$\mathcal{D}-$ Complement and $\mathcal{D}(i)-$ Complement of a Graph

Sabitha D'Souza, Shankar Upadhyay, Swati Nayak* and Pradeep G. Bhat

***Abstract*—A dominating set for a graph $\mathcal{G} = (V, E)$ is a subset $\mathcal{D}$ of V such that every point not in $\mathcal{D}$ is adjacent to at least one member of $\mathcal{D}$. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of point set $V(G)$. For all V_i and V_j in $\mathcal{P}$ of order $k \geq 2$, $i \neq j$, delete the lines between V_i and V_j in G and include the lines between V_i and V_j which are not in $\mathcal{G}$. The resultant graph thus obtained is $k-$*complement* of $\mathcal{G}$ with respect to the partition $\mathcal{P}$ and is denoted by $\mathcal{G}_k^{\mathcal{P}}$. For each set V_r in $\mathcal{P}$ of order $k \geq 1$, delete the lines of $\mathcal{G}$ inside V_r and insert the lines of $\overline{\mathcal{G}}$ joining the points of V_r. The graph $\mathcal{G}_{k(i)}^{\mathcal{P}}$ thus obtained is called the $k(i)-$*complement of* $\mathcal{G}$ with respect to the partition $\mathcal{P}$. In this paper, we define $\mathcal{D}-$complement and $\mathcal{D}(i)-$complement of a graph $\mathcal{G}$. Further we study various properties of $\mathcal{D}$ and $\mathcal{D}(i)$ complements of a given graph.**

***Index Terms*—$\mathcal{D}-$complement, $\mathcal{D}(i)-$complement, $\gamma-$complement and $\gamma(i)-$complement.**

I. Introduction

The graphs considered in this paper are finite, undirected and simple graphs. Any term or concept or symbol not defined here, follows from [2]. Let $V(\mathcal{G}), E(\mathcal{G}), n, m$ and $\overline{\mathcal{G}}$ denote point set, line set, order, size and complement of $\mathcal{G}$ respectively. We begin with a brief survey of some related concepts. The problems in graph partition arise in popular areas like computer science, engineering and related fields. The importance of graph partition has increased in recent years in the field of route planning, cluster, detection of cliques in social, pathological and biological networks and high performance computing. The graph partition problems are defined on the data which can be represented in the form of a graph $\mathcal{G} = (V, E)$. Let $V_1, V_2, \ldots, V_k$ be non-empty disjoint subsets of V such that their union equal to V. Then $\{V_1, V_2, \ldots, V_k\}$ is called partition of point set V. There are many ways of partitioning a given graph. One can partition $\mathcal{G}$ into smaller components arbitrarily or with respect to some specific properties. For example, Uniform graph partition is a type of graph partitioning problem that consists of dividing a graph into components such that the components are of same size and there exist few connections between the components. Equal degree partition of a graph is a partition of the point set of $\mathcal{G}$ such that every point of equal degree belongs to same set. For more on complements of graphs we refer [1] and [4].

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E. Sampathkumar and Pushpalatha [5] observed that the complement of a graph $\mathcal{G} = (V, E)$ of order n can also be defined in the following way.

Definition 1. *Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of the point set $V(\mathcal{G})$ into sets of order 1. For all pairs V_i and V_j in $\mathcal{P}, i \neq j$, if there is a line between V_i and V_j exclude it and include the line if it is not in $\mathcal{G}$. The graph thus obtained is indeed the complement of $\mathcal{G}$ and $\mathcal{G}$ is self-complementary if $\mathcal{G} \cong \overline{\mathcal{G}}$.*

The authors in [5] introduced the generalization of the complement of a graph by viewing definition 1 in broader sense.

Definition 2. *[5] Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 2$. For all V_i and V_j in $\mathcal{P}$, $i \neq j$, if there is a line between V_i and V_j exclude it and include the lines which are not in $\mathcal{G}$. The resultant graph thus obtained is denoted by $\mathcal{G}_k^{\mathcal{P}}$ is called $k-$complement of $\mathcal{G}$ with respect to $\mathcal{P}$ and $\mathcal{G}$ is k-self complementary (k-s.c) if $\mathcal{G}_k^{\mathcal{P}} \cong \mathcal{G}$ for some partition $\mathcal{P}$ of order k.*

The authors of [6] defined another generalization of the complement of a graph in slightly different way.

Definition 3. *[6] For each set V_r in the partition $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ of $V(\mathcal{G})$, delete the lines of $\mathcal{G}$ inside V_r, and insert the lines of $\overline{\mathcal{G}}$ joining the points of V_r. The graph $\mathcal{G}_{k(i)}^{\mathcal{P}}$ thus obtained is called the $k(i)-$complement of $\mathcal{G}$ with respect to the partition $\mathcal{P}$ of V. The graph $\mathcal{G}$ is $k(i)-$self complementary (k(i)-s.c) if $\mathcal{G}_{k(i)}^{\mathcal{P}} \cong \mathcal{G}$ for some partition $\mathcal{P}$ of order k.*

Theory of dominations in graph is one of the main emerging field of graph theory. Domination arises in facility location problem and has a variety of applications in linear algebra and optimization. It plays major role in online social networks, computer communication networks and wireless sensor networks,

Definition 4. *[2] For any graph $\mathcal{G}(V, E)$, a subset $\mathcal{D}$ of $V(G)$ is called dominating set if for every $w_i \in V - \mathcal{D}$, there exists $w_j \in D$ with $w_i \sim w_j$.*

Minimum cardinality of dominating set is the domination number $\gamma(\mathcal{G})$.

Definition 5. *[7] A dominating set $\mathcal{D}$ of $\mathcal{G}$ is a global dominating (g.d. set) set of $\mathcal{G}$ if $\mathcal{D}$ also dominates $\overline{\mathcal{G}}$. Minimum cardinality of g.d. set of $\mathcal{G}$ is called global dominating number and is denoted by γ_g.*

Definition 6. *[3] A total dominating set T of $\mathcal{G}$ is a dominating set such that the induced subgraph $\langle T\rangle$ has no isolates.*

Definition 7. *[3] A set T of $\mathcal{G}$ is total global dominating (t.g.d.) set if T is total dominating set of both $\mathcal{G}$ and $\overline{\mathcal{G}}$. Total global domination number $\gamma_{tg}(\mathcal{G})$ of $\mathcal{G}$ is the minimum cardinality of a $t.g.d.$ set.*

Motivated by definitions 2 and 3, we now define new complements called $\mathcal{D}-$complement and $\mathcal{D}(i)-$complement of $\mathcal{G}$. We also study different types of self complementation with respect to the $\mathcal{D}-$complement of a graph.

II. $\mathcal{D}-$COMPLEMENT OF A GRAPH

Definition 8. *Let $\mathcal{D}\subseteq V$ be a dominating set of a graph $\mathcal{G}(V,E)$ and $\mathcal{P}=\{V_1,V_2\}$ be a partition of V such that $V_1=\mathcal{D}$ and $V_2=V-\mathcal{D}$. Then $2-$complement of a graph $\mathcal{G}$ with respect to every set $\mathcal{D}\subseteq V$ is said to be $\mathcal{D}-$complement of $\mathcal{G}$ and is denoted by $\mathcal{G}_{\mathcal{D}}(V,E_{\mathcal{D}})$.*

A graph $\mathcal{G}$ is $\mathcal{D}-$self complementary($\mathcal{D}-s.c$) if $\mathcal{G}\cong\mathcal{G}_{\mathcal{D}}$.

Example 9. *The smallest $\mathcal{D}-s.c$ graph $\mathcal{G}$ is shown in Fig 1.*

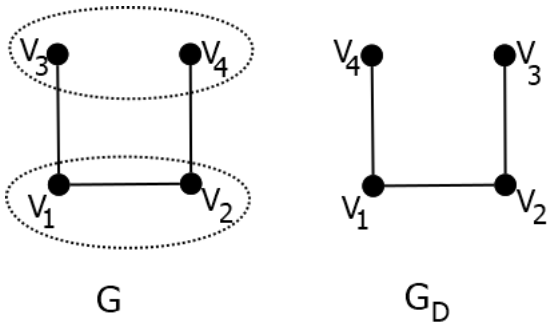

Fig. 1. $\mathcal{G}\cong\mathcal{G}_{\mathcal{D}}$ with respect to $\mathcal{D}=\{v_1,v_2\}$.

Remark 10. *For any graph $\mathcal{G}$ and its complement $\overline{\mathcal{G}}$, we note that $\mathcal{G}_{\mathcal{D}}\cong\overline{\mathcal{G}}$ if and only if $\mathcal{D}=V$.*

Proposition 11. *For any graph $\mathcal{G}(V,E),\overline{\mathcal{G}_{\mathcal{D}}}\cong(\overline{\mathcal{G}})_{\mathcal{D}}$.*

Proof: Let $\mathcal{D}\subseteq V$ be any dominating set of $\mathcal{G}(V,E)$.

Let u and v be any two points of $\mathcal{G}$. Then $u\sim v$ in $\overline{\mathcal{G}_{\mathcal{D}}}$ if and only if $u\nsim v$ in in $\mathcal{G}_{\mathcal{D}}$. Then the following three cases exist.

i. $u,v\in\mathcal{D}$ and are adjacent in $\mathcal{G}$
   $\leftrightarrow u\sim v$ in $\mathcal{G}_{\mathcal{D}}$
   $\leftrightarrow u\nsim v$ in $\overline{\mathcal{G}_{\mathcal{D}}}$ and $(\overline{\mathcal{G}})_{\mathcal{D}}$.
ii. $u,v\in V-\mathcal{D}$ and are adjacent in $\mathcal{G}$
   $\leftrightarrow u\sim v$ in $\mathcal{G}_{\mathcal{D}}$
   $\leftrightarrow u\nsim v$ in $\overline{\mathcal{G}_{\mathcal{D}}}$ and $(\overline{\mathcal{G}})_{\mathcal{D}}$.
iii. $u\in\mathcal{D},v\in V-\mathcal{D}$ and are adjacent in $\mathcal{G}$
   $\leftrightarrow u\nsim v$ in $\mathcal{G}_{\mathcal{D}}$
   $\leftrightarrow u\sim v$ in $\overline{\mathcal{G}_{\mathcal{D}}}$ and $(\overline{\mathcal{G}})_{\mathcal{D}}$.

Thus in either of three cases $\overline{\mathcal{G}_{\mathcal{D}}}\cong(\overline{\mathcal{G}})_{\mathcal{D}}$. ■

Definition 12. *[2] In any graph $G(V,E)$, a dominating set $\mathcal{D}\subseteq V$ is minimal dominating set if no proper subset $\mathcal{D}'\subset\mathcal{D}$ is dominating.*

Proposition 13. *The $\mathcal{D}-$complement of star graph $K_{1,n},n\geq 1$ is completely disconnected if $\mathcal{D}$ is minimal dominating.*

Proof: Consider a star graph $K_{1,n},n\geq 1$. Let $\mathcal{D}$ be minimal dominating set. Then $\mathcal{D}$ is either singleton or set of pendant points of $\mathcal{G}$, in either of the cases, by definition, $\mathcal{G}_{\mathcal{D}}$ is disconnected. ■

Proposition 14. *Let $\mathcal{G}(V,E)$ be a complete bipartite graph K_{r_1,r_2} with $1\leq r_1\leq r_2$ and $|E|=m$. Let $\mathcal{D}\subseteq V$ be minimal dominating set of $\mathcal{G}$. Then*

i. $|E_{\mathcal{D}}|=\phi$ if $\mathcal{D}$ is independent.
ii. $|E_{\mathcal{D}}|=m-r_1-r_2+2$ if $\mathcal{D}$ is not independent.

Proof: Let $\mathcal{G}(V,E)$ be a complete bipartite graph K_{r_1,r_2} with $1\leq r_1\leq r_2$ and $|E|=m$. Let $\mathcal{D}$ be a minimal dominating set. Then there are two cases.

i. $\mathcal{D}$ is independent.
   In this case, $\mathcal{D}$ is equal to either V_1 or V_2 of $\mathcal{P}$.
   Let $\mathcal{D}=V_1$. Then every point of $\mathcal{D}$ is adjacent to each point of $V-\mathcal{D}=V_2$.
   Hence by definition of $\mathcal{G}_{\mathcal{D}},|E_{\mathcal{D}}|=\phi$.
   If $\mathcal{D}=V_2$, then also the result follows.
ii. $\mathcal{D}$ is not independent.
   In this case, $\mathcal{D}$ contains exactly two points v_1 and v_2 such that $v_1\in V_1$ and $v_2\in V_2$. Then v_1 is adjacent to r_2 points of V_2 and v_2 is adjacent to r_1 points of V_1. Thus by definition of $\mathcal{G}_{\mathcal{D}}$, $|E_{\mathcal{D}}|=m-r_1-r_2+2$.

■

Definition 15. *[2] The corona product of two graphs $\mathcal{G}_1,\mathcal{G}_2$, denoted by $\mathcal{G}_1\odot\mathcal{G}_2$ is obtained by taking a copy of $\mathcal{G}_1$ and $|V(\mathcal{G}_1)|$ copies of $\mathcal{G}_2$ and hence connecting the i^{th} point of $\mathcal{G}_1$ to every point in the i^{th} copy of $\mathcal{G}_2$ for each i.*

Proposition 16. *Let $\mathcal{D}$ be the minimal dominating set of corona $C_q\odot K_1$. Then $|E(C_q\odot K_1)_{\mathcal{D}}|=q^2$.*

Proof: Consider a corona graph $\mathcal{G}=C_q\odot K_1$. It has same order and size equal to $2q$. Let $\mathcal{D}\subseteq V$ be minimal dominating set of $\mathcal{G}$. Then $|\mathcal{D}|=q$.
We consider three exclusive possibilities:
Case (i). $\mathcal{D}$ contains all points of cycle C_q. Every point d of $\mathcal{D}$ is adjacent to exactly one point in $V-\mathcal{D}$ and non adjacent to $q-1$ points in $\mathcal{G}$. Thus in $\mathcal{G}_{\mathcal{D}}$, each of q points of $\mathcal{D}$ is adjacent to $q-1$ points of $V-\mathcal{D}$. Also $\langle\mathcal{D}\rangle$ has q lines.
Thus $|E(\mathcal{G}_{\mathcal{D}})|=|E(C_q\odot K_1)_{\mathcal{D}}|=2q-q+q(q-1)=q^2$.
Case (ii). $\mathcal{D}$ contains all the pendant points of $\mathcal{G}$.
In this case too, the result holds as in Case (i).
Case (iii). $\mathcal{D}$ is union of some points of C_q and pendant points. In this case also we can prove the result as in Case (i). Hence the Proposition. ■

Definition 17. *[2] A graph $\mathcal{G}(V,E)$ is Eulerian graph, if $\mathcal{G}$ contains a closed trail containing every line of the graph.*

If a graph $\mathcal{G}$ is Eulerian, then every point v in $\mathcal{G}$ is of even degree.

Proposition 18. *Let $\mathcal{G}$ be an Eulerian graph. Then $\mathcal{G}_{\mathcal{D}}$ is non Eulerian if there is at least one point in $V-\mathcal{D}$ which is adjacent to odd number of points in $\mathcal{D}$.*

Proof: Let $\mathcal{G}$ be any Eulerian graph. Let $\mathcal{D}$ be any dominating set of $\mathcal{G}$ with $|\mathcal{D}|$ odd. Let $v\in V-\mathcal{D}$ be the point adjacent to odd number of points of $\mathcal{D}$. Suppose $|\mathcal{D}|=d$, $deg_{\mathcal{G}}(v)=r$ and s is the number of points in $\mathcal{D}$ to which v is adjacent. Since, d and s are odd numbers and r is even, the degree of v in $\mathcal{G}_{\mathcal{D}}$ is $r-s+(d-s)=r+d-2s$ which is odd. Hence $\mathcal{G}_{\mathcal{D}}$ is non Eulerian. ∎

Definition 19. *Let $\mathcal{G}(V,E)$ be any graph and $\mathcal{D}$ be any $\gamma-$set of $\mathcal{G}$. Then $\mathcal{G}_{\mathcal{D}}$ is called $\gamma-$complement of $\mathcal{G}$ and is denoted by $\mathcal{G}_{\gamma}$.*

A graph $\mathcal{G}(V,E)$ is $\gamma-$self complementary ($\gamma-$s.c.) if $\mathcal{G}_{\gamma}\cong\mathcal{G}$.

Proposition 20. *If a graph $\mathcal{G}(V,E)$ with $|V|=n$ is $\gamma-$self complementary, then $\mathcal{G}$ must have a point of degree at least $\frac{1}{2}(n-\gamma)$.*

Proof: Let $\mathcal{G}(V,E)$ be any graph with domination number γ and $|V|=n$. Let $\mathcal{D}$ be the γ set. Now for a point $v\in\mathcal{D}$, $deg_{\mathcal{G}}(v)+deg_{\mathcal{G}_{\gamma}}(v)\geq n-\gamma$. This implies that the degree of v is at least $\frac{1}{2}(n-\gamma)$ in $\mathcal{G}$ or $\mathcal{G}_{\gamma}$. ∎

Corollary 21. *If a graph $\mathcal{G}$ is $\gamma-$s.c., then $\mathcal{G}$ has a point of degree at least $\frac{\Delta(\mathcal{G})}{2}$, where $\Delta(\mathcal{G})$ is maximum degree of $\mathcal{G}$.*

Proof: For any graph $\mathcal{G}$, $\gamma(\mathcal{G})\leq n-\Delta(\mathcal{G})$ [2]. Substituting this in Proposition 20, the result follows. ∎

Corollary 22. *If a graph $\mathcal{G}$ with no isolated points is $\gamma-$s.c., then $\mathcal{G}$ has a point of degree at least $\frac{n+\delta(\mathcal{G})-2}{4}$, where δ is minimum degree of $\mathcal{G}$.*

Proof: For any graph $\mathcal{G}$ with no isolated points, $\gamma(\mathcal{G})\leq\frac{n+2-\delta(\mathcal{G})}{2}$. Substituting this in Proposition 20, the result follows. ∎

III. $\mathcal{D}(i)-$COMPLEMENT OF A GRAPH

In this section, we introduce another generalization of 2− complement, which we call, the $\mathcal{D}(i)-$complement of a graph.

Definition 23. *Let $\mathcal{D}\subseteq V$ be any dominating set of a graph $\mathcal{G}(V,E)$. Then $\mathcal{D}(i)-$complement of $\mathcal{G}$ denoted by $\mathcal{G}_{\mathcal{D}(i)}$ is the graph $(V,E_{\mathcal{D}(i)})$ obtained by deleting the lines of $\langle\mathcal{D}\rangle$ in $\mathcal{G}$ and including the lines which are not in $\langle\mathcal{D}\rangle$ of $\mathcal{G}$.*

A graph $\mathcal{G}$ is $\mathcal{D}(i)-$self complementary ($\mathcal{D}(i)-$ s.c) if $\mathcal{G}_{\mathcal{D}(i)}\cong\mathcal{G}$.

Example 24.

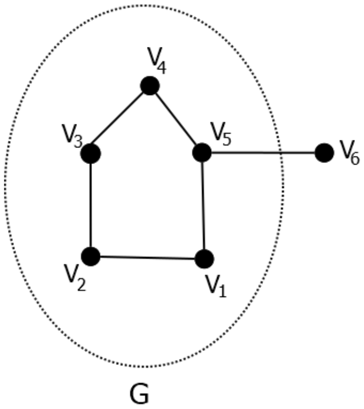

Fig. 2. $\mathcal{G}$ is $\mathcal{D}(i)-$self complementary for $\mathcal{D}=\{v_1,v_2,v_3,v_4,v_5\}$.

Remark 25. *For any graph $\mathcal{G}$ and its complement $\overline{\mathcal{G}}$, we note that $\mathcal{G}_{\mathcal{D}(i)}\cong\overline{\mathcal{G}}$ if $\mathcal{D}$ is singleton.*

Proposition 26. *For a graph $\mathcal{G}$ of size m, $\mathcal{G}_{\mathcal{D}(i)}$ has $m-2m_{\mathcal{D}}+\binom{d}{2}$ lines, where $d=|\mathcal{D}|$ and $m_{\mathcal{D}}$ is the number of lines in $\langle\mathcal{D}\rangle$.*

Proposition 27. *Let $\mathcal{D}$ be any dominating set of graph $\mathcal{G}(V,E)$. Then $\overline{\mathcal{G}_{\mathcal{D}(i)}}\cong(\overline{\mathcal{G}})_{\mathcal{D}(i)}$.*

Proof: Let $\mathcal{D}\subseteq V$ be any dominating set of $\mathcal{G}(V,E)$.

Let u and v be any two points of $\mathcal{G}$. Then $u\sim v$ in $\mathcal{G}_{\mathcal{D}(i)}$ if and only if $u\nsim v$ in $\overline{\mathcal{G}_{\mathcal{D}(i)}}$. Here we consider two exclusive cases.

i. $u,v\in\mathcal{D}$ and are adjacent in $\mathcal{G}$
   $\leftrightarrow u\nsim v$ in $\mathcal{G}_{\mathcal{D}(i)}$
   $\leftrightarrow u\sim v$ $\overline{\mathcal{G}_{\mathcal{D}(i)}}$ and $(\overline{\mathcal{G}})_{\mathcal{D}(i)}$.

ii. $u,v\in V-\mathcal{D}$ and are adjacent in $\mathcal{G}$
   $\leftrightarrow u\sim v$ in $\mathcal{G}_{\mathcal{D}(i)}$
   $\leftrightarrow u\nsim v$ in $\overline{\mathcal{G}_{\mathcal{D}(i)}}$ and $(\overline{\mathcal{G}})_{\mathcal{D}(i)}$.

Hence in either of the cases $\overline{\mathcal{G}_{\mathcal{D}(i)}}\cong(\overline{\mathcal{G}})_{\mathcal{D}(i)}$. ∎

Proposition 28. *Let $\mathcal{D}$ be the dominating set of graph $\mathcal{G}(V,E)$. Then $\mathcal{D}$ is also a dominating set of $\mathcal{G}_{\mathcal{D}(i)}$.*

Proof: Let $\mathcal{D}$ be any dominating set of graph $\mathcal{G}$. Then for every point v of $\mathcal{G}$, v is either in $\mathcal{D}$ or adjacent to a point of $\mathcal{D}$. Now by definition of $\mathcal{G}_{\mathcal{D}(i)}$, point v is either in $\mathcal{G}$ or adjacent to a point of $V-\mathcal{D}$. Hence $\mathcal{D}$ is also a dominating set of $\mathcal{G}_{\mathcal{D}(i)}$. ∎

Remark 29. *In a graph $\mathcal{G}(V,E)$, if $\mathcal{D}$ is any dominating set of $\mathcal{G}$, then $\mathcal{D}$ need not be a dominating set of $\mathcal{G}_{\mathcal{D}}$. Suppose v is any point in $V-\mathcal{D}$ such that v is adjacent to every point of $\mathcal{D}$, then v is not adjacent to any point of $\mathcal{D}$ in $\mathcal{G}_{\mathcal{D}}$. Therefore $\mathcal{D}$ is not a dominating set of $\mathcal{G}_{\mathcal{D}}$.*

Proposition 30. *Let $\mathcal{G}(V,E)$ be any disconnected graph. Then $\mathcal{G}_{\mathcal{D}(i)}$ is connected for every minimal dominating set $\mathcal{D}$.*

Proof: Let $\mathcal{G}(V,E)$ be any disconnected graph with components $\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_n$. Let $\mathcal{D}\subseteq V$ be any minimal dominating set of $\mathcal{G}$. Then $\mathcal{D}$ contains at least one point from each component of $\mathcal{G}$. Then by definition of $\mathcal{G}_{\mathcal{D}(i)}$, it is connected. ∎

Proposition 31. *Let $\mathcal{G}(V,E)$ be any Eulerian graph. Then $\mathcal{G}_{\mathcal{D}(i)}$ is non Eulerian for any dominating set $\mathcal{D}$ with $|\mathcal{D}|=2$.*

Proof: Let $\mathcal{G}(V,E)$ be any Eulerian graph. Then degree of every point of $\mathcal{G}$ is even. Let $\mathcal{D}$ be any dominating set of order 2. Let $u,v\in\mathcal{D}$. If u and v are adjacent in $\mathcal{G}$, then they will be non adjacent in $\mathcal{G}_{\mathcal{D}(i)}$ and vice versa. In any case degree of u and v will increase or decrease by 1 resulting into odd degree in $\mathcal{G}_{\mathcal{D}(i)}$. Hence $\mathcal{G}_{\mathcal{D}(i)}$ is non Eulerian. ∎

Definition 32. *Let $\mathcal{G}(V,E)$ be any graph and $\mathcal{D}$ be any $\gamma-$set of $\mathcal{G}$. Then $\mathcal{G}_{\mathcal{D}(i)}$ is called $\gamma(i)-$complement of $\mathcal{G}$ and is denoted by $\mathcal{G}_{\gamma(i)}$.*

A graph $\mathcal{G}(V,E)$ is $\gamma(i)-$self complementary ($\gamma(i)-$s.c.) if $\mathcal{G}_{\gamma(i)}\cong\mathcal{G}$.

Proposition 33. *Let $\mathcal{G}$ be any disconnected graph with domination number γ such that every component of $\mathcal{G}$ be Eulerian. Then $\mathcal{G}_{\gamma(i)}$ is Eulerian if and only if γ is odd.*

Proof: Let $\mathcal{G}$ be any disconnected graph with domination number γ such that every component of $\mathcal{G}$ be Eulerian. Let $\mathcal{D}$

be the $\gamma-$set. Then $|\mathcal{D}|=\gamma$. Since $\mathcal{D}$ is a $\gamma-$set, $\mathcal{D}$ contains at least one point from each component of $\mathcal{G}$. Therefore $\mathcal{G}_{\gamma(i)}$ is connected graph. Suppose $v \in \mathcal{D}$ such that $deg(v) = d$ in $\mathcal{G}$ and $deg(v) = d_i$ in $\langle\mathcal{D}\rangle$. Then degree of v in $\mathcal{G}_{\gamma(i)}$ is $(\gamma - 1 - d_i) + (d - d_i)$. Since every component of $\mathcal{G}$ is Eulerian, d is even. Suppose $\mathcal{G}_{\gamma(i)}$ is Eulerian and if possible, assume that γ is even. Then if d_i is even, $\gamma - 1 - d_i$ is odd and $d - d_i$ is even resulting into $(\gamma - 1 - d_i) + (d - d_i)$ odd. If d_i is odd, then $\gamma - 1 - d_i$ is even and $d - d_i$ is odd. Thus, we see that $(\gamma - 1 - d_i) + (d - d_i)$ is odd. In either of the cases degree of v in $\mathcal{G}_{\gamma(i)}$ is odd. It contradicts that $\mathcal{G}_{\gamma(i)}$ is Eulerian. Therefore γ must be odd.

Conversely, suppose γ is odd. Then for every point $v \in \mathcal{D}$ irrespective of d_i is even or odd, the degree of v in $\mathcal{G}_{\gamma(i)}$, $(\gamma - 1 - d_i) + (d - d_i)$ is even. Since every component of $\mathcal{G}$ is Eulerian, every point v of $V - \mathcal{D}$ is of even degree. Thus $\mathcal{G}_{\gamma(i)}$ is Eulerian. ∎

IV. Global and Total Domination in $k-$ complement of a graph.

We now introduce the concept of $k-$global and $k-$total global dominating set of a graph $\mathcal{G}$.

Definition 34. *A dominating set $\mathcal{D}$ of $\mathcal{G}$ is called $k-$global dominating set ($k - g.d.$ set) of $\mathcal{G}$ if $\mathcal{D}$ is also a dominating set of $\mathcal{G}_k^{\mathcal{P}}$. Minimum cardinality of $k-$global dominating set is called $k-$global domination number and is denoted by $\gamma_{kg} = \gamma_{kg}(\mathcal{G})$.*

Example 35.

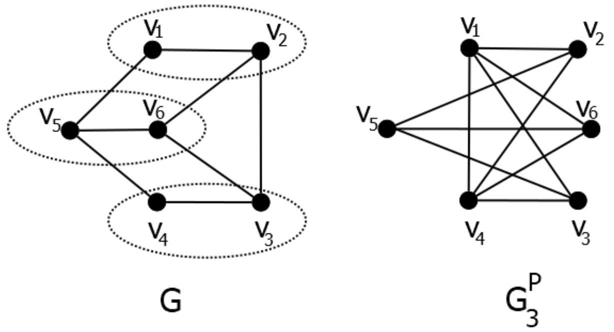

Fig. 3. $\mathcal{D} = \{v_5, v_6\}$ is a 3−global dominating set of graph $\mathcal{G}$.

Theorem 36. *Let $\mathcal{D}$ be a dominating set of a graph $\mathcal{G}$. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 2$ such that $V_i = \mathcal{D}$ for some $i = 1, 2, 3, \ldots, k$. Then $\mathcal{D}$ is $k - g.d.$ set if and only if for every $v \in V_j, i \neq j$, there is a point $u \in V_i$ not adjacent to v.*

Proof: Let $\mathcal{P}=\{V_1, V_2, \ldots, V_k\}$ be a partition of V of $\mathcal{G}$ of order $k \geq 2$ such that $V_i = \mathcal{D}$ for some $i = 1, 2, 3, \ldots, k$. Suppose for every $v \in V_j, i \neq j$, there is a point $u \in V_i$ not adjacent to v. Then by definition of $\mathcal{G}_k^{\mathcal{P}}$, $\mathcal{D}$ is also a dominating set of $\mathcal{G}_k^{\mathcal{P}}$. Thus $\mathcal{D}$ is a $k - g.d.$ set.

Conversely, suppose $\mathcal{D}$ is a $k - g.d.$ set. If a point $v \in V_j, i \neq j$ is adjacent to every point of V_i. Then v is non-adjacent to any of the points of $V_i = \mathcal{D}$ in $\mathcal{G}_k^{\mathcal{P}}$. Thus $\mathcal{D}$ is not a dominating set of $\mathcal{G}_k^{\mathcal{P}}$, which is a contradiction. Therefore for every $v \in V_j, i \neq j$, there exists a point $u \in V_i$ not adjacent to v.

∎

Proposition 37. *Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of complete graph K_n of order $k \geq 2$, such that V_i be singleton for every $i = 1, 2, 3, \ldots, k$. Then $\gamma_{kg}(K_n) = n$.*

Proof: Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of complete graph K_n of order $k \geq 2$, such that V_i be singleton for every $i = 1, 2, 3, \ldots, k$. Then $(K_n)_k^P \cong \overline{K_n}$. Let $\mathcal{D}$ be any minimum $k - g.d.$ set. Then $\mathcal{D}$ contains all the points of $\mathcal{G}$. Therefore $\gamma_{kg} = n$. ∎

Remark 38. *Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of $\overline{K_n}$, $k \geq 2$, such that V_i be singleton for every $i = 1, 2, \ldots, k$. Then $\gamma_{kg}(\overline{K_n}) = n$.*

Remark 39. *Suppose $\mathcal{P} = \{V_1, V_2\}$ is the partition of V of $K_{p,q} = \{V_1 \cup V_2, E\}$. Then $\gamma_{2g} = p + q = n$.*

Proposition 40. *Let $\mathcal{D}$ be a minimum dominating set of $\mathcal{G}$. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 2$ such that $V_i = \mathcal{D}$ for some $i = 1, 2, 3, \ldots, k$. If there exists a point $v \in V - V_i$ which is adjacent to every point of V_i, then $\gamma_{kg} \geq \gamma + 1$.*

Proof: Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of $\mathcal{G}$ of order $k \geq 2$ such that $V_i = \mathcal{D}$ for some $i = 1, 2, 3, \ldots, k$. Suppose there exists a point $v \in V - V_i$ which is adjacent to every point of V_i, then $\mathcal{D} \cup \{v\}$ is a $k - g.d.$ set. Hence $\gamma_{kg} \geq \gamma + 1$. ∎

Definition 41. *A total dominating set T of $\mathcal{G}$ is called $k-$total global dominating set ($k - t.g.d.$ set) if it is also a total dominating set of $\mathcal{G}_k^{\mathcal{P}}$. Minimum cardinality of $k.t.g.d.$ set is called $k-$total domination number of a graph $\mathcal{G}$ and is denoted by $\gamma_{ktg} = \gamma_{ktg}(\mathcal{G})$.*

Remark 42. *(i) $\gamma_{ktg}(\mathcal{G}) = \gamma_{ktg}(\mathcal{G}_k^{\mathcal{P}})$.*
(ii) γ_{ktg} exists only if $\delta(\mathcal{G}) \geq 1$ and $\delta(\mathcal{G}_k^{\mathcal{P}}) \geq 1$, where δ stands for minimum degree of a graph.

Proposition 43. *Let T be a total dominating set of $\mathcal{G}$. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 2$ such that $V_i = T$ for some $i = 1, 2, 3, \ldots, k$. Then T is $k - t.g.d.$ set if and only if for every $v \in V_j, i \neq j$, there exists a point $u \in V_i$ not adjacent to v.*

Proof: Let $\mathcal{P}=\{V_1, V_2, \ldots, V_k\}$ be a partition of V of $\mathcal{G}$ of order $k \geq 2$ such that $V_i = T$ for some $i = 1, 2, 3, \ldots, k$. Suppose for every $v \in V_j, i \neq j$, there is a point $u \in V_i$ not adjacent to v. Then by definition of $\mathcal{G}_k^{\mathcal{P}}$, T is also a total dominating set of $\mathcal{G}_k^{\mathcal{P}}$. Thus T is a $k - t.g.d.$ set.

Conversely, suppose T is a $k-t.g.d.$ set. If a point $v \in V_j$, $i \neq j$ is adjacent to every point of V_i, then in $\mathcal{G}_k^{\mathcal{P}}$, v is non adjacent to any of the points of $V_i = T$. Thus T is not a total dominating set of $\mathcal{G}_k^{\mathcal{P}}$, which is a contradiction. Therefore for every $v \in V_j, i \neq j$, there is a point $u \in V_i$ not adjacent to v.

∎

V. Global and Total Domination in $k(i)-$ complement of a graph.

Definition 44. *A dominating set $\mathcal{D}$ of $\mathcal{G}$ is $k(i)-$global dominating set ($k(i)-g.d.$ set) of $\mathcal{G}$, if $\mathcal{D}$ is also a dominating set of $\mathcal{G}_{k(i)}^{\mathcal{P}}$. Minimum cardinality of $k(i)-$global dominating set is called $k(i)-$global domination number and is denoted by $\gamma_{k(i)g} = \gamma_{k(i)g}(\mathcal{G})$.*

Example 45.

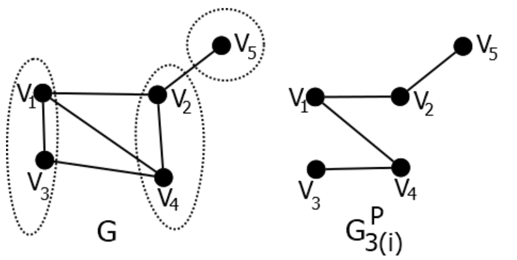

Fig. 4. A dominating set $\mathcal{D} = \{v_2, v_4\}$ of graph $\mathcal{G}$ is $3(i)-$global dominating set.

Remark 46. *Let $\mathcal{D}$ be a dominating set of a graph $\mathcal{G}$. Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of order $k \geq 2$. Then $\mathcal{D}$ is a $k(i) - g.d.$ set if $V_i = \mathcal{D}$ for some $i = 1, 2, 3, \ldots, k$.*

Proposition 47. *Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of complete graph K_n of order $k \geq 2$, such that V_i be singleton for every $i = 1, 2, 3, \ldots, k$. Then $\gamma_{k(i)g}(K_n) = 1$.*

Proof: Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of complete graph K_n of order $k \geq 2$, such that V_i be singleton for every $i = 1, 2, 3, \ldots, k$. Then $(K_n)^P_{k(i)} \cong K_n$. Let $\mathcal{D}$ be any minimum $k(i) - g.d.$ set. Then $|\mathcal{D}| = 1$. Therefore $\gamma_{k(i)g}(K_n) = 1$. ∎

Remark 48. *Let $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ be a partition of V of $\overline{K_n}$, $k \geq 2$, such that V_i be singleton for every $i = 1, 2, \ldots, k$. Then $\gamma_{k(i)g}(\overline{K_n}) = n$.*

Remark 49. *Suppose $\mathcal{P} = \{V_1, V_2\}$ is the partition of V of $K_{p,q} = \{V_1 \cup V_2, E\}$. Then $\gamma_{2(i)g} = 1$.*

Definition 50. *A total dominating set T of $\mathcal{G}$ is $k(i)-$total global dominating set ($k(i) - t.g.d.$ set), if it is also a total dominating set of $\mathcal{G}^{\mathcal{P}}_{k(i)}$. Minimum cardinality of $k(i) - t.g.d.$ set is called $k(i)-$total domination number of a graph $\mathcal{G}$ and is denoted by $\gamma_{k(i)tg} = \gamma_{k(i)tg}(\mathcal{G})$.*

Remark 51.
- $\gamma_{k(i)tg}(\mathcal{G}) = \gamma_{k(i)tg}(\mathcal{G}^{\mathcal{P}}_{k(i)})$.
- *$\gamma_{k(i)tg}$ exists only if $\delta(\mathcal{G}) \geq 1$ and $\delta(\mathcal{G}^{\mathcal{P}}_{k(i)}) \geq 1$, where δ stands for minimum degree of a point.*

VI. Conclusion

This paper presents new generalization of 2 complements called $\mathcal{D}-$complement and $\mathcal{D}(i)-$complement in graphs. It gives a new insight and analysis of various properties of this generalization. Also, we have defined and studied $k-$global and $k-$total global domination set for generalised complements of a graph.

References

[1] T. Gangopadhyay, Hebbare, S. P. Rao, "r-partite self-complementary graphs-diameters," *Discrete Math.*, vol. 32, no.3, pp. 245-255, 1980.
[2] F. Harary, *Graph Theory*, Addison Wesley, Academic Press, 1972.
[3] V. R. Kulli and B. Janakiram, "The total global domination number pf a graph," *Indian J. pure appl. Math.*, vol. 27, no. 6, pp. 537-542, 1996.
[4] A. Mishra and K. Patra, "Domination and Independence Parameters in the Total Graph of Zn with respect to Nil Ideal," *IAENG International Journal of Applied Mathematics*, vol. 50, no. 3, pp. 707-712, 2020.
[5] E. Sampathkumar and L. Pushpa Latha, "Complement of a graph: A generalization," *Graphs and Comb.*, vol. 14, no. 4, pp. 377-392, 1998.
[6] E. Sampathkumar, L. Pushpa Latha, C. V. Venkatachalam and P. G. Bhat, "Generalized complements of a graph," *Indian J. pure appl. Math.*, vol. 29, no. 6, pp. 625-639, 1998.
[7] E. Sampathkumar, "The global domination number of a graph," *Journal of Mathematical and Physical Sciences*, vol. 23, no. 5, pp. 377–385, 1989.

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Description of changes: Symbolic representation of partition is being changed to $\mathcal{P} = \{V_1, V_2, \ldots, V_k\}$ throughout the paper.

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