

$$\begin{aligned} &\phi_{10}(x) = x - \frac{1}{2}cosx + \frac{22}{28}, \\ &x_0 = 0.2, \quad \alpha \approx 0.30946613920821465141." \end{aligned}$$

We have compared the new proposed method with the methods given in (5), (6), (7), (8) and (9) denoted by MAM, MHM, YAH, DSM, and KHM respectively. We denotes the newly proposed method as NPM which defined in equation (17). All the comparied results are given in Table I to Table X. In the respective tables, we have presented the absolute residual error of the corresponding functions (i.e $|\phi(x_n)|$), error in the consecutive iterations $|x_n - x_{n-1}|$ and the approximate roots $|x_n|$, the approximated computational order of convergence (ACOC) after completion of four full iterations. The ACOC is calculated by the following formula [4]:

$$ACOC = \frac{log \mid \frac{(x_{n+1}-x_n)}{(x_n-x_{n-1})} \mid}{log \mid \frac{(x_n-x_{n-1})}{(x_{n-1}-x_{n-2})} \mid}.$$

We have also included the CPU running time in seconds for each method in the Table I to Table X. The elapsed CPU-time is computed by selecting $\mid f(x_n) \mid \leq 10^{-1000}$ as the stopping criterion. Note that CPU running time is not unique and depends entirely on the computer's specification, but here we present an average of three performances to ensure the robustness of the methods. In the last two columns of Table I to Table X, we have also given the number of iteration (IT) and total number of functions evaluation (TNFE) required to satisfy the stopping criterion. The results have been carried out with Mathematica 12.2 software on a CPU 2.30 GHz with 4GB of RAM running on the windows 10 on Intel(R) Core(TM) i3-8145U. Form the results available in the respective tables, we have found that the newly proposed iterative method give better estimate of simple roots in less CPU-time as compared to other existing methods.

Table I
CONVERGENCE BEHAVIOUR FOR ϕ_1

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\|\phi(x_n)\|$	ACOC	CPU Time	IT	TNFE
ϕ_1	MAM	0.58840177650099628	5.8793×10^{-11}	4.1133×10^{-21}	2.0000	24.022300	8	32
	MHM	0.58840177650099628	4.5008×10^{-189}	2.0218×10^{-1129}	6.0000	0.5496900	5	30
	YAM	0.58840177650099628	8.8166×10^{-1035}	$5.7863 \times 10^{-10340}$	10.0000	17.496000	8	40
	DSM	0.58840177650099628	2.2769×10^{-929}	4.2825×10^{-9284}	10.0000	19.135100	5	30
	KHM	0.58840177650099628	2.4090×10^{-1025}	$4.2825 \times 10^{-10245}$	10.0000	20.413600	5	30
	NPM	0.58840177650099628	8.5212×10^{-1196}	$3.8516 \times 10^{-11952}$	10.0000	0.3525300	4	20

Table II
CONVERGENCE BEHAVIOUR ON ϕ_2

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\|\phi(x_n)\|$	ACOC	CPU Time	IT	TNFE
ϕ_2	MAM	1.1594672726420347	7.6931×10^{-10}	4.9932×10^{-19}	2.0000	26.403000	8	32
	MHM	1.1594672726420347	1.0710×10^{-148}	1.6068×10^{-887}	6.0000	0.4508300	6	36
	YAM	1.1594672726420347	3.9638×10^{-670}	1.4507×10^{-6694}	10.0000	17.320700	6	30
	DSM	1.1594672726420347	1.9006×10^{-653}	2.2131×10^{-6525}	10.0000	22.719500	5	30
	KHM	1.1594672726420347	1.9006×10^{-653}	2.2131×10^{-6525}	10.0000	16.819600	5	30
	NPM	1.1594672726420347	1.4594×10^{-944}	5.3853×10^{-9441}	10.0000	0.3446470	5	25

Table III
CONVERGENCE BEHAVIOUR ON ϕ_3

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\|\phi(x_n)\|$	ACOC	CPU Time	IT	TNFE
ϕ_3	MAM	1.0000002500096525	6.7512×10^{-4}	3.7501×10^{-7}	1.9791	1.4288300	8	32
	MHM	7.4641016151377546	1.7149×10^{-50}	3.8357×10^{-303}	6.0000	0.0291786	6	36
	YAM	0.91253413652291706	1.2765×10^{-0}	2.1554×10^{-3}	4.6020	3.4547600	8	40
	DSM	1.0000000000000003	8.6040×10^{-2}	5.4338×10^{-16}	7.8577	3.6224000	7	42
	KHM	0.39740380521880609	2.0574×10^{-245}	8.7358×10^{-2455}	3.4500	0.4676800	5	30
	NPM	7.4641016151377546	4.2081×10^{-288}	1.4208×10^{-2883}	10.0000	0.0246871	5	25

Table IV
CONVERGENCE BEHAVIOUR ON ϕ_4

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\|\phi(x_n)\|$	ACOC	CPU Time	IT	TNFE
ϕ_4	MAM	1.1528937224350386	4.4047×10^{-19}	2.7341×10^{-38}	2.0000	11.872000	8	32
	MHM	1.1528937224350386	3.7752×10^{-172}	2.9533×10^{-1028}	6.0000	0.1390940	7	42
	YAM	1.1528937224350386	1.4625×10^{-1014}	$1.2458 \times 10^{-10141}$	10.0000	75.914400	5	30
	DSM	1.1528937224350386	4.9187×10^{-451}	1.4728×10^{-4500}	10.0000	13.135100	6	36
	KHM	1.1528937224350386	1.3916×10^{-872}	6.7979×10^{-8719}	10.0000	5.4488000	5	30
	NPM	1.1528937224350386	1.2422×10^{-936}	1.9453×10^{-9359}	10.0000	0.1015200	5	25

Table V
CONVERGENCE BEHAVIOUR ON ϕ_5

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\|\phi(x_n)\|$	ACOC	CPU Time	IT	TNFE
ϕ_5	MAM	0.78489598766119938	2.0469×10^{-7}	3.7061×10^{-14}	2.0000	124.06800	8	32
	MHM	0.78489598766121254	2.2623×10^{-149}	2.6876×10^{-892}	6.0000	1.2173300	7	42
	YAM	0.78489598766121254	9.3274×10^{-221}	2.0009×10^{-2200}	10.0000	75.914400	5	25
	DSM	0.78489598766121254	0	2.2562×10^{-3944}	10.0000	19.135100	5	30
	KHM	0.78489598766121254	8.0375×10^{-457}	1.5699×10^{-4561}	10.0000	66.748800	5	30
	NPM	0.78489598766121254	7.3026×10^{-811}	3.0094×10^{-8102}	10.0000	0.8331250	5	25

Table VI
CONVERGENCE BEHAVIOUR ON ϕ_6

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\|\phi(x_n)\|$	ACOC	CPU Time	IT	TNFE
ϕ_6	MAM	*div.*	*div.*	*div.*	*div.*	–	–	–
	MHM	1.2076478271309189	1.0906×10^{-6}	9.2903×10^{-34}	5.9503	1.6348200	7	42
	YAM	*div.*	*div.*	*div.*	*div.*	–	–	–
	DSM	*div.*	*div.*	*div.*	*div.*	–	–	–
	KHM	1.2076478271309189	5.8962×10^{-59}	1.9197×10^{-579}	10.0000	61.134800	6	36
	NPM	1.2076478271309189	5.3981×10^{-87}	1.4018×10^{-862}	10.0000	1.0217170	6	30

Table VII
CONVERGENCE BEHAVIOUR ON ϕ_7

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\lvert\phi(x_n)\rvert$	ACOC	CPU Time	IT	TNFE
ϕ_7	MAM	$div.$	$div.$	$div.$	$div.$	–	–	–
	MHM	4.5860×10^{-252}	1.0230×10^{-42}	9.1720×10^{-252}	6.0000	0.0220036	6	36
	YAM	$div.$	$div.$	$div.$	$div.$	–	–	–
	DSM	1.2332×10^{-142}	3.0730×10^{-29}	1.2332×10^{-142}	5.0000	0.5761010	7	42
	KHM	$7.71637 \times 10^{-2427}$	2.0148×10^{-243}	1.5433×10^{-2426}	10.0000	0.2454000	5	30
	NPM	1.5752×10^{-4955}	3.4358×10^{-451}	1.5753×10^{-4955}	10.0000	0.0170060	5	25

Table VIII
CONVERGENCE BEHAVIOUR ON ϕ_8

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\lvert\phi(x_n)\rvert$	ACOC	CPU Time	IT	TNFE
ϕ_8	MAM	$div.$	$div.$	$div.$	$div.$	–	–	–
	MHM	2.4587×10^{-458}	4.2800×10^{-77}	1.4753×10^{-457}	6.0000	0.0198278	6	36
	YAM	2.4246×10^{-268}	1.4066×10^{-27}	7.2740×10^{-268}	9.9997	0.3587970	6	36
	DSM	$div.$	$div.$	$div.$	$div.$	–	–	–
	KHM	1.5928×10^{-4014}	3.4333×10^{-402}	9.5569×10^{-4014}	10.0000	0.2429390	5	30
	NPM	4.6912×10^{-7695}	3.0775×10^{-700}	2.8147×10^{-7694}	10.9940	0.0128573	5	25

Table IX
CONVERGENCE BEHAVIOUR ON ϕ_9

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\lvert\phi(x_n)\rvert$	ACOC	CPU Time	IT	TNFE
ϕ_9	MAM	4.9651142316608617	0.000044851	1.6101×10^{-11}	2.0035	9.7335700	8	32
	MHM	4.9651142317442763	1.2384×10^{-128}	7.3997×10^{-775}	6.0000	0.2084970	5	30
	YAM	$div.$	$div.$	$div.$	$div.$	–	–	–
	DSM	4.9651142317442763	3.0386×10^{-526}	3.6566×10^{-5268}	10.0000	7.9587200	5	30
	KHM	4.9651142317442763	2.7087×10^{-582}	4.9915×10^{-5830}	10.0000	0.6082100	5	30
	NPM	4.9651142317442763	1.1545×10^{-608}	1.6115×10^{-6091}	10.0000	0.1567100	5	25

Table X
CONVERGENCE BEHAVIOUR ON ϕ_{10}

$\phi(x)$	Methods	$\mid x_n \mid$	$\mid x_n - x_{n-1} \mid$	$\lvert\phi(x_n)\rvert$	ACOC	CPU Time	IT	TNFE
ϕ_{10}	MAM	0.30946613912659715	0.000013547	6.9189×10^{-11}	2.0017	15.998100	8	32
	MHM	0.30946613920821465	4.9357×10^{-188}	2.4196×10^{-1127}	6.0000	0.3040520	5	30
	YAM	0.30946613920821465	4.6579×10^{-867}	1.8885×10^{-8668}	10.0000	9.8006900	5	30
	DSM	0.30946613920821465	4.5010×10^{-628}	5.7566×10^{-6278}	10.0000	18.186100	5	25
	KHM	0.30946613920821465	9.0501×10^{-667}	1.1465×10^{-6665}	10.0000	10.458600	5	30
	NPM	0.30946613920821465	2.2536×10^{-882}	1.2685×10^{-8822}	10.0000	0.2386380	5	25

IV. CONCLUSIONS

A new iterative scheme has been developed using Ostrowski's methods and Taylor series expansion. The efficiency index for the new iterative scheme is 1.5848, and it involves four evaluations function and one evaluation of its first order derivative. We have also proved that the new method preserves the tenth order of convergence with the help of convergence analysis. The primary goal of developing this method is to provide a higher order convergence technique that gives better results than other existing well-known methods. Extensive experimentation has shown that the absolute residual error of the proposed method is highly efficient and competitive as compared to other existing tenth-order methods. Also, from the last two columns of the tables, we obsrved show that the newly proposed method reached the stopping criterion in fewer or similar number of iterations and the number of functions evaluations required is lesser than the other existing well known methods. The elapsed low CPU-time also confirms the highly efficient nature of the proposed method as compared to the existing methods of the same nature.

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