Determination of Pi Terms by the Method of Repeating Variables and Rayleigh Method: a Comparative Study

M. Z. I. Bangalee, Muntasir Shohrab

Abstract— **Dimensional Analysis is a simple and qualitative technique used to predict physical parameters. It is important to find out dimensionless terms from a set of variables that are used in an equation describing a natural phenomenon. Two methods are used to determine the dimensionless terms, also called the Pi terms. They are known as the method of repeating variables and the Rayleigh method (also known as the row reduced echelon form method). The method of repeating variables, which is commonly used to determine the Pi terms, uses the Buckingham Pi Theorem. The Rayleigh method was introduced by Lord Rayleigh to develop a theory of sound. The computational algorithms for both methods are developed and comparative analysis between the two methods is also discussed. The complexity analysis of both algorithms shows which algorithm performs faster for a particular number of variables and reference dimensions. These methods have also been tested on several problems. The analysis of these problems describes the stability of these algorithms along with their limitations. It is found that the method of repeating variables can lead to an error if the number of reference dimensions differs from the number of basic dimensions, unlike the Rayleigh method. However, the method of repeating variables can be faster than the Rayleigh method in some cases.**

Index Terms— **Dimensional analysis, Buckingham Pi theorem, the method of repeating variables, the Rayleigh method**

I. INTRODUCTION

MANY practical real-world engineering problems cannot be solved analytically as there are no equations developed for these problems to find their solutions. However, experimentally obtained data can be collected to understand the nature of these problems. The concept of dimensional analysis provides a step-by-step method to find the solutions to problems for which analytical solutions are infeasible. The main assumption of dimensional analysis is that any equation representing a universal law has to be independent of the unit system. This rule frequently makes it possible to develop the form of the equation. If a certain physical phenomenon is governed by an equation where all/some variables are dimensional, then the above phenomena can be represented as another equation where all the variables are dimensionless. The characteristics of the two equations can be obtained from experiments. The advantage of using dimensional analysis is that the number of dimensionless quantities is smaller than that of the actual equation [10].

The complexity of the model depends on the number of variables and material parameters. With the growing number of parameters, it is difficult to establish the dependencies among them. Dimensional analysis can help to select appropriate groups of parameters known as the π terms [14]. The dimensionless products are referred to as "pi terms". Edgar Buckingham used the symbol Π to represent a dimensionless product which is commonly used [2].

There are several methods for the determination of Pi terms. This paper only focuses on the method of repeating variables and the Rayleigh method to determine the Pi terms by Matlab or any other programming language. The main objective of this paper is to develop algorithms for these two methods. This paper also deals with the complexity of these algorithms. It discusses the limitations of using these algorithms, depending on the number of variables and reference dimensions for any given problem that is being solved. Subsequently, some problems are solved to show the applicability of the algorithms.

II. DIMENSIONAL ANALYSIS

Dimensional Analysis makes use of the principle that all terms of a physical equation must have the same dimension [1]. It is used to understand the behavior of a physical system without the need for complex mathematics. If a certain physical phenomenon is governed by the equation $f(x_1, x_2, ..., x_n) = 0$ where all/ some variables are dimensional, then the above phenomenon can be represented as $\psi(\Pi_1, \Pi_2, ..., \Pi_k) = 0$ where all/some variables are dimensional and $0 < k < n$ [8, 13]. The nature of the two functions is obtained from experiments. The total number of Pi terms is known in advance by the Buckingham Pi theorem which states that,

"If an equation involving n variables is dimensionally homogeneous, it can be reduced to a relationship among $n-r$ independent dimensionless products, where r is the minimum number of reference dimensions required to describe the variables." [5, 7, 9, 12, 16, 17, 18, 21]

Manuscript received October 1, 2020; revised October 25, 2021.

This project is funded by the University Grant Commission (UGC) and Bose Centre for Advanced Study and Research in Natural Sciences, University of Dhaka, Bangladesh.

M.Z.I Bangalee is an associate professor at the University of Dhaka in the department of Applied Mathematics, Bangladesh. (E-mail: zavid@du.ac.bd).

Muntasir Shohrab is a master's student at the University of Dhaka in the department of Applied Mathematics, Bangladesh. (E-mail: muntasir.tiash@gmail.com).

If an equation truly expresses a proper relationship between variables in a physical process, it will be dimensionally homogeneous. That is, each of its additive terms will have the same dimensions [4, 6].

A. The method of repeating variables

The method of repeating variables is a powerful tool to find the dimensionless Pi terms. If a problem contains n number of variables whose reference dimension can be represented by m number of variables, then a table is created such that,

TABLE I
DIMENSIONS OF THE VARIABLES

Variables \ Dimensions	M	L	…	T
v_1	a_{11}	a_{12}	…	a_{1m}
v_2	a_{21}	a_{22}	…	a_{2m}
…	…	…	…	…
v_n	a_{n1}	a_{n2}	…	a_{nm}

For simplicity, the dimensions are assumed to be M, L, and T. From this table, a $n \times m$ dimension matrix, A is created such that,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

The total number of Pi terms is determined by the Buckingham Pi theorem, $k = n - m$ [20]. For the method of repeating variables, m variables have to be selected from the total n variables. The dimensions of these variables are stored in a $m \times m$ matrix B, such that,

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ b_{m1} & b_{m2} & \dots & b_{mm} \end{pmatrix}$$

Since B is a square matrix. For a valid result, the determinant of the B matrix has to be non-zero. Otherwise, this method fails to find any solution. From the dimensions of each of the $n-m$ non-repeating variables, the Pi terms are calculated. The dimension of the first non-repeating variable is stored in a $m \times 1$ matrix, C.

Since each of the Pi terms is dimensionless, their equation must also be dimensionless. So,

$$\Pi_1 = v_{nr1}(v_{r1})^{x_{11}}(v_{r2})^{x_{12}}\dots(v_{rm})^{x_{1m}}$$

$$\Leftrightarrow M^0L^0\dots T^0 = (M^{c_{11}}L^{c_{12}}\dots T^{c_{1m}})$$

$$(M^{b_{11}}L^{b_{12}}\dots T^{b_{1m}})^{x_{11}}$$

$$(M^{b_{21}}L^{b_{22}}\dots T^{b_{2m}})^{x_{12}}\dots$$

$$(M^{b_{m1}}L^{b_{m2}}\dots T^{b_{mm}})^{x_{1m}}$$

$$\Leftrightarrow M^0L^0\dots T^0 =$$

$$M^{c_{11}+b_{11}x_{11}+b_{21}x_{12}+\dots+b_{m1}x_{1m}}$$

$$L^{c_{12}+b_{12}x_{11}+b_{22}x_{12}+\dots+b_{m2}x_{1m}}\dots$$

$$T^{c_{1m}+b_{1m}x_{11}+b_{2m}x_{12}+\dots+b_{mm}x_{1m}}$$

Equating the coefficient on both sides,

$$\Leftrightarrow \begin{pmatrix} c_{11}+b_{11}x_{11}+b_{21}x_{12}+\dots+b_{m1}x_{1m}=0 \\ c_{12}+b_{12}x_{11}+b_{22}x_{12}+\dots+b_{m2}x_{1m}=0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ c_{1m}+b_{1m}x_{11}+b_{2m}x_{12}+\dots+b_{mm}x_{1m}=0 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} b_{11}x_{11}+b_{21}x_{12}+\dots+b_{m1}x_{1m}=-c_{11} \\ b_{12}x_{11}+b_{22}x_{12}+\dots+b_{m2}x_{1m}=-c_{12} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_{1m}x_{11}+b_{2m}x_{12}+\dots+b_{mm}x_{1m}=-c_{1m} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} b_{11} & b_{21} & \dots & b_{m1} \\ b_{12} & b_{22} & \dots & b_{m2} \\ \dots & \dots & \dots & \dots \\ b_{1m} & b_{2m} & \dots & b_{mm} \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{12} \\ \dots \\ x_{1m} \end{pmatrix} = \begin{pmatrix} -c_{11} \\ -c_{12} \\ \dots \\ -c_{1m} \end{pmatrix}$$

$$\Leftrightarrow B^T X = -C^T \quad \dots\dots \quad (1)$$

After solving for X, the first Pi term is found. It indicates the power of each of the repeating variables. The power of the non-repeating variable is always 1. Repeating the same process for all the other non-repeating variables, all the other Pi terms are found.

B. The Rayleigh Method

The Rayleigh method (also known as the row reduced echelon form method) is a basic dimensional analysis method and can be simplified to yield dimensionless groups controlling the phenomenon. It was invented by Lord Rayleigh and introduced in 1871. In 1915 [11] he discussed the principle of the method for the first time.

The Rayleigh method is established on the relation $v_1 = Kv_2^{x_1} v_3^{x_2} \ldots v_n^{x_{n-1}}$ [15]. Then each of the variables is represented in terms of their basic dimensions. So,

$$\Leftrightarrow M^{a_{11}} L^{a_{12}} \ldots T^{a_{1m}} = K(M^{a_{21}} L^{a_{22}} \ldots T^{a_{2m}})^{x_1} (M^{a_{31}} L^{a_{32}} \ldots T^{a_{3m}})^{x_2} \ldots (M^{a_{n1}} L^{a_{n2}} \ldots T^{a_{nm}})^{x_{n-1}}$$

$$\Leftrightarrow M^{a_{11}} L^{a_{12}} \ldots T^{a_{1m}} = K(M^{a_{21}x_1 + a_{31}x_2 + \ldots + a_{n1}x_{n-1}}) (L^{a_{22}x_1 + a_{32}x_2 + \ldots + a_{n2}x_{n-1}}) \ldots (T^{a_{2m}x_1 + a_{3m}x_2 + \ldots + a_{nm}x_{n-1}})$$

Now equating coefficients on both sides,

$$\Leftrightarrow \begin{pmatrix} a_{21}x_1 + a_{31}x_2 + \ldots + a_{n1}x_{n-1} = a_{11} \\ a_{22}x_1 + a_{32}x_2 + \ldots + a_{n2}x_{n-1} = a_{12} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{2m}x_1 + a_{3m}x_2 + \ldots + a_{nm}x_{n-1} = a_{1m} \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a_{21} & a_{31} & \ldots & a_{11} \\ a_{22} & a_{32} & \ldots & a_{12} \\ \ldots & \ldots & \ldots & \ldots \\ a_{2m} & a_{3m} & \ldots & a_{1m} \end{pmatrix} = A$$

Now, expressing this system of equations in terms of reduced row-echelon form a unique solution is found. There are some pivot elements and some free variables. If a matrix is in row-echelon form, then the first nonzero entry of each row is called a pivot, and the columns in which pivots appear are called pivot columns. Let, the number of pivot elements is m . Then the system can be expressed as,

$$\begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_m \end{pmatrix} = x_{m+1} \begin{pmatrix} r_{(m+1)1} \\ r_{(m+1)2} \\ \ldots \\ r_{(m+1)m} \end{pmatrix} + \ldots + x_{n-1} \begin{pmatrix} r_{(n-1)1} \\ r_{(n-1)2} \\ \ldots \\ r_{(n-1)m} \end{pmatrix} + \begin{pmatrix} r_{n1} \\ r_{n2} \\ \ldots \\ r_{nm} \end{pmatrix}$$

Here, each vector represents the power of the pivot variables which acts like the repeating variables. Each of the free variables acts like the non-repeating variables and their power is always 1. The last vector represents the solution of the first variable v_1 .

III. Problem Identification

Many real-life problems, which cannot be solved analytically have to rely on the results of experimentally obtained data. To perform experiments for each of the variables can be very costly, depending on the number of variables included in the problem. Pi terms reduce the number of experiments needed, by using the Buckingham Pi theorem. The method of repeating variables and the Rayleigh method, both can be used to determine the pi terms. This paper develops algorithms for these two methods. Moreover, there is no comparative study done to find the effectiveness of these algorithms.

A comparative study is necessary to understand the complexity of these two algorithms. Any algorithm can perform very well or poorly, depending on the total number of variables and the total number of reference dimensions.

This paper, first of all, develops step-by-step algorithms of the two methods. Then a comparative analysis is shown between the two algorithms that discuss their complexity analysis and the effectiveness of using the algorithms in different problem sets involving varying numbers of variables and reference dimensions.

The Rayleigh method (also known as the row reduced echelon form method) is used to find the Pi terms of a given problem [19]. But there are cases when the method of repeating variables is more effective in finding the Pi terms. Though there are some limitations to the method of repeating variables. When the number of reference dimensions differs from the number of basic dimensions (Total number of reference dimensions needed to express each of the variables in their respective dimensions), the method of repeating variables leads to an error [3].

IV. Time Complexity Analysis

A. The method of repeating variables:

In this method, firstly, a loop is executed that is equal to the number of variables n. It has a complexity of $O(n)$. Then, to find the determinant of a $m \times m$ matrix,

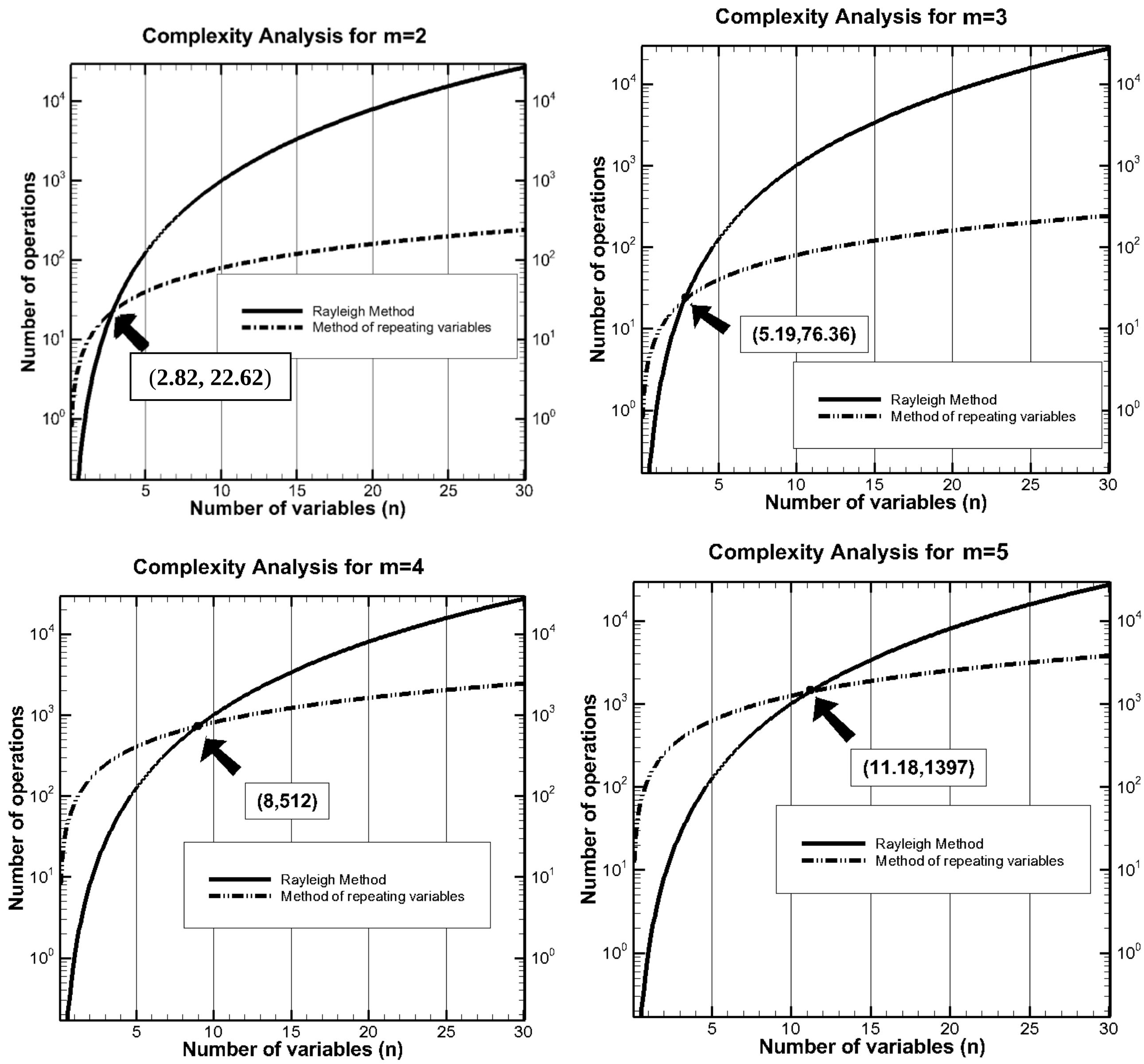

Fig 1. The complexity analysis for the both methods when the reference dimension is m=2, m=3, m=4 and m=5.

$O(m^3)$ operations are executed. To find the output, a loop for n values is executed. For each value in that loop, a linear solver is executed that has a complexity of $O(m^3)$, which results in a total complexity of $O(nm^3)$. So, the total complexity for the algorithm is $n+m^3+nm^3$, which can be denoted by $O(nm^3)$.

B. The Rayleigh method:

To initialize the matrix, a loop is executed which has complexity n. To convert the B matrix into a row reduced echelon form $O(n^3)$ operations are executed. To find the output, a loop for n values is executed. For each value in that loop, another loop is executed that has a value less than or equal to m, which results in a complexity of $O(nm)$. So, the total complexity for the algorithm is $n+n^3+nm$, which can be denoted by $O(n^3)$.

The Rayleigh methods and the method of repeating variables perform differently for a different number of reference dimensions and different number of variables. The complexity of the Rayleigh method is dependent only on the number of variables. The change in reference dimensions does not affect the Rayleigh method. With the increasing number of total variables, the number of operations needed is increasing rapidly for the Rayleigh method. However, the complexity of the method of repeating variables is dependent on both the total number of variables and the total number of reference dimensions. The number of operations needed for the method of repeating variables does not increase rapidly with the increase of the total number of variables.

For the reference dimension m=2, as the number of variables has to be greater than 3 the method of repeating variables is always faster than the Rayleigh method. For the

TABLE II
NUMBER OF OPERATIONS NEEDED FOR BOTH ALGORITHMS

Number of reference dimensions, m	Total number of variables, n	The Rayleigh method	The method of repeating variables	Operation difference (Ray-MRV)
m=2	5	125	40	85
	10	1000	80	920
	15	3375	120	3255
	20	8000	160	7840
	25	15625	200	15425
	30	27000	240	26760
m=3	5	125	135	-10
	10	1000	270	730
	15	3375	405	2970
	20	8000	540	7460
	25	15625	675	14950
	30	27000	810	26190
m=4	5	125	320	-195
	10	1000	640	360
	15	3375	960	2415
	20	8000	1280	6720
	25	15625	1600	14025
	30	27000	1920	25080
m=5	5	125	625	-500
	10	1000	1250	-250
	15	3375	1875	1500
	20	8000	2500	5500
	25	15625	3125	12500
	30	27000	3750	23250

reference dimension m=3, the Rayleigh method is faster than the method of repeating variables when the number of variables is less than 6. If the number of variables is greater than or equal to 6 then the method of repeating variables performs faster. For the reference dimension m=4, the Rayleigh method is faster than the method of repeating variables when the total number of variables is less than 8. When the total number of variables is 8 both methods performs at an equal level of efficiency. If the total number of variables exceeds 8 then the method of repeating variable performs faster. For the reference dimension m=5, the Rayleigh method is faster when the total number of variables is less than 12. When the total number of variables is 12 or more then the method of repeating variables performs faster.

The table shows that the method of repeating variables is faster than the Rayleigh method in most cases. However, the Rayleigh method is more effective for a larger value of m (the total number of reference dimensions) and a smaller value of n (the total number of variables).

V. Memory Complexity Analysis

A. *The method of repeating variables:*

In this method, 4 bytes of memory is used to store the variables n and m which represent the total number of variables and the total number of reference dimensions respectively. Then an array of characters is used to store the variable names which take $3\times 2n$ bytes. An array $A(n,m)$ is used to store the dimension matrix which take up to $2nm$ bytes. An array of $r(m)$ is used to store the repeating variables which take $2m$ bytes. Again, an array of $B(m,m)$ takes $2mm$ bytes and an array of $C(m)$ takes $2m$ bytes. To store the variables like result, i and k a total of 2+2+2 bytes is used. To store the dimensions of the Pi terms an array $X(m)$ is used which take up to $2m$ bytes. To display the results the total memory needed is $3\times 2(m+1)$ and $2(m+1)$. The total memory needed for this method is $2nm+2mm+6n+14m+18$ which has $O(nm)$.

B. *The Rayleigh method:*

In this method, 4 bytes of memory is used to store the variables n and m which represent the total number of variables and the total number of reference dimensions respectively. Then an array of characters is used to store the variable names which takes $3\times 2n$ bytes. An array $A(n,m)$ is used to store the dimension matrix which takes up to $2nm$ bytes. An array of $D(n,m)$ is used to store the row reduced echelon form which takes up to $2nm$ bytes. To store the pivot elements an array $jb(m)$ is used which takes $2m$ bytes. To store variables like the result, i and k total 2+2+2 bytes is used. To store the dimensions of the Pi terms an array $X(length(jb))$ is used which takes up to $2m$ bytes. To display the results the total memory needed is $3\times 2(m+1)$ and $2(m+1)$. The total memory needed for this method is $4nm+6n+12m+18$ which has $O(nm)$.

The analysis shows that, if n is greater than $m+1$ the Rayleigh method will take more memory. But if n is equal to $m+1$ then both algorithms will take the same memory. $m+1$ cannot be greater than n, because there will be no Pi terms in this case.

VI. Results And Discussion

The Pi terms are determined for few problems using both the method of repeating variables and the Rayleigh method. Their limitations are also discussed here.

Problem 1: [3]

A thin rectangular plate having a width *w* and a height *h* is located so that it is normal to a moving stream of fluid as shown in figure. Assume the drag, *D*, that the fluid exerts on the plate is a function of *w* and *h,* the fluid viscosity and density, *μ* and *ρ*, respectively, and the velocity *V* of the fluid approaching the plate. Find a suitable set of Pi terms to study this problem experimentally.

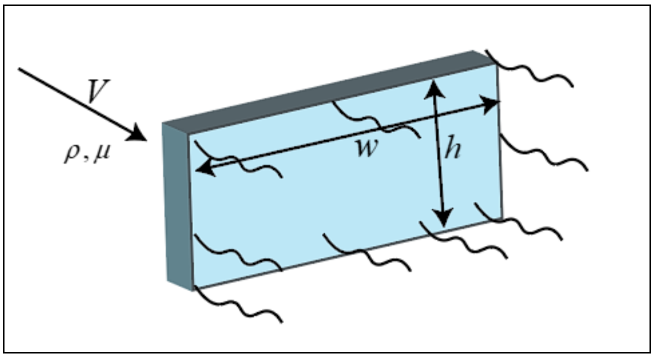

Fig 2. A unit rectangular plate located normal to a moving stream

Step 1: First, find all the variables related to this problem. So, $D = f(w,h,\mu,\rho,V)$

Step 2: Express each of the variables in terms of the basic dimensions.

TABLE III
VARIABLES IN TERMS OF BASIC DIMENSIONS

Variables	D	w	h	μ	ρ	V
Unit	$kgms^{-2}$	m	m	$kgm^{-1}s^{-1}$	kgm^{-3}	ms^{-1}
Dimension	MLT^{-2}	L	L	$ML^{-1}T^{-1}$	ML^{-3}	LT^{-1}

TABLE IV
DIMENSION MATRIX FOR PROBLEM 1 IN MLT SYSTEM

	M	L	T
D	1	1	-2
w	0	1	0
h	0	1	0
μ	1	-1	-1
ρ	1	-3	0
V	0	1	-1

Step 4: Choose the repeating variables. The repeating variables chosen are w, ρ and V which are in order 2, 5 and 6.

INPUT:

The input file has to be created with the number of variables (n), the number of reference dimensions (m), the dimension matrix, A and the indices of repeating variables.

OUTPUT:

For the method of repeating variables, the B matrix has to be created first from the dimension matrix. The repeating variables are 2, 5 and 6. So,

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

And,

$$B^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Now, the non-repeating variables are 1, 3 and 4. From (1),

$B^T X = -C^T$

For the first non-repeating variable,

$$C^T = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Now, solving for X,

$$X = \begin{pmatrix} -8 \\ 2 \\ -3 \end{pmatrix}$$

This is the power for each of the repeating variables. The power of non-repeating variable is always 1. So, the first π term is, $\Pi_1 = Dw^{-8}\mu^2\rho^{-3}$

Similarly, the other π terms are,

$\Pi_2 = hw^{-1}\mu^0\rho^0$

$\Pi_3 = Vw^{-3}\mu^1\rho^{-1}$

The average run time for this problem is 0.002808 seconds.

Now, for the Rayleigh method, a matrix B has to be created where the first dimension matrix A will be transposed and the first column will move to the last column. The order of the variables will be w h μ ρ V and D. So,

$$B = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -2 \end{pmatrix}$$

The reduced row echelon form of the matrix B is,

$$rref(B) = \begin{pmatrix} 1 & 1 & 0 & 0 & 3 & 8 \\ 0 & 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{pmatrix}$$

And the pivot variables are $jb = \begin{pmatrix} 1 & 3 & 4 \end{pmatrix}$.

So, it can be expressed as,

$$\begin{pmatrix} x_1 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix} x_5 + \begin{pmatrix} -8 \\ 2 \\ -3 \end{pmatrix}$$

Each of the matrices will be the power of the repeating variables which is 1, 3 and 4 corresponding to each of the non-repeating variable which will have the power of 1.

So, the π terms will be,

$\Pi_1 = hw^{-1}\mu^0\rho^0$

$\Pi_2 = Vw^{-3}\mu^1\rho^{-1}$

$\Pi_3 = Dw^{-8}\mu^2\rho^{-3}$

The average run time for this problem is 0.029320 seconds. These results are consistent with the result found in the reference book.

Problem 2: [3 pp {343}]

An open, cylindrical paint can having a diameter D is filled to a depth h with paint having a specific weight γ .The vertical deflection δ, at the center of the bottom is a function of D, h, d, γ and E, where d is the thickness of the bottom and E is the modulus of elasticity of the bottom material.

Firstly, solve this problem with the FLT system.
Step 1: Find all the variables related to this problem.
So, $\delta = f(D, h, d, \gamma, E)$
Step 2: Express each of the variables in terms of basic dimensions.

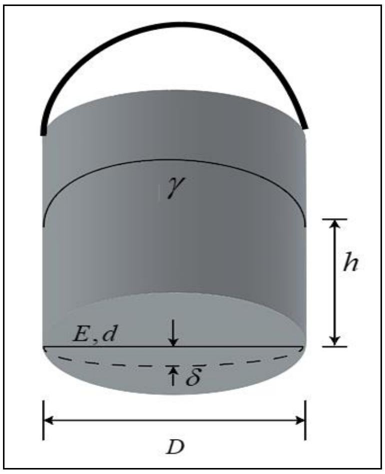

Fig 3. An open cylinder can

TABLE V
VARIABLES IN TERMS OF BASIC DIMENSIONS

Variables	δ	D	h	d	γ	E
Unit	m	m	m	m	$kgm^{-2}s^{-2}$	$kgm^{-1}s^{-2}$
Dimension	L	L	L	L	FL^{-3}	FL^{-2}

TABLE VI
DIMENSION MATRIX FOR PROBLEM 2 IN FLT SYSTEM

	F	L
δ	0	1
D	0	1
h	0	1
d	0	1
γ	1	-3
E	1	-2

Step 3: Here are 6 variables and 2 reference dimensions.

Step 4: Choose the repeating variables. The repeating variables that are chosen are D and γ which are in order 2 and 5.

INPUT:

The input file has to be created with the number of variables (n), the number of reference dimensions (m), the dimension matrix, A and the indices of repeating variables.

OUTPUT:

For the method of repeating variables, the B matrix has to be created first from the dimension matrix. The repeating variables are 2 and 5. So,

$$B = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \text{ and } B^T = \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}$$

Now, the non-repeating variables are 1, 3, 4 and 6. From (1),

$$B^T X = -C^T$$

For the first non-repeating variable,

$$C^T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So, solving for X,

$$X = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

This is the power of each of the non-repeating variables. The power of repeating variables is always 1. So, the first π term is, $\Pi_1 = \delta D^{-1}\gamma^0$

Similarly, the other π terms are,

$$\Pi_2 = hD^{-1}\gamma^0$$

$$\Pi_3 = dD^{-1}\gamma^0$$

$$\Pi_4 = ED^{-1}\gamma^{-1}$$

The average run time for this problem is 0.002232 seconds.

Now, for the Rayleigh method, a matrix B has to be created where the first dimension matrix A will be transposed and the first column will move to the last column. The order of the variables will be D, h, d, γ, E and δ. So,

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -3 & -2 & 1 \end{pmatrix}$$

The reduced row echelon form of the matrix B is,

$$rref(B) = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and the pivot variables are $jb = \begin{pmatrix} 1 & 4 \end{pmatrix}$

So, it can be expressed as,

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -1 \\ -1 \end{pmatrix} x_5 + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Each of the matrices will be the power of the repeating variables which is 1 and 4 corresponding to each of the non-repeating variables which will have the power of 1.
So, the π terms are,

$$\Pi_1 = \delta D^{-1}\gamma^0 \qquad \Pi_3 = dD^{-1}\gamma^0$$

$$\Pi_2 = hD^{-1}\gamma^0 \qquad \Pi_4 = ED^{-1}\gamma^{-1}$$

The average run time for this problem is 0.023603 seconds. These results are consistent with the result found in the reference book.

Solution (Alternative):

Now, solve this problem with the MLT system.

Step 1: Find all the variables related to this problem.

So, $\delta = f(D, h, d, \gamma, E)$

Step 2: Express each of the variables in terms of basic dimensions.

TABLE VII
VARIABLES IN TERMS OF BASIC DIMENSIONS

Variables	δ	D	h	d	γ	E
Unit	m	m	m	m	$kgm^{-2}s^{-2}$	$kgm^{-1}s^{-2}$
Dimension	L	L	L	L	$ML^{-2}T^{-2}$	$ML^{-1}T^{-2}$

TABLE VIII
DIMENSION MATRIX FOR PROBLEM 2 IN MLT SYSTEM

	M	L	T
δ	0	1	0
D	0	1	0
h	0	1	0
d	0	1	0
γ	1	-2	-2
E	1	-1	-2

Step 4: Choose the repeating variables. The repeating variables that are chosen are *D*, *γ* and *E* which are in order 2, 5 and 6.

INPUT:

The input file has to be created with the number of variables (*n*), number of reference dimensions (*m*), the dimension matrix, and the indices of repeating variables.

OUTPUT:

For the method of repeating variables, the *B* matrix has to be created first from the dimension matrix. The repeating variables are 2, 5 and 6. So,

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & -2 \\ 1 & -1 & -2 \end{pmatrix} \text{ and } B^T = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -2 & -2 \end{pmatrix}$$

Now, the determinant of B^T is,

$$\det(B^T) = -1\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 0 & -2 \end{vmatrix}$$

$$= (-1)\times 2 + 1\times 2 = 0$$

So, B^T is a singular matrix. There is no solution for $B^T X = -C^T$ .Therefore, this problem cannot be solved by the method of repeating variables. The average run time for this problem is 0.003046 seconds.

Now, for the Rayleigh method, a matrix *B* has to be created where the first dimension matrix *A* will be transposed and the first column will move to the last column. The order of the variables will be *D, h, d, γ* , *E* and *δ* . So,

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & -2 & -1 & 1 \\ 0 & 0 & 0 & -2 & -2 & 0 \end{pmatrix}$$

The reduced row echelon form of the matrix *B* is,

$$rref(B) = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the pivot variables are $jb = \begin{pmatrix} 1 & 4 \end{pmatrix}$.

Here, the last row has become all zero. That means, there will be only 2 repeating variables. So, the required number of π terms does not decrease by simply stating it in MLT system of basic dimensions. A closer look reveals only 2 reference dimensions are required for this problem which are ML^{-2} and *T*.

TABLE IX
RESULT ANALYSIS

	Problem 1		Problem 2		Problem 2 (Alternative)	
Method name	Method of repeating variables	Rayleigh method	Method of repeating variables	Rayleigh method	Method of repeating variables	Rayleigh method
Number of operations	195	240	62	234	195	240
Memory needed (MB)	770	774	768	772	770	774
Average time needed (seconds)	0.002808	0.029320	0.002232	0.023603	0.003046	0.028623
Successful execution	Yes	Yes	Yes	Yes	No	Yes

So, it can be expressed as,

$$\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -1 \\ -1 \end{pmatrix} x_5 + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Each of the matrices will be the power of the repeating variables which is 1 and 4 corresponding to each of the non-repeating variables which will have the power of 1.
So, the π terms will be,

$$\Pi_1 = \delta D^{-1}\gamma^0 \qquad \Pi_3 = dD^{-1}\gamma^0$$

$$\Pi_2 = hD^{-1}\gamma^0 \qquad \Pi_4 = ED^{-1}\gamma^{-1}$$

The average run time for this problem is 0.02862 seconds.

These results are consistent with the result found in the reference book.

The machine used to determine the average run time of those problems is Intel(R) Core i5-7500 CPU @ 3.40 GHz with 16 GB RAM and 64-bit Operating System.

The following table (Table-IX) shows a comprehensive comparison between the two algorithms for the aforementioned problems.

VII. Conclusion

The method of repeating variables and the Rayleigh method both can be used to find the Pi terms. However, their effectiveness depends on the total number of variables and the total number of reference dimensions needed to express each of the variables. Their effectiveness also depends on the dimensions of the repeating variables. If the dimensions of variables are such that the dimensions of one variable are the scalar multiple of the others, then the method of repeating variables fails to determine a solution. This happens when the number of reference dimensions differs from the number of basic dimensions needed to express each of the variables.

However, the complexity of the two methods depends on the total number of variables and the total number of reference dimensions. For a fixed number of reference dimensions, the Rayleigh method is usually faster than the method of repeating variables for a certain number of variables. When the number of variables is more than that certain number, the method of repeating variables becomes faster. The method of repeating variables needs less memory than the Rayleigh method. That is why it is important to evaluate both algorithms to understand their effectiveness. Both methods can be useful to solve real-life problems depending on the number of variables and the number of reference dimensions.

VIII. Nomenclature

TABLE X
Units of Variables Used

Symbol	Quantity	Units
D	drag force	$kgms^{-2}$
w	width	*m*
h	height	*m*
μ	fluid viscosity	$kgm^{-1}s^{-1}$
ρ	fluid density	*kgm*
V	velocity	ms^{-1}
δ	vertical deflection	*m*
D	diameter	*m*
h	depth	*m*
d	thickness	*m*
γ	specific weight	$kgm^{-2}s^{-2}$
E	modulus of elasticity	$kgm^{-1}s^{-2}$

Appendix

Algorithm: To find the π terms given the dimension matrix by the method of repeating variables.

INPUT: the number of variables *n*; the number of reference dimensions required to describe these variables *m*; the names of the variables *var(i)*,$1 \le i \le n$; the entries *a(i,j)*, $1 \le i \le n$ and $1 \le j \le m$; the index of the repeating variables *r(i)*, $1 \le i \le m$.

OUTPUT: the variables with their respective powers to form all the π terms.

Step 1: Set *k=1*.

Step 2: For *i=1,…,n* do Steps 3-4

Step 3: If the matrix *r* contains *i* then set *B(k,:) = A(i,:)*; set *k=k+1*;

Step 4: Set *k=1*. If the determinant of B^T is zero then OUTPUT (“No solution exists”).

Step 5: For *i=1,…,n* do Steps 6-10

Step 6: If matrix *r* does not contain *i* then do the following:

Step 7: set *C=A(i,:)*;

Step 8: Linear solve $B^T X = -C$ for *X*. OUTPUT(“Number %d π term is:\n”,*k*). OUTPUT (“var(i) ‘1’ ”).

Step 9: For *j=1,…,m* do Step 10

Step 10: OUTPUT (“var(r(j)) X(j)”)

END

Algorithm: To find the π terms given the dimension matrix by Rayleigh method.

INPUT: the number of variables n; the number of reference dimensions required to describe these variables m; the names of the variables $var(i), 1 \le i \le n$; the entries $a(i,j)$, $1 \le i \le n$ and $1 \le j \le m$;

OUTPUT: the variables with their respective powers to form all the π terms.

Step 1: Set $k=1$.

Step 2: For $i=2,\ldots,n$ do Steps 3-4

Step 3: Set $D(k,:)=A(i,:)$;

Step 4: set $k=k+1$;

Step 5: Set $D(k,:)=A(1,:)$ and then set $B = D^T$. Set $k=1$.

Step 6: Convert B into reduced row echelon form R and set jb as the pivot elements of R.

Step 7: For $i=1,\ldots,n$ do Step 8-12

Step 8: If the matrix jb does not contain i then do the following:

Step 9: set $X=R(:,i)$;

Step 10: OUTPUT ("Number %d π term is:\n",k). Set $k=k+1$.

If ($i==n$) OUTPUT ("*var(1) '1'* ").

Else OUTPUT ("*var(i+1) '-1'* ").

End if

Step 11: For $j=1,\ldots,length(jb)$ do Step 10

Step 12: OUTPUT ("*var(jb(j)+1) -X(j)*")

END

REFERENCES

[1] E. Shaughnessy, I. Katz and J. Schaffer, Introduction to fluid mechanics. New York: Oxford University Press, 2005, pp. 554-555.

[2] B. Massey and A. Ward-Smith, Mechanics of fluids, 9th ed.

[3] B. Munson, T. Okiishi and D. Young, Fundamentals of fluid mechanics, 6th ed. New York: John Wiley & Sons, Ltd.

[4] F. White and H. Xue, Fluid mechanics, 8th ed. McGraw-Hill Education.

[5] E. Taylor, Dimensional analysis for engineers, 5th ed. Oxford [England]: Clarendon Press, 1974.

[6] R. De A. Martins, "The origin of dimensional analysis", Journal of the Franklin Institute, vol. 311, no. 5, pp. 311 (5) -331 (7). Available: 10.1016/0016-0032(81)90475-0.

[7] E. Buckingham, "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations", Physical Review, vol. 4, no. 4, 1914. Available: 10.1103/physrev.4.345.

[8] J. Gibbings, Dimensional analysis. [Place of publication not identified]: Springer, 2014.

[9] QM Tan, Dimensional analysis, Springer-Verlag Berlin Heidelberg, 2011. Available: 10.1007/978-3-642-19234-0.

[10] G. Barenblatt, "Self-similarity: Similarity and intermediate asymptotic form", Radiophysics and Quantum Electronics, vol. 19, no. 6, 1976. Available: 10.1007/bf01043552.

[11] Rayleigh, "The Principle of Similitude", Nature, vol. 95, no. 66, 1915. Available: 10.1038/095202d0.

[12] J. Palacios, Dimensional analysis. London: Macmillan, 1964.

[13] B. Massey, Units, dimensional analysis and physical similarity. London: Van Nostrand Reinhold, 1971.

[14] C. Mendoza, "A theorem for Rayleigh's method of dimensional analysis and its proof", Mechanics Research Communications, vol. 21, no. 2, pp. 103-107, 1994. Available: 10.1016/0093-6413(94)90081-7.

[15] L. Sedov, Similarity and dimensional methods in mechanics. Boca Raton, Fla.: CRC Press, 1993.

[16] L. Brand, "The Pi theorem of dimensional analysis", Archive for Rational Mechanics and Analysis, vol. 1, no. 1, pp. 35-45, 1957. Available: 10.1007/bf00297994.

[17] R. Kurth, "A Note on Dimensional Analysis", The American Mathematical Monthly, vol. 72, no. 9, pp. 965-969, 1965. Available: 10.1080/00029890.1965.11970652.

[18] R. Kruth, Dimensional Analysis and Group Theory in Astrophysics, Long Island City, N.Y.: Pergammon, 1972.

[19] D. Calvetti and E. Somersalo, Computational mathematical modeling. Philadelphia [USA]: Society for Industrial and Applied Mathematics, 2012.

[20] R. Laskowski, "The application of the Buckingham theorem to modeling high-pressure regenerative heat exchangers in changed conditions." Journal of Power Technologies 91.4, pp. 198-205, 2011.

[21] S. Maxemow, "That's a Drag: The Effects of Drag Forces", Undergraduate Journal of Mathematical Modeling: One + Two, vol. 2, no. 1, 2009. Available: 10.5038/2326-3652.2.1.4.