Generalized Adjacency, Laplacian Spectra and Signless Laplacian Spectra of the Weighted Neighbourhood Coronae Networks

Zhibin Zhu, Qun Liu

Abstract—**Spectra of weighted networks have received increasing attention from scientific community, such as mathematical chemistry, computer science, coding theory. The real networks behavior is completely differently, not only in the degree distribution, but also in the weight distribution. In this paper, we determine the generalized adjacency (resp., Laplacian and signless Laplacian) spectra of the weighted neighbourhood corona networks with different structures. As applications, the two important indices of the weighted neighbourhood corona networks are computed.**

Index Terms—**Weighted neighbourhood coronae networks; Generalized adjacency matrix; Kirchhoff index; Spanning trees**

I. Introduction

In the past decade, the new researchs on complex network have drawn attentions of scholars in many fields, such as computer science, physics and chemistry and so on. The key issues in the field of complex networks is to uncover the topological characteristics and dynamic process of complex networks. For example, Qi and Zhang et al. [1] studied the spectra and their applications for extended Sierpiski graphs and their applications. Dai et al [2] obtained the recursive relationship of its eigenvalues at two successive generation of the Markov matrix. Meanwhile, a fundamental issue in the study of complex networks is to uncover how the structure properties affect different dynamics, many of which are related to the exact knowledge of the spectra. In recently years, spectra of weighted networks have attracted a great deal of attention by some researchers $[1-6]$, since various dynamical processes and structural aspects of complex networks are related to the spectra of the matrix. The wide applications of the spectra of the matrix have Kirchhoff index, spanning trees, eigentime identity, expected hitting time, and so on.

Real networks behave quite differently, not only in the aspect of degree distribution but also in the context of weight distribution. Dai et.al [7] gave a complete description of the eigenvalues and the eigenvectors of graphs with the weighted corona networks. Liu et.al [8] presented a completely characterization of generalized adjacency(resp.,Laplacian and signless Laplacian) spectra of graphs with the weighted edge corona networks. In $[7, 8]$, they pointed that it is natural and interesting to study the other weighted networks. The impacts of weight factors are vital in analyzing some properties of networks. Practical realizations of weights in real networks range from the number of passengers travelling yearly between two airports in airport networks [9] the traffic measured in packets per unit time between routers in the Internet [10] or the intensity of predator-prey interactions in ecosystems [11]. In spectra graph theory, spectra of the adjacency and the Laplacian matrix plays an important role in recognizing the graph properties[9-15]. In [15], the authors used the adjacency matrix to prove the Aanderaa-Rosenberg conjecture. Qi and Zhang [15-16] found the normalized Laplacian spectra has important applications in exploring relevant structural properties of the weighted fractals. Motivated by these works, we consider the spectral properties of the weighted neighbourhood corona networks.

In this paper, firstly, we give the generalized adjacency spectra of the weighted neighbourhood corona graphs with two different initial graphs. Then the spectral analysis of the Laplacian spectra are given. Finally, the signless Laplacian spectra of the weighted neighbourhood corona graphs with two different structures are derived, which methods used are similar in those in Section 3. The number of spanning trees and Kirchhoff index of the weighted neighbourhood corona are computed as an application of these tesults.

II. Generalized adjacency spectra of the weighted $G_1 \star G_2$

The adjacency, Laplacian and signless Laplacian spectra of the neighbourhood corona graph [19] was given. Next the weighted neighbourhood corona graphs is defined as follows.

(i) For a positive real number $0 < r \leq 1$, we call r the weight factor.

(ii) Initial graph: G_1 is a simple connected graph with n vertices. And every edge has a unitary weight.

(iii) Attaching copy graph: G_2 is a simple connected graph with n_2 vertices. And every edge has a unitary weight.

(iv) G_2^i is the copy of $G_2 (i = 1, 2, \cdots, k)$, its weighted edges have been fixed by a factor r. Joining every neighbour of the ith vertex of G_1 to every vertex in the ith copy of G_1 by a new edge, the newly generated edges carry the weight r.

We have constructed the weighted neighbourhood corona graphs $G_1 \star G_2$ with the weight factor. As instance, the weighted neighbourhood corona graph $P_4 \star P_3$ is as illustrated in Fig.1.

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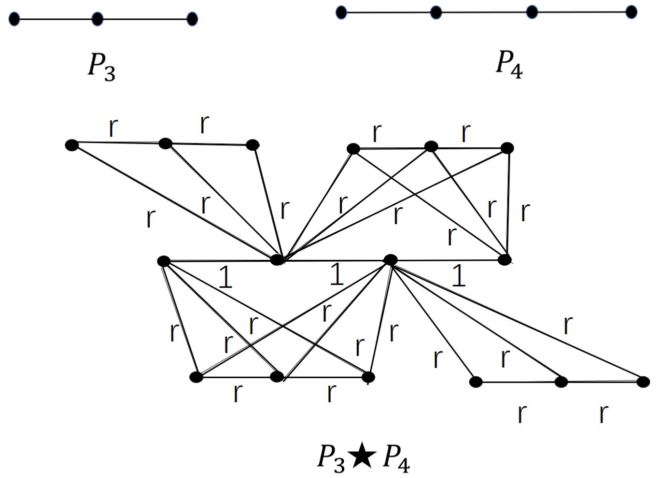

$Fig.1$ the $weighted$ $neighbourhood$ $corona$ $graph$

We use $W(G)$ to express the generalized adjacency matrix(weight matrix) of G, the entries $W_{i,j}$ of $W(G)$ are defined as follows: if vertices i and j are adjacent in $W(G)$, then $W_{i,j} = \omega_{i,j}$, otherwise $W_{i,j} = 0$, where $\omega_{i,j}$ is the weight of edge linking vertices i and j.

According to the construction of $G_1 \star G_2$, the generalized adjacency matrix is as follows:
$W(G_1 \star G_2) =$

$$\begin{pmatrix} W(G_1) & rj_{n_2}^T \otimes W(G_1) \\ r(j_{n_2} \otimes W(G_1))^T & rW(G_2) \otimes I(n_1) \end{pmatrix}, \quad (1)$$

where $j_{n_2}^T$ denote the row vector with order n_2 and all elements are 1.

A. G_2 is a d_2-regular graph

Let G_1 be a connected graph with n_1 vertices attaching copy graph G_2 which is a d_2-regular graph with n_2 vertices. Let $X = [X_1 X_2 \cdots X_{n_2+1}]^T$ be the eigenvector corresponding to the eigenvalue λ of $W(G_1 \star G_2)$. Then

$$W(G_1 \star G_2)X = \lambda X. \quad (2)$$

Next, we obtain the eigenvector of $W(G_1 \star G_2)$. It divides into the following two cases with $\lambda \neq rd_2$.

Case 1 : X_1 is nonzero vector

From Equations (1) and (2), it follows that

$$W(G_1)X_1 + rW(G_1)(X_2 + X_3 + \cdots X_{n_2+1}) = \lambda X_1, \quad (3)$$

and

$$\begin{cases} rW^T(G_1)X_1 + rE_1[W(G_2) \otimes I(n_1)][X_2 X_3 \cdots X_{n_2+1}]^T \\ = \lambda X_2, \\ rW^T(G_1)X_1 + rE_2[W(G_2) \otimes I(n_1)][X_2 X_3 \cdots X_{n_2+1}]^T \\ = \lambda X_3, \\ \vdots \\ rW^T(G_1)X_1 + rE_1[W(G_2) \otimes I(n_1)][X_2 X_3 \cdots X_{n_2+1}]^T \\ = \lambda X_{n_2+1}, \end{cases}$$

where $E_i = (\underbrace{0, \cdots, 0}_{\text{i-1}}, I_{n_1}, \underbrace{0, \cdots, 0}_{\text{k-i}})$.

Since G_2 is a d_2-regular graph, there are d_2-nonzero entries in each row of matrix $W(G_2)$. By adding all equation in the above equation, it gives

$$\begin{aligned} rn_2 W(G_1)X_1 + rd_2(X_2 + X_3 + \cdots X_{n_2+1}) \quad (4) \\ = \lambda(X_2 + X_3 + \cdots X_{n_2+1}), \end{aligned}$$

which is

$$(X_2 + X_3 + \cdots X_{n_2+1}) = \frac{rn_2}{\lambda - rd_2} W^T(G_1)X_1. \quad (5)$$

Substituting Equation (5) to Equation (3), we have

$$W(G_1)X_1 + \frac{r^2 n_2}{\lambda - rd_2} W(G_1)W^T(G_1)X_1 = \lambda X_1. \quad (6)$$

Notice that $\sigma(G_1) = \{\lambda_1(G_1), \lambda_2(G_1) \cdots \lambda_{n_1}(G_1)\}$. Based on Eq.(7), we have

$$\lambda^2 - (rd_2 + \lambda_i(G_1))\lambda + rd_2\lambda_i(G_1) - r^2 n_2 \lambda_i(G_1) = 0, \quad (7)$$

$i = 1, 2, \ldots, n_1$.

Solving the equation (7), we obtain
$\lambda_{1,2} =$

$$\frac{rd_2 + \lambda_i(G_1) \pm \sqrt{(rd_2 - \lambda_i(G_1))^2 + 4r^2 n_2(\lambda_i(G_1))^2}}{2}, \quad (8)$$

$i = 1, 2, \cdots, n_1$.

Case 2 : X_1 is zero vector

From Equation (3) and Equation (4), we obtain

$$rA(G_2)(X_2 + X_3 + \cdots X_{n_2+1}) = 0,$$

and

$$r[A(G_2) \otimes I(n_1)][X_2 X_3 \cdots X_{n_2+1}]^T = \lambda[X_2 X_3 \cdots X_{n_2+1}]^T.$$

Notice that the spectra of $A(G_2)$ is $\sigma(G_2) = \{\lambda_1(G_2), \lambda_2(G_2) \cdots \lambda_{n_1}(G_2) = d_2\}$, we can straightforward get that

$$\lambda = r\lambda_j(G_2), j = 1, 2, \cdots, n_2 - 1. \quad (9)$$

According to Equation (8) and Equation (9), we can easily get that the multiplicity of $\lambda = r\lambda_j(G_2)$ is n_1. Through the above steps, we present the following results.

Theorem 2.1 Let G_1 be a graph on n_1 vertices and G_2 be an d_2-regular graph on n_2 vertices, where $n_1 \geq 1$, $n_2 \geq 1$ and $d_2 \geq 1$. Suppose $\sigma(G_i) = \{\lambda_1(G_i) \leq \lambda_2(G_i), \cdots \leq \lambda_{n_2}(G_i)\}(i = 1, 2)$. Then the generalized adjacency spectra of $W(G_1 \star G_2)$ are as follows.

(i)$\frac{rd_2 + \lambda_i(G_1) \pm \sqrt{(rd_2 - \lambda_i(G_1))^2 + 4r^2 n_2(\lambda_i(G_1))^2}}{2} \in \sigma(G_1 \star G_2)$ with multiplicity 1 for $i = 1, 2, \cdots, n_1$.
(ii)$r\lambda_j(G_2) \in \sigma(G_1 \star G_2)$ with multiplicity n_1 for $j = 1, 2, \cdots, n_2 - 1$.

B. G_2 is a complete bipartite graph

In this section, we will focus on finding out the spectra of $W(G_1 \star G_2)$ when G_2 is a complete bipartite graph, it may be a regular graph or not. As we known, a complete bipartite graph $G = K_{p,q}$ is defined as a graph that any vertex in X has a unique edge with each vertex in Y, where $|X| = p$ and $|Y| = q$.

Let G_2 be a complete bipartite graph on $k = p + q$ vertices. Then the generalized adjacency matrix $W(G_1 \star G_2)$ associated with $G_1 \star G_2$ is given by
$A(G_1 \star G_2) =$

$$\begin{pmatrix} A & rA & rA & \cdots & rA & rA & \cdots & rA \\ rA^T & 0 & 0 & \cdots & 0 & rI & \cdots & rI \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ rA^T & 0 & 0 & \cdots & 0 & rI & \cdots & rI \\ rA^T & rI & rI & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ rA^T & rI & rI & \cdots & rI & 0 & \cdots & 0 \end{pmatrix}.$$

Let $X = [X_1, X_2, \cdots, X_{p+1}, X_{p+2}, \cdots X_{k+1}]^T$ be the eigenvector corresponding to the eigenvalue λ of $W(G_1 \star G_2)$. Then $W(G_1 \star G_2)X = \lambda X$.
It follows that

$$A(G_1)X_1 + rA(G_1)\sum_{j=2}^{p+1} X_j + rA(G_1)\sum_{j=p+2}^{k+1} X_j = \lambda X_1, \quad (10)$$

$$rA^T(G_1)X_1 + r\sum_{j=p+2}^{k+1} X_j = \lambda X_l, l = 2, 3, \cdots, p+1, \quad (11)$$

$$rA^T(G_1)X_1 + r\sum_{j=2}^{p+1} X_j = \lambda X_l, \quad l = p+2, p+3, \cdots, k+1. \quad (12)$$

Next, two cases will be discussed.
Case 1 : $\lambda \neq 0$ From Eq.(11) and Eq.(12), it follows $X_2 = X_3 = \cdots X_{p+1}$ and $X_{p+2} = X_{p+3} = \cdots X_{k+1}$. Consequently, from Eq.(12) and Eq.(13), we derive

$$X_2 = \frac{r(\lambda + rq)}{\lambda^2 - r^2pq} A^T(G_1)X_1 \quad and$$

$$X_{p+2} = \frac{r(\lambda + rq)}{\lambda^2 - r^2pq} A^T(G_1)X_1. \quad (13)$$

Then from Eq.(11) and Eq.(13), we obtain

$$A(G_1)X_1 + rA(G_1)p\frac{r(\lambda + rq)}{\lambda^2 - r^2pq} A^T(G_1)X_1 + rA(G_1)q\frac{r(\lambda + rq)}{\mu^2 - r^2pq} A^T(G_1)X_1 = \lambda X_1.$$

Notice that $\sigma(G_1) = \{\lambda_1(G_1), \lambda_1(G_2), \cdots, \lambda_{n_1}(G_1)\}$, so we can obtain

$$\lambda^3 - \lambda_i(G_1)\lambda^2 - (r^2pq + r^2p\lambda_i^2(G_1) + r^2q\lambda_i^2(G_1))\lambda - r^2pq(2\lambda_i^2(G_1)) - \lambda_i(G_1)) = 0,$$

which implies that λ is a root of the cubic polynomial $x^3 - \lambda_i(G_1)x^2 - (r^2pq + r^2p\lambda_i^2(G_1) + r^2q\lambda_i^2(G_1))x - r^2pq(2\lambda_i^2(G_1)) - \lambda_i(G_1)) = 0, i = 1, 2, \cdots, n_1$.

Case 2 : $\lambda = 0$
It is easily known that 0 is one of the generalized adjacency eigenvalues of $W(G_1 \star G_2)$. Moreover, as the total number of nonzero is $3n_1$, we can obtain that 0 is the eigenvalue with multiplicity $n_1(p+q-2)$. Then Theorem 2.2 is obtained.

Theorem 2.2 Let G_1 be a graph on n_1 vertices, G_2 be a complete bipartite graph on $k = p + q$ vertices. Suppose $\sigma(G_1) = \{\lambda_1(G_1), \lambda_2(G_1), \cdots, \lambda_{n_1}(G_1)\}$. Then the generalized adjacency spectra of $W(G_1 \star G_2)$ are as follows.

(i) $0 \in \sigma(G_1 \star G_2)$ with multiplicity $n_1(p+q-2)$.
(ii) $\lambda \in \sigma(G_1 \star G_2)$ with multiplicity 1 where λ is a root of the cubic polynomial $x^3 - \lambda_i(G_1)x^2 - (r^2pq + r^2p\lambda_i^2(G_1) + r^2q\lambda_i^2(G_1))x - r^2pq(2\lambda_i^2(G_1)) - \lambda_i(G_1)) = 0, i = 1, 2, \cdots, n_1$.

III. Laplacian spectra of the weighted $G_1 \star G_2$

In this section, we will obtain the Lapalcian spectra of the weighted graph $G_1 \star G_2$. Let $s_n(i)$ denote the strength of vertex i in G. It is defined by the sum of its linked edges weight. S_n denotes the diagonal strength matrix of G with its ith diagonal entry being the strength $s_n(i)$ of vertex i. The Laplacian matrix of G is defined by $L(G) = S(G) - W(G)$. The Laplacian spectra of G is denoted by $l(G)$ which are the eigenvalues of $L(G)$.

According to the construction of the weighted $G_1 \star G_2$, the Lapalcian matrix is as follows:
$L(G_1 \star G_2) =$

$$\begin{pmatrix} r_1n_2d_1I_{n_1} + L(G_1) & -rj_{n_2}^T \otimes A(G_1) \\ -r(j_{n_2} \otimes A(G_1))^T & r(L(G_2) + rd_1I_{n_2}) \otimes I(n_1) \end{pmatrix}, \quad (14)$$

where $j_{n_2}^T$ denote the row vector with order n_2 and all elements are 1.

The eigenvalue μ of $L(G_1 \star G_2)$ with corresponding eigenvector is $X = [X_1X_2 \cdots X_{k+1}]^T$. Then

$$L(G_1 \star G_2)X = \mu X. \quad (15)$$

Next, we determine the Laplacian spectra of $G_1 \star G_2$ for the following two cases.

Case1 : X_1 is nonzero vector
From Eqs.(14) and (15), one knows

$$(L(G_1) + rn_2d_1I_{n_1})X_1 - rA(G_1)(X_2 + X_3 + \cdots X_{n_2+1}) = \mu X_1. \quad (16)$$

Meanwhile, set $E_i = (\underbrace{0, \cdots, 0}_{\text{i-1}}, I_n, \underbrace{0, \cdots, 0}_{\text{k-i}})$, it also satisfies that

$$\begin{cases} -rA^T(G_1)X_1 + rE_1[(L(G_2) + d_1I_{n_2}) \otimes I(n_1)][X_2 \cdots X_{n_2+1}]^T = \mu X_2, \\ -rA^T(G_1)X_1 + rE_2[(L(G_2) + d_1I_{n_2}) \otimes I(n_1)][X_2 \cdots X_{n_2+1}]^T = \mu X_3, \\ \vdots \\ -rA^T(G_1)X_1 + rE_{n_2}[(L(G_2) + d_1I_{n_2}) \otimes I(n_1)][X_2 \cdots X_{n_2+1}]^T = \mu X_{n_2+1}. \end{cases}$$

By the above equation, it gives

$$-rn_2A(G_1)^TX_1 + rd_1(X_2 + X_3 + \cdots X_{n_2+1}) = \mu(X_2 + X_3 + \cdots X_{n_2+1}), \quad (17)$$

namely,

$$(X_2 + X_3 + \cdots X_{n_2+1}) = \frac{-rn_2}{\mu - rd_1} A^T(G_1)X_1. \quad (18)$$

Substituting (18) to (17), we have

$$[L(G_1) + rn_2 d_1 I_{n_1}]X_1 + \frac{r^2 n_2}{\mu - rd_1} A(G_1)A^T(G_1)X_1$$

$$= \mu X_1 \quad (19)$$

Assume that the spectra of $L(G_1)$ is $l(G_1) = \{\mu_1(G_1), \mu_2(G_1), \cdots, \mu_{n_1}(G_1)\}$. Based on Eq.(20), one has

$$\mu^2 - (rd_1 + \mu_j(G_1) + rn_2 d_1)\mu + rd_1\mu_j(G_1) + 2r^2 n_2 \mu_j(G_1)d_1$$

$$-r^2 n_2 \mu_j^2(G_1) = 0, j = 1, 2, \ldots, n_1.$$

Evidently, the above equation yields

$$\lambda_{1,2} = \frac{rd_1 + \mu_j(G_1) + rn_2 d_1 \pm \sqrt{\Delta_1}}{2},$$

where $\Delta_1 = (rd_1 + \mu_j(G_1) + rn_2 d_1)^2 - 4\mu_j(G_1)(rd_1 + 2r^2 n_2 d_1 - r^2 n_2 \mu_j(G_1))$, $j = 1, 2, \ldots, n_1$.

Case 2 : X_1 is zero vector

The similarly consideration of Eq.(17) and Eq.(18), one gets

$$rn_2 d_1 I_{n_1} X_1 - rA(G_1)(X_2 + X_3 + \cdots X_{n_2+1}) = 0. \quad (20)$$

$$r[(L(G_2) + d_1 I_{n_2}) \otimes I(n_1)][X_2 X_3 \cdots X_{n_2+1}]^T$$

$$= \mu[X_2 X_3 \cdots X_{n_2+1}]^T. \quad (21)$$

Notice that the spectra of $L(G_2)$ is $l(G_2) = \{\mu_1(G_2), \mu_2(G_2), \cdots, \mu_{n_2}(G_2)\}$, then we can straightforward obtain that

$$\mu = r(\mu_j(G_2) + d_1), j = 2, 3, \cdots, n_2.$$

From Eq.(22), we can easily know that the multiplicity of $\mu = r(\mu_j(G_2) + d_1)$ is n_1. Then the following result is obtained.

Theorem 3.1 Let G_1 be the d_1-regular graph with n_1 vertices and m_1 edges, G_2 be any graph with n_2 vertices and m_2 edges, respectively. Assume that the Laplacian spectrum of G_1 and G_2 are $l(G_1) = \{\mu_1(G_1), \mu_2(G_1), \cdots, \mu_{n_1}(G_1)\}$ and $l(G_2) = \{\mu_1(G_2), \mu_2(G_2), \cdots, \mu_{n_2}(G_2)\}$. Then the Laplacian eigenvalues of $G_1 \star G_2$ are as follows:

(i)

$$\frac{rd_1 + \mu_j(G_1) + rn_2 d_1 \pm \sqrt{\Delta_1}}{2} \in l(G_1 \star G_2),$$

where $\Delta_1 = (rd_1 + \mu_j(G_1) + rn_2 d_1)^2 - 4\mu_j(G_1)(rd_1 + 2r^2 n_2 d_1 - r^2 n_2 \mu_j(G_1))$ with multiplicity 1 for $i = 1, 2, \cdots, n_1$

(ii)$\mu_j(G_2) + d_1 \in l(G_1 \star G_2)$ with multiplicity n_1 for $j = 2, 3, \cdots, n_2$.

IV. Signless Laplacian spectra of the weighted $G_1 \star G_2$

A. G_2 is a d_2-regular graph

The signless Laplacian matrix of $G_1 \star G_2$ is as follows.
$Q(G_1 \star G_2) =$

$$\begin{pmatrix} r_1 n_2 d_1 I_{n_1} + Q(G_1) & -rj_{n_2}^T \otimes A(G_1) \\ -r(j_{n_2} \otimes A(G_1))^T & r(Q(G_2) + rd_1 I_{n_2}) \otimes I(n_1) \end{pmatrix}.$$

Next we pay attention to determine the signless Laplacian spectra of $G_1 \star G_2$. Given the eigenvalue δ, its corresponding eigenvector of $Q(G_1 \star G_2)$ is $X = [X_1 X_2 \cdots X_{k+1}]^T$. Next, we obtain the desired result from the below two cases with $\delta \neq r(2d_2 + d_1)$.

Case 1 : X_1 is nonzero vector

Since the definitions of eigenvalues and eigenvectors, we obtain

$$(Q(G_1) + rn_2 d_1 I_{n_1})X_1 + rA(G_1)(X_2 + X_3 + \cdots X_{n_2+1})$$

$$= \delta X_1. \quad (22)$$

Meanwhile, set $E_i = (\underbrace{0, \cdots, 0}_{\text{i-1}}, I_n, \underbrace{0, \cdots, 0}_{\text{k-i}})$, it also satisfies

$$\begin{cases} rA^T(G_1)X_1 + rE_1[(Q(G_2) + d_1 I_{n_2}) \otimes I(n_1)][X_2 \cdots X_{n_2+1}]^T = \delta X_2, \\ rA^T(G_1)X_1 + rE_2[(Q(G_2) + d_1 I_{n_2}) \otimes I(n_1)][X_2 \cdots X_{n_2+1}]^T = \delta X_3, \\ \vdots \\ rA^T(G_1)X_1 + rE_{n_2}[(Q(G_2) + d_1 I_{n_2}) \otimes I(n_1)][X_2 \cdots X_{n_2+1}]^T = \delta X_{n_2+1}. \end{cases}$$

By the above equation, it gives

$$rn_2 A(G_1)^T X_1 + r(2d_2 + d_1)(X_2 + X_3 + \cdots X_{n_2+1})$$

$$= \delta(X_2 + X_3 + \cdots X_{n_2+1}), \quad (23)$$

namely,

$$(X_2 + \cdots X_{n_2+1}) = \frac{rn_2}{\delta - r(2d_2 + d_1)} A^T(G_1)X_1. \quad (24)$$

Substituting (24) to (23), we have

$$[Q(G_1) + rn_2 d_1 I_{n_1}]X_1 + \frac{r^2 n_2}{\delta - r(2d_2 + d_1)} A(G_1)$$

$$A^T(G_1)X_1 = \delta X_1. \quad (25)$$

Assume that the spectrum of $Q(G_1)$ is $q(G_1) = \{\delta_1(G_1), \delta_2(G_2), \cdots, \delta_{n_2}(G_1)\}$. Based on Eq.(25), we have

$$\delta^2 - [r(1 + n_2)d_1 + 2rd_2 + \delta_i(G_1)]x + [r(2d_2 + d_1 + 2rn_2 d_1)$$

$$\delta_i(G_1) + 2r^2 n_2 d_1 d_2 - r^2 n_2 \delta_i^2(G_1)] = 0, i = 1, 2, \ldots, n_1.$$

Evidently, the above equation yields

$$\delta_{1,2} = \frac{[r(1 + n_2)d_1 + 2rd_2 + \delta_i(G_1)] \pm \sqrt{\Delta_2}}{2},$$

where $\Delta_2 = [r(1 + n_2)d_1 + 2rd_2 + \delta_i(G_1)]^2 - 4[r(2d_2 + d_1 + 2rn_2 d_1)\delta_i(G_1) + 2r^2 n_2 d_1 d_2 - r^2 n_2 \delta_i^2(G_1)]$.

Case 2 : X_1 is zero vector

The similarly consideration of Eqs.(23) and (24), one gets

$$rA(G_1)(X_2 + X_3 + \cdots X_{n_2+1}) = \delta X_1. \quad (26)$$

$$r[(Q(G_2) + d_1 I_{n_2}) \otimes I(n_1)][X_2 X_3 \cdots X_{n_2+1}]^T \quad (27)$$

$$= \delta[X_2 X_3 \cdots X_{n_2+1}]^T.$$

Notice that the spectra of $Q(G_2)$ is $q(G_2) = \{\delta_1(G_2), \delta_2(G_2), \cdots, \delta_{n_2}(G_2)\}$, then we can straightforward obtain that

$$\mu = r(\delta_j(G_2) + d_1), j = 2, 3, \cdots, n_2.$$

From Eq.(27), we can determine that the multiplicity of $\mu = r(\mu_j(G_2) + d_1)$ is n_1. Then the following result is obtained.

Theorem 4.1 Let G_1 be the d_1-regular graph with n_1 vertices and m_1 edges, G_2 be the d_2-regular graph with n_2 vertices ,respectively. Assume that the Laplacian spectra of G_1 and G_2 are $q(G_1) = \{\mu_1(G_1), \mu_2(G_1), \cdots, \mu_{n_1}(G_1)\}$ and $q(G_2) = \{\mu_1(G_2), \mu_2(G_2), \cdots, \mu_{n_2}(G_2)\}$. Then the signless Laplacian spectra of $G_1 \star G_2$ are as follows:

(i)$\frac{[r(1+n_2)d_1+2rd_2+\delta_i(G_1)]\pm\sqrt{\Delta}}{2} \in q(G_1 \star G_2)$, where $\Delta = [r(1+n_2)d_1 + 2rd_2 + \delta_i(G_1)]^2 - 4[r(2d_2 + d_1 + 2rn_2d_1)\delta_i(G_1) + 2r^2n_2d_1d_2 - r^2n_2\delta_i^2(G_1)]$, with multiplicity 1 for $i = 1, 2, \cdots, n_1$.
(ii)$\mu_j(G_2) + d_1 \in q(G_1 \star G_2)$ with multiplicity n_1 for $j = 2, 3, \cdots, n_2$.

V. Applications

The main purpose of this section is to determine the Kirchhoff index and the number of spanning trees of the weighted neighbourhood corona graphs as applications of Theorem 3.1. First, we gave the definition for the Kirchhoff index and the number of spanning trees.

Kirchhoff index is an important index. In physics, the Kirchhoff index characterizes the average power consumed by a resistance network when current is arbitrarily injected into it. If the Kirchhoff index is large, the electrical energy consumed by the resistor network per unit time is large.
The Kirchhoff index [25] is defined as the sum of the resistance distance r_{ij} of G, see $[22-24]$ for more detail. The Kirchhoff index is denoted by the reciprocal of the Laplacian eigenvalues of G, namely $Kf(G) = \sum_{i<j} r_{ij} = |N(G)| \prod_{i=2}^{|N(G)|} \frac{1}{\mu_i}$.

Among the instruction of networks, spanning trees are one of the most important and fundamental indices. They are related to diverse aspects of networks, including resistor networks, self-organized criticality [26], and standard random walks [27]. For example, the number of spanning trees in a network is closely related to the effective resistance between two nodes in the network. The mean first-passage time between the two nodes is a fundamental quantity for random walks, which have wide distinct applications in various theoretical and applied fields, such as physics, chemistry, and computer science, among others. The number of spanning trees [16] of a given graph G can be expressed by $\tau(G) = \frac{1}{|N(G)|} \prod_{i=2}^{|N(G)|} \mu_i$.

Applying Theorem 3.1, we consider those two parameters by two cases.

Theorem 5.1 Let G_1 be the d_1-regula graph with order n_1 and size m_1, G_2 be any graph with order n_2 and size m_2, respectively. Assume that the Laplacian spectrum of G_1 and G_2 are $\sigma(G_1) = \{\mu_1(G_1), \mu_2(G_1), \cdots, \mu_{n_1}(G_1)\}$ and $\sigma(G_2) = \{\mu_1(G_2), \mu_2(G_2), \cdots, \mu_{n_2}(G_2)\}$. Then one has :

(i) $\tau(G_1 \star G_2)$

$$= \frac{1}{n_1 + n_1n_2} \prod_{i=1}^{n_1} \mu_i(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1))$$

$$\prod_{j=2}^{n_2} (r(d_1 + \mu_j(G_2))^{n_2}.$$

(ii)

$$Kf(G_1 \star G_2) = (n_1 + n_1n_2) \left[\sum_{i=2}^{n_2} \frac{n_2}{r(d_1 + \mu_j(G_2)} \right.$$

$$\left. + \sum_{i=1}^{n_1} \frac{rd_1 + \mu_j(G_1) + rn_2d_1}{\mu_j(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1))} \right].$$

Proof The order of the weighted neighbourhood corona graph $G_1 \star G_2$ is $N(G_1 \star G_2) = n_1 + n_1n_2$.
Let

$$x = \sqrt{y},$$

where $y = (rd_1 + \mu_j(G_1) + rn_2d_1)^2 - 4\mu_j(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1))$. Then

$$\prod_{i=1}^{n_1} \frac{rd_1 + \mu_j(G_1) + rn_2d_1 + x}{2} \prod_{i=1}^{n_1} \frac{rd_1 + \mu_j(G_1) + rn_2d_1 - x}{2}$$

$$= \prod_{i=1}^{n_1} \mu_j(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1)).$$

This gives

$$\prod_{i=1}^{|N(G)|} \mu_i = \prod_{j=1}^{n_1} \mu_j(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1))$$

$$\prod_{j=2}^{n_2} (r(d_1 + \mu_j(G_2))^{n_2}. \quad (28)$$

Combining Eq.(28) and Eq.(29), we obtain the desired result of the number of spanning trees as below.

$\tau(G_1 \star G_2)$

$$= \frac{1}{n_1 + n_1n_2} \prod_{j=1}^{n_1} \mu_j(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1))$$

$$\prod_{j=2}^{n_2} (r(d_1 + \mu_j(G_2))^{n_2}.$$

For the Kirchhoff index, one obtains

$$\sum_{i=1}^{n_1} \frac{2}{rd_1 + \mu_j(G_1) + rn_2d_1 + x} +$$

$$\sum_{i=1}^{n_1} \frac{2}{rd_1 + \mu_j(G_1) + rn_2d_1 - x}$$

$$= \sum_{i=1}^{n_1} \frac{rd_1 + \mu_j(G_1) + rn_2d_1}{\mu_j(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1))}.$$

This leads

$$\sum_{i=2}^{|N(G)|} \frac{1}{\mu_i} = \sum_{i=1}^{n_1} \frac{rd_1 + \mu_j(G_1) + rn_2d_1}{\mu_j(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1))} + \sum_{i=2}^{n_2} \frac{n_2}{r(d_1 + \mu_j(G_2)}.$$

Thus, one gets

$Kf(G_1 \star G_2)$

$$= (n_1 + n_1n_2) \left[\sum_{i=1}^{n_1} \frac{rd_1 + \mu_j(G_1) + rn_2d_1}{\mu_j(G_1)(rd_1 + 2r^2n_2d_1 - r^2n_2\mu_j(G_1))} + \sum_{i=2}^{n_2} \frac{n_2}{r(d_1 + \mu_j(G_2)} \right].$$

The desired results holds.

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