Axiomatic Analysis for Scaled Allocating Rule

Ruey-Rong Huang, Hui-Chuan Wei, Ching-Yun Huang and Yu-Hsien Liao

***Abstract*—Due to the tide of cross-disciplinary investigation among different areas, it seems imperative to improve resource-allocating situations by analyzing different perspectives and thinking. Many researchers have shown that there are various critical characteristics, such as changes in allocating patterns, reactive behavior, and the interplay and operational usefulness of decisions to be executed. Different from the expert assemblies, the regulations of thumb, or other pre-existing conceptions, this article aims to simulate, construct, derive and analyze the most effective resource allocating notion by applying game-theoretical methods under allocating analysis. The main investigative steps are as follows: (1) Different from existing allocation concepts, a new allocating rule is proposed by both focusing on the members and its energetic grades. (2) By using the axioms of consistency, grade completeness, grade synchronization, scaled criterion for models and unmixed excesses equality, several axiomatic results are adopted to dissect the applied rationality and the mathematical accuracy of this allocating notion. (3) These axioms and related axiomatic results would be further endowed with applied interpretations under real-world situations. (4) These axiomatic results and associated interpretations would be used to present that this allocating rule is a useful resource allocating mechanism. Correlative applications and comparisons are also mentioned.**

***Index Terms*—Member; energetic grade, allocating notion, axiomatic result.**

I. Introduction

Transformations and relevant calibration related to allocating notions arise an important impact on the sustainable development as a whole. For a long period, it has been the purpose of many investigators to evaluate how to properly operate under resource allocating situations. In this case, a suitable resource allocating mechanism for causing the proper adoption of restricted utility to the most required links and to reach the most prompt goals. Related topics have been investigated wildly. On the flip side, game-theoretical notions can be applied to resolve many procedures with interactive phenomenon by using numerous mathematical fields, and then suitable outcomes arise with acceptability, correctness, feasibility and rationality simultaneously. It further involves the establishment and analysis for how to engage the allocating notion under interactive procedures, such as the proportion making of decision implementation, the operation of utility distributing and so on. Thus, game-theoretical outcomes are diffusely used among numerous fields, such as management engineering, environmental analysis, economic sustainability, orientation formulation, biochemical science and so on. Game-theoretical allocating rules have been adopted to examine the ability of each member under a situation. For example, the *core* is the collection of associated outcomes matching coalitional rationality and efficiency under utility distributing procedures. Ransmeier [13] introduced the *equal allocation non-separable costs* (EANSC) to assess the peak yield for dams engaged by the Tennessee Valley Authority. Shapley [14] proposed the *Shapley value* to resolve the utility distributing procedures by collecting the entirly operating expected value for each member. Based on the operating notion of the EANSC, members first obtain its marginal influences and then allocate equally the rest of assets. Further, Hsieh and Liao [4] firstly applied the individual index to introduce the *pseudo equal allocation non-separable costs* (PEANSC), and then defined a reduced model and associated consistency to illustrate that the PEANSC is a equitable allocating notion matching several practical axioms. Based on the operating notion of the PEANSC, members first obtain its individual influences and then allocate equally the rest of utility. The principal differentiation is that the EANSC is emanated from "marginal influences" of members, and the PEANSC is stemmed from "individual influences" of members.

Based on traditional side-payment consideration, a distributing concept is formed by focusing on all the coalitions generated by participated members. This implies that the behaviors available for every member are either to operate completely under a process or not to operate at all. Under real-world situations, but, allocating concepts often vary relatively to each other in response to the promptly changing interplaies among members, coalitions, and environmental situations. Each member will be offered with a particular amount of energetic grades, and thus its capability might be distinct. Thus, a *multi-choice side-payment consideration* could be pondered as an extended analogue of a traditional side-payment consideration in which each member applies numerous energetic grades to participate. Several allocating notions further have been analyzed under multi-choice side-payment considerations. By evaluating entire affects for a given member under multi-choice clan considerations, Hwang and Liao [6] proposed a generalized core concept by considering *duplicate behavior among members and its energetic grades*; Liao [8] defined an extended EANSC by adopting *maximal pure affects of all members among its energetic grades*; Nouweland et al. [12] pondered an extension of the Shapley value [14] by evaluating *replicated behavior among members among its energetic grades*. Furthermore, Hwang and Li [5] consider an extended core by both evaluating the members and its energetic grades under multi-choice consideration. Related researches also could be found in Cheng et al. [2], Hwang and Liao [7], Liao et al.

R.R. Huang is a lecturer of General Education Center, Chung-Jen Junior College of Nursing, Health Sciences and Management, 600 Chiayi, Taiwan. (e-mail: m021@cjc.edu.tw)

H.C. Wei is a professor of Department of Adult and Continuing Education, National Chung Cheng University, 600 Chiayi, Taiwan. (e-mail: aduhcw@ccu.edu.tw)

C.Y. Huang is a professor of Department of Product and Interior Design, Nanhua University, 600 Chiayi, Taiwan. (e-mail: cyh@nhu.edu.tw)

Y.H. Liao is a professor of Department of Applied Mathematics, National Pingtung University, 900 Pingtung, Taiwan. (Corresponding author. e-mail: twincos@ms25.hinet.net)

[9] and so on.

The influence arisen by members might vary basing on numerous objective and subjective characteristics under real-world situations, such as the scale of the electoral district indicated by a member of the congress, the contribution arisen from a member of a company, and the haggling ability of a business staff might vary. Also, lack of symmetry might produce when different haggling abilities for distinct members are molded. In line with the pre-existing statements, one would hope that the resource might be partaken by the members and its energetic grades in proportion to *scales*. Scales yield involuntary under utility allocating situations. For instance, one might be dealing with resource allocating among investing projects. Therefore, the scales could be sent to the profitability of the various choices of all projects. In the discussion of allotting journey costs among numerous places seeked, the scales could be the volume of days spent at each one (cf. Shapley [14]).

On the strength of the mentioned statements, one motivation for this paper could be arisen:

- As stated above, the core, the EANSC and the Shapley value have been generalized to multi-choice side-payment considerations. Whether the PEANSC could be extended to be the most prompt utility allocating notion by both applying multi-choice notion and scales.

This paper is aimed to resolving this motivation. The major consequences are as follows.

- By building on the allocating rules due to Hwang and Liao [5] and Hsieh and Liao [4] under multi-choice side-payment considerations, the *cumulative scaled-single rule* (CSSR) is pondered by simultaneously focusing on the members, its energetic grades and scales in Section 2.
- To dissect the mathematical accuracy and the applied rationality of the CSSR, in Section 3, some game-theoretical axioms are adopted to represent that the CSSR is the unique allocating rule matching consistency, grade completeness, grade synchronization, scaled criterion for models and unmixed excesses equality.
- By applying these axiomatic processes to utility allocating conditions, these axioms and related axiomatic results would be further endowed with applied interpretations to present the applicability and the plausibility of the CSSR in Section 4. Related applications and comparisons would be also submitted throughout this paper.

II. Preliminaries

Let EG be the universe of members, for instance, the collection formed by all members in a country. Each $a \in EG$ is said to be a member of EG, for instance, a national of a country. For $a \in EG$ and $p_a \in \mathbb{N}$, $P_a = \{0, 1, \cdots, p_a\}$ could be treated as the energetic grade space of member a and $P_a^+ = P_a \setminus \{0\}$, where 0 indicates no partaking. Suppose that $G \subseteq EG$ is the maximal set of all members of an interactive procedure in EG, for instance, all members of a university in a country. Let $P^p = \prod_{a \in G} P_a$ be the product collection of the energetic grade (decision) expanses of all members of G. Indicate 0_G to be the zero vector throughout $\mathbb{R}^G$.

A **multi-choice model** is denoted by (G, p, S), where $G \neq \emptyset$ is a finite collection of participating members, $p = (p_a)_{a\in G}$ is the vector that shows the maximal capacity of total energetic grades for each member, and $S : P^G \to \mathbb{R}$ is a map with $S(0_G) = 0$ which assorts to every $\zeta = (\zeta_a)_{a\in G} \in P^G$ the utility that the members can generate when every member a takes energetic grade ζ_a. As $p \in \mathbb{R}^G$ is fixed over this article, one may denote (G, S) rather than (G, p, S).

Indicate the family of all multi-choice models to be $\mathcal{MCM}$. Taken $(G, S) \in \mathcal{MCM}$ and $\zeta \in P^G$, one may denote $L(\zeta) = \{a \in G|\ \zeta_a \neq 0\}$, ζ_H to be the restriction of ζ at H for each $H \subseteq G$ and $\|\zeta\| = \sum_{a\in G} \zeta_a$.

Taken $(G, S) \in \mathcal{MCM}$, let $M^p = \{(a, k_a) \mid a \in G, k_a \in P_a^+\}$. An **allocating rule** on $\mathcal{MCM}$ is a function ϕ assorting to every $(G, S) \in \mathcal{MCM}$ a vector

$$\phi(G, S) = \Big(\phi_{a,k_a}(G, S)\Big)_{(a,k_a)\in M^G} \in \mathbb{R}^{M^G}.$$

In briefly, $\phi_{a,k_a}(G, S)$ is the influence or the payoff of the member a when it revolves with grade k_a in (G, S). For convenience, one may suppose that $\phi_{a,0}(G, S) = 0$ for every $a \in G$.

As stated above, scales appear involuntarily under resource allocating processes. For example, one may be dealing with a matter of utility allocating for investing projects. Hence, the scales might be sent to the profitability of the distinct alternatives of all projects. Scales might be set in contracts approved by the proprietors of a townhouse and applied to allot the expenses of building or managing common facilities. d is called a **scale map for grades** if $d : \cup_{a\in G} P_a^+ \to \mathbb{R}^+$ is a positive function. Given $(G, S) \in \mathcal{MCM}$, scale map for grades d and $\zeta \in P^G$, one might define that $\|\zeta\|_d = \sum_{a\in G} \sum_{k_a=1}^{\zeta_a} d(k_a)$.

A multi-choice extension of the PEANSC (Hsieh and Liao [4]) is provided as follows.

Definition 1: The **cumulative scaled-single rule (CSSR)** of multi-choice models, $\overline{\Phi^d}$, is the map on $\mathcal{MCM}$ which associates to each $(G, S) \in \mathcal{MCM}$, each scale map for grades d, each member $a \in G$ and each $k_a \in P_a^+$ the effect or the payoff

$$\begin{aligned} &\overline{\Phi^d}_{a,k_a}(G, S) \\ = \quad & \Phi_{a,k_a}(G, S) + \tfrac{d(k_a)}{\|p\|_d} \cdot \Big[S(p) - \sum_{t\in G} \sum_{k_t=1}^{p_t} \Phi_{t,k_t}(G, S)\Big], \end{aligned}$$

where $\Phi_{a,k_a}(G, S) = S\big(k_a, 0_{G\setminus\{a\}}\big)$ is the **cumulative single-grade affect** of the member a from its grade 0 to k_a. Based on the allocating rule $\overline{\Phi^d}$, members firstly partake its cumulative single-grade affects with corresponding grades, and then distribute proportionally the rest of resource by scales for grades.

Remark 1: Chang et al. [1] introduced the multi-choice weighted-individual index as follows. The **multi-choice weighted-individual index** (MWII), $\overline{\Psi^d}$ over multi-choiceconsiderations, is the map on $\mathcal{MCM}$ which assorts to every $(G, S) \in \mathcal{MCM}$, every scale map for grades d, every member $a \in G$ and each $k_a \in P_a^+$ the affect

$$\begin{aligned} &\overline{\Psi^d}_{a,k_a}(G, S) \\ = \quad & \Psi_{a,k_a}(G, S) + \tfrac{d(k_a)}{\|p\|_d} \cdot \Big[S(p) - \sum_{t\in G} \sum_{k_t=1}^{p_t} \Psi_{t,k_t}(G, S)\Big], \end{aligned}$$

where $\Psi_{a,k_a}(G,S) = S\big(k_a, 0_{G\setminus\{a\}}\big) - S\big(k_a - 1, 0_{G\setminus\{a\}}\big)$ is the **individual-level variation** of the member a from its grade $k_a - 1$ to k_a. By applying the power index $\overline{\Psi^d}$, members get its individual-level variations with relevant grades respectively, and then distribute the rest of resource proportionally by scales for grades.

The major differentiation is that the MWII due to Chang et al. [1] is emanated from "individual-level variation", and the CSSR is stemmed from "cumulative single-grade effect". Surely, related axioms and axiomatic techniques among this two allocating rules are different.

To appear how the notions of multi-choice model and the CSSR can be used and to let its meaning more visible, a motivating example is provided as follows.

Example 1: The kinds of resources held by a corporation are multiplex, and they present distinct shapes, such as, brand image, capital, equipment, human resources, etc. Furthermore, there are numerous operative sections in a biological technique corporation, for instance, the board of trustees is in charge of the operational trend of the corporation; the accounting section confines and operates the capital management for the corporation; the research and development section provides new development notion or revises and updates developments as needed; the human resource section enlists staffs, downsizes or expands the operation force according to the corporation needs; the market investigative section evaluates current market tendency; the production section is responsible for outcome manufacturing and quality-related assignments; the marketing section establishes marketing programs for objective consumers; the risk control section analyzes operational risks. Therefore, one could assume that G is the set of all operational sections of this corporation. The operating level of every section is not set in stone, and there exist distinct operating levels in response to distinct conditions. Thai is, every section $a \in G$ might possess distinct operating levels p_a Moreover, the operating levels among sections also influence one another as a result of distinct conditions. For instance, the market investigative section provides the popular tendency and demands of biopharmaceuticals, then the research and development section introduces product development notions and a products mix, the marketing section establishes relevant sales programs and market prospect, the accounting section, the human resources section and the production section carry out capital allocation, output capacity evaluation and related rightsizing, and finally, the project is presented to the board of trustees to determine the operational trend and proceed necessary adjustment. To put it another way, each section will interrelate over the framework of the condition, submit and assess distinct implementation projects; Thus, there would be distinct combination of operating levels and relevant advantages. Therefore, each section would interflow with other sections for distinct conditions, and take distinct operating levels $\gamma_a \in P_a$ for distinct conditions and other distinct sections. Hence, a map S can be applied to compute the benefits of operating combination $\gamma \in P^G$ given by total operational sections (i.e. $S(\gamma)$). Hence, the resource-allocating procedure of a biological technique corporation can be generalized to be a multi-choice model (G, S). Moreover, the importance of distinct sections is indeed different from the qualities of its operations. For instance, the board of trustees is necessarily more important than other sections in policy-making conferences, and the marketing section is surely more significant than other sections in fund-raising activities. It is rational that one could apply a scale map d to weight each operating level of each section against different conditions.

To evaluate the influence of each section, applying the allocating rules defined in this paper, one would first assess the single-grade contribution that every section has accumulated over former projects based on various and alternative allocating levels, which is the cumulative single-grade contribution Φ proposed in Definition 1. The rest of shared effects should be also alloted proportionally by scales for grades of all sections, which is the cumulative individual rule $\overline{\Phi^d}$ proposed in Definition 1. Hence, it is very considerable to effectually consort sections to apply relevant operating levels for allocating utility, so that limited utility can arise the most optimal productivity. It is expected that an assessment condition for the utility-allocating processes would eventually be exploited by considering real-world situations with the game-theoretical outcomes of the CSSR under the mode that operators exert multiple operating levels.

Subsequently, a numerical instance is presented as follows.

Example 2: Let $(G, S) \in \mathcal{MCM}$ be a utility-distributing condition, where $U = \{a, b, c\}$ is the collection of members, $p = (1, 2, 1)$ is the operational grade vector that shows the maximal amount of total energetic grades for each member, $S : P^G \to \mathbb{R}$ is a utility function with $S(0, 0, 0) = 0$ which allots to every $\zeta = (\zeta_x)_{x\in G}$ the utility that the members can produce if every member a manipulates at energetic grade ζ_x, and $d(1_a) = 6$, $d(1_b) = 4$, $d(2_b) = 2$, $d(1_c) = 8$ are the corresponding scales of all members under this situations. Further, assume that $S(1, 2, 1) = 24$, $S(1, 1, 1) = 4$, $S(1, 2, 0) = -6$, $S(1, 0, 1) = 8$, $S(0, 2, 1) = 18$, $S(0, 1, 1) = -10$, $S(1, 1, 0) = 14$, $S(1, 0, 0) = -4$, $S(0, 1, 0) = 6$, $S(0, 2, 0) = 8$, $S(0, 0, 1) = -2$ and $S(0, 0, 0) = 0$ be the resource that the members can produce under all operational behavior. By adpoting Definition 1,

$$\begin{array}{rclrcl}
\Phi_{a,1}(G,S) &=& -4, & \Phi_{b,1}(G,S) &=& 6,\\
\Phi_{b,2}(G,S) &=& 8, & \Phi_{c,1}(G,S) &=& -2,\\
\overline{\Phi^d_{a,1}}(G,S) &=& 0.8, & \overline{\Phi^d_{b,1}}(G,S) &=& 9.2,\\
\overline{\Phi^d_{b,2}}(G,S) &=& 9.6, & \overline{\Phi^d_{c,1}}(G,S) &=& 4.4.
\end{array}$$

By Definition 1, it is clear to have the affect of each member if it takes specific grade in (G, S). For example, the affect of member b is $\overline{\Phi^d_{b,2}}(G, S) = 9.6$ when b takes its grade 2 in (G, S).

III. Axiomatic processes

In order to appear the rationality of the CSSR, this section would demonstrate that the CSSR can be characterized by some meaningful axioms. Therefore, some useful axioms should be needed. Let ϕ be an allocating rule on $\mathcal{MCM}$.

- ϕ matches **grade completeness (GCLS)** if $\sum_{a\in G} \sum_{k_a=1}^{p_a} \phi_{a,k_a}(G, S) = S(p)$ for all $(G, S) \in \mathcal{MCM}$. GCLS states that all members allot whole resource entirely.
- ϕ matches **scaled criterion for models (SCM)** if $\phi(G, S) = \overline{\Phi^d}(G, S)$ for all $(G, S) \in \mathcal{MCM}$ with $|G| \leq 2$ and for all scale maps for grades d. SCM

is a self-sufficient condition if there exists unique one member under the situation, but if there exist two members under the situation, each of them first receives what they could have produced alone, and at the last of the reacting procedure, they share whole the rest of profits and losses.

- Given $(G,S) \in \mathcal{MCM}$ and $(a,k_a) \in M^G$, the **normalized excesses** of k_a is defined to be $e^{k_a}_{\phi}(G,S) = \phi_{a,k_a}(G,S) - S(k_a, 0_{G\setminus\{a\}})$. ϕ matches **unmixed excesses equality (GEE)** if for all scale maps for grades d, for all $(G,S) \in \mathcal{MCM}$ and for all $\zeta \in P^G$ with $S(k_a, 0, \zeta_{N\setminus\{a,b\}}) = S(0, k_b, \zeta_{N\setminus\{a,b\}})$ for some $(a,k_a),(b,k_b) \in M^G$, it holds that $\frac{1}{d(k_a)} \cdot e^{k_a}_{\phi}(G,S) = \frac{1}{d(k_b)} \cdot e^{k_b}_{\phi}(G,S)$. UEE states that the excesses of two energetic grades should be the same if the cumulative single-grade effects of these two members are equal.
- ϕ matches **grade synchronization (GSRN)** if for all $(G,S),(G,D) \in \mathcal{MCM}$ with $S(\zeta) = D(\zeta) + \sum_{a\in L(\zeta)} \mu_{a,\zeta_a}$ for some $\mu \in \mathbb{R}^{M^G}$ and for all $\zeta \in P^G$, $\phi(G,S) = \phi(G,D) + \mu$. GSRN can be asserted as a mighty weakness of *additivity*.

The interaction among above axioms and utility-allocating procedures will be stated in Section 4.

A multi-choice case of the reduction due to Hsieh and Liao [4] would be pondered as follows. Taken $(G,S) \in \mathcal{MCM}$, $H \subseteq G$ and an allocating rule ϕ, the **reduced model** (H, S^{ϕ}_H) **related to** H **and** ϕ is defined by for all $\zeta \in P^H$,

$$S^{\phi}_H(\zeta) = \begin{cases} 0 & \zeta = 0_H, \\ S(\zeta_a, 0_{G\setminus\{a\}}) & H \geq |2|,\ L(\zeta) = \{a\} \text{ for some } a, \\ S(\zeta, p_{G\setminus H}) - \sum\limits_{a\in G\setminus H}\sum\limits_{k_a=1}^{p_a} \phi_{a,k_a}(G,S) & \text{otherwise..} \end{cases}$$

The *consistency* axiom may be asserted as follows. Let ϕ be an allocating rule on $\mathcal{MCM}$. For any couple of two members over a situation, one would propose a "reduced consideration" among them by analyzing the amounts remaining after the rest of the members are given the affects allotted from ϕ. Thus, ϕ is *consistent* if it usually emerges the same effects as in the original situation if it is applied to arbitrary reduced consideration. Formally, an allocating rule ϕ matches **consistency (CSY)** if for every $(G,S) \in \mathcal{MCM}$ with $|G| \geq 3$, for every $H \subseteq G$ with $|H| = 2$ and for every $(a,k_a) \in M^H$, $\phi_{a,k_a}(G,S) = \phi_{a,k_a}(H, S^{\phi}_H)$.

In the following, some results are provided by applying the CSY axiom.

Lemma 1: The CSSR $\overline{\Phi^d}$ matches CSY.

Proof: Let $(G,S) \in \mathcal{MCM}$ with $|G| \geq 3$ and $H \subseteq G$ with $|H| = 2$. Assume that $H = \{a,h\}$. By the definition of $\overline{\Phi^d}$, for all $(a,k_a) \in M^H$,

$$\begin{aligned} &\overline{\Phi^d}_{a,k_a}(H, S^{\overline{\Phi^d}}_H) \\ = \ & \Phi_{a,k_a}(H, S^{\overline{\Phi^d}}_H) \\ &+ \frac{d(k_a)}{\|p_H\|_d}\Big[S^{\overline{\Phi^d}}_H(p_H) - \sum_{t\in H}\sum_{k_t=1}^{p_t} \Phi_{t,k_t}(H, S^{\overline{\Phi^d}}_H)\Big]. \end{aligned} \tag{1}$$

By definitions of Φ and $S^{\overline{\Phi^d}}_H$, for all $k_a \in P^+_a$,

$$\begin{aligned} \Phi_{a,k_a}(H, S^{\overline{\Phi^d}}_H) &= S^{\overline{\Phi^d}}_H(k_a, 0) \\ &= S\big(k_a, 0_{G\setminus\{a\}}\big) \\ &= \Phi_{a,k_a}(G,S). \end{aligned} \tag{2}$$

Hence, by equations (1), (2) and definitions of $S^{\overline{\Phi^d}}_H$ and $\overline{\Phi^d}$,

$$\begin{aligned} &\overline{\Phi^d}_{a,k_a}(H, S^{\overline{\Phi^d}}_H) \\ = \ & \Phi_{a,k_a}(G,S) + \frac{d(k_a)}{\|p_H\|_d}\Big[S^{\overline{\Phi^d}}_H(p_H) - \sum_{t\in H}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big] \\ = \ & \Phi_{a,k_a}(G,S) + \frac{d(k_a)}{\|p_H\|_d}\Big[S(p) - \sum_{t\in G\setminus H}\sum_{k_t=1}^{p_t}\overline{\Phi^d}_{t,k_t}(G,S) \\ &\qquad - \sum_{t\in H}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big] \\ = \ & \Phi_{a,k_a}(G,S) + \frac{d(k_a)}{\|p_H\|_d}\Big[\sum_{t\in H}\sum_{k_t=1}^{p_t}\overline{\Phi^d}_{t,k_t}(G,S) \\ &\qquad - \sum_{t\in H}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big] \\ = \ & \Phi_{a,k_a}(G,S) + \frac{d(k_a)}{\|p_H\|_d}\Big[\frac{\|p_H\|_d}{\|p\|_d}\big[S(p) - \sum_{t\in G}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\big]\Big] \\ = \ & \Phi_{a,k_a}(G,S) + \frac{d(k_a)}{\|p\|_d}\Big[S(p) - \sum_{t\in G}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big] \\ = \ & \overline{\Phi^d}_{a,k_a}(G,S). \end{aligned}$$

Similarly, $\overline{\Phi^d}_{h,k_h}(H, S^{\overline{\Phi^d}}_H) = \overline{\Phi^d}_{h,k_h}(G,S)$ for all $k_h \in P^+_h$. So, the CSSR matches CSY. ■

Lemma 2: If an allocating rule ϕ matches SCM and CSY then it also matches GCLS.

Proof: Let ϕ be an allocating rule on $\mathcal{MCM}$ matching SCM and CSY, and $(G,S) \in \mathcal{MCM}$. It is trivial for $|G| \leq 2$ by SCM. Assume that $|G| \geq 3$. Let $t \in G$, consider the reduced model $\big(\{t\}, S^{\phi}_{\{t\}}\big)$. By definition of $S^{\phi}_{\{t\}}$,

$$E^{\phi}_{\{t\}}(p_t) = S(p) - \sum_{a\in G\setminus\{t\}}\sum_{k_a=1}^{p_a}\phi_{a,k_a}(G,S).$$

Since ϕ matches CSY, $\phi_{t,k_t}(G,S) = \phi_{t,k_t}\big(\{t\}, S^{\phi}_{\{t\}}\big)$ for all $k_t \in p_t$. Especially, $\phi_{t,p_t}(G,S) = \phi_{t,p_t}\big(\{t\}, S^{\phi}_{\{t\}}\big)$. On the other hand, by SCM of ϕ, $\sum\limits_{k_t=1}^{p_t}\phi_{t,k_t}(G,S) = E^{\phi}_{\{t\}}(p_t)$. Hence, $\sum\limits_{a\in G}\sum\limits_{k_a=1}^{p_a}\phi_{a,k_a}(G,S) = S(p)$, i.e., ϕ matches GCLS. ■

Remark 2: Based on definition of SCM and Definition 1, it is easy to see that the CSSR matches SCM. By applying Lemmas 1 and 2, the CSSR matches GCLS.

Inspired by Hart and Mas-Colell [3], the CSSR would be characterized by SCM and CSY.

Theorem 1: An allocating rule ϕ on $\mathcal{MCM}$ matches SCM and CSY if and only if $\phi = \overline{\Phi^d}$.

Proof: By Lemma 1, $\overline{\Phi^d}$ matches CSY. Clearly, $\overline{\Phi^d}$ matches SCM.

To analyze uniqueness, suppose that ϕ matches SCM and CSY on $\mathcal{MCM}$. By Lemma 2, ϕ matches GCLS absolutely. Let $(G,S) \in \mathcal{MCM}$. If $|G| \leq 2$, then by SCM of ϕ, $\phi(G,S) = \overline{\Phi^d}(G,S)$. The case $|G| > 2$: Let $a \in G$ and

$H=\{a,b\}$ for some $b\in G\setminus\{a\}$. For all $k_a\in P_a^+$,

$$\begin{array}{ll}\Phi_{a,k_a}(H,S_H^{\phi}) & =S_H^{\phi}(k_a,0)\\ & =S(k_a,0_{G\setminus\{a\}})\\ & \quad\text{(By definition of } S_H^{\phi})\\ & =S_H^{\overline{\Phi^d}}(k_a,0)\\ & \quad\text{(By definition of } S_H^{\overline{\Phi^d}})\\ & =\Phi_{a,k_a}(H,S_H^{\overline{\Phi^d}}).\end{array}\tag{3}$$

Further,

$$\begin{array}{ll} & \phi_{a,k_a}(G,S)-\overline{\Phi^d}_{a,k_a}(G,S)\\ = & \phi_{a,k_a}(H,S_H^{\phi})-\overline{\Phi^d}_{a,k_a}(H,S_H^{\overline{\Phi^d}})\\ & \text{(By CSC of } \phi \text{ and } \overline{\Phi^d})\\ = & \overline{\Phi^d}_{a,k_a}(H,S_H^{\phi})-\overline{\Phi^d}_{a,k_a}(H,S_H^{\overline{\Phi^d}})\\ & \text{(By SCM of } \phi \text{ and } \overline{\Phi^d})\\ = & \frac{d(k_a)}{\|p_H\|_d}\Big[S_H^{\phi}(p_H)-S_H^{\overline{\Phi^d}}(p_H)\Big].\\ & \text{(By equation (3) and SCM of } \phi \text{ and } \overline{\Phi^d})\end{array}\tag{4}$$

Similarly, for every $k_b\in P_b^+$,

$$\begin{array}{ll} & \phi_{b,k_b}(G,S)-\overline{\Phi^d}_{b,k_b}(G,S)\\ = & \frac{d(k_b)}{\|p_H\|_d}\Big[S_H^{\phi}(p_H)-S_H^{\overline{\Phi^d}}(p_H)\Big].\end{array}\tag{5}$$

By (4) and (5),

$$\begin{array}{ll} & \phi_{a,k_a}(G,S)-\overline{\Phi^d}_{a,k_a}(G,S)\\ = & \frac{d(k_a)}{d(k_b)}\Big[\phi_{b,k_b}(G,S)-\overline{\Phi^d}_{b,k_b}(G,S)\Big].\end{array}$$

This implies that $\phi_{a,k_a}(G,S)-\overline{\Phi^d}_{a,k_a}(G,S)=\alpha$ for all (a,k_a). It remains to demonstrate that $\alpha=0$. By GCLS of ϕ and $\overline{\Phi^d}$,

$$\begin{array}{ll}0 & =S(p)-S(p)\\ & =\sum_{a\in G}\sum_{k_a=1}^{p_a}\Big[\phi_{a,k_a}(G,S)-\overline{\Phi^d}_{a,k_a}(G,S)\Big]\\ & =\frac{\alpha\cdot\|p\|_d}{d(k_b)}\Big[\phi_{a,k_a}(G,S)-\overline{\Phi^d}_{a,k_a}(G,S)\Big].\end{array}$$

That is, $\alpha=0$. ■

Similar to Maschler and Owen [10], one would characterize the CSSR by means of consistency, grade completeness, grade synchronization and unmixed excesses equality.

Lemma 3: If an allocating rule ϕ on $\mathcal{MCM}$ matches GCLS, UEE and GSRN, then ϕ matches SCM.

Proof: Assume that an allocating rule ϕ matches GCLS, UEE and GSRN. Let $(G,S)\in\mathcal{MCM}$. The proof of $|G|=1$ could be done by GCLS of ϕ. Let $G=\{a,h\}$ for some $a\neq h$. A model (G,D) is defined to be that for every $\zeta\in P^G$,

$$D(\zeta)=S(\zeta)-\sum_{t\in L(\zeta)}S(\zeta_t,0_{G\setminus\{t\}}).$$

Clearly, for every $k_a\in P_a^+$ and for every $k_b\in P_b^+$,

$$\begin{array}{ll}D(k_a,0,0_{G\setminus\{a,h\}}) & =D(k_a,0)\\ & =S(k_a,0)-S(k_a,0)\\ & =0\\ & =S(0,k_b,0)-S(0,k_b,0)\\ & =D(0,k_b)\\ & =D(0,k_b,0_{G\setminus\{a,h\}}).\end{array}$$

That is, $D(k_a,0,\zeta_{G\setminus\{a,b\}})=D(0,k_b,\zeta_{G\setminus\{a,b\}})$ for every $\zeta\in P^G$. Further,

$$\begin{array}{ll}\frac{1}{d(k_a)}\cdot\phi_{a,k_a}(G,D) & =\frac{1}{d(k_a)}\cdot[\phi_{a,k_a}(G,D)-0]\\ & =\frac{1}{d(k_a)}\cdot[\phi_{a,k_a}(G,D)-D(k_a,0)]\\ & =\frac{1}{d(k_a)}\cdot e^{k_a}(G,D)\\ & =\frac{1}{d(k_b)}\cdot e^{k_b}(G,D)\\ & \quad\textbf{(by UEE of } \phi\textbf{)}\\ & =\frac{1}{d(k_b)}\cdot[\phi_{b,k_b}(G,D)-D(0,k_b)]\\ & =\frac{1}{d(k_b)}\cdot[\phi_{b,k_b}(G,D)-0]\\ & =\frac{1}{d(k_b)}\cdot\phi_{a,k_a}(G,D).\end{array}$$

By GCLS of ϕ,

$$\begin{array}{ll}D(p) & =\sum_{k_a=1}^{p_a}\phi_{a,k_a}(G,D)+\sum_{k_h=1}^{p_h}\phi_{h,k_h}(G,D)\\ & =\frac{\|p\|_d}{d(k_a)}\cdot\phi_{a,k_a}(G,D)\end{array}\tag{6}$$

for every $k_a\in P_a^+$. By GSRN of ϕ, equation (6) and definition of D,

$$\begin{array}{ll} & \phi_{a,k_a}(G,S)\\ = & S(k_a,0)+\frac{d(k_a)}{\|p\|_d}\cdot\Big[S(p)-\sum_{t\in G}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big]\\ = & \Phi_{a,k_a}(G,S)+\frac{d(k_a)}{\|p\|_d}\cdot\Big[S(p)-\sum_{t\in G}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big].\end{array}$$

Hence, ϕ matches SCM. ■

Lemma 4: The CSSR matches UEE.

Proof: Let $(G,S)\in\mathcal{MCM}$. Assume that

$$S(\zeta,k_a,0)=S(\zeta,0,k_b)$$

for some $(a,k_a),(b,k_b)\in M^G$ and for every $\zeta\in P^{G\setminus\{a,b\}}$. By taking $\zeta=0_{G\setminus\{a,b\}}$,

$$\begin{array}{ll} & S(k_a,0_{G\setminus\{a\}})\\ = & S(\zeta,k_a,0)\\ = & S(\zeta,0,k_b)\\ = & S(k_b,0_{G\setminus\{b\}}),\end{array}$$

i,e,.

$$\begin{array}{ll} & e_{\Phi}^{k_a}(G,S)\\ = & \Phi_{a,k_a}(G,S)-S(k_a,0_{G\setminus\{a\}})\\ = & \frac{d(k_a)}{\|p\|_d}\cdot\Big[S(p)-\sum_{t\in G}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big].\end{array}$$

Similarly,

$$\begin{array}{ll} & e_{\Phi}^{k_b}(G,S)\\ = & \frac{d(k_b)}{\|p\|_d}\cdot\Big[S(p)-\sum_{t\in G}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big].\end{array}$$

So,

$$\begin{array}{ll} & \frac{1}{d(k_a)}\cdot e_{\phi}^{k_a}(G,S)\\ = & \frac{1}{\|p\|_d}\cdot\Big[S(p)-\sum_{t\in G}\sum_{k_t=1}^{p_t}\Phi_{t,k_t}(G,S)\Big]\\ = & \frac{1}{d(k_b)}\cdot e_{\phi}^{k_b}(G,S).\end{array}$$

Thus, the CSSR $\overline{\Phi^d}$ matches UEE. ■

Lemma 5: The CSSR matches GSRN.

Proof: Let $(G,S),(G,D)\in\mathcal{MCM}$ with

$$S(\zeta)=D(\zeta)+\sum_{t\in L(\zeta)}\mu_{t,\zeta_t}$$

for some $\mu \in \mathbb{R}^{M^G}$ and for every $\zeta \in P^G$. For every $(a, k_a) \in M^G$,

$$\begin{aligned} & \Phi_{a,k_a}(G,S) \\ = \; & S(k_a, 0_{G\setminus\{p\}}) \\ = \; & D(k_a, 0_{G\setminus\{p\}}) + \mu_{a,k_a} \\ = \; & \Phi_{a,k_a}(G,D) + \mu_{a,k_a}. \end{aligned}$$

So,

$$\begin{aligned} & \overline{\Phi^d}_{a,k_a}(G,S) \\ = \; & \Phi_{a,k_a}(G,S) + \tfrac{d(k_a)}{\|p\|_d}\Big[S(p) - \sum_{b\in G}\sum_{k_b=1}^{p_b}\Phi_{b,k_b}(G,S)\Big] \\ = \; & \Phi_{a,k_a}(G,D) + \mu_{a,k_a} + \tfrac{d(k_a)}{\|p\|_d}\Big[D(p) + \sum_{t\in G}\sum_{k_t=1}^{p_t}\mu_{t,k_t} \\ & - \sum_{b\in G}\sum_{k_b=1}^{p_b}\Phi_{b,k_b}(G,D) - \sum_{b\in G}\sum_{k_b=1}^{p_b}\mu_{b,k_b}\Big] \\ = \; & \Phi_{a,k_a}(G,D) + \mu_{a,k_a} + \tfrac{d(k_a)}{\|p\|_d}\Big[D(p) \\ & - \sum_{b\in G}\sum_{k_b=1}^{p_b}\Phi_{b,k_b}(G,D)\Big] \\ = \; & \overline{\Phi^d}_{a,k_a}(G,D) + \mu_{a,k_a}. \end{aligned}$$

Thus, the CSSR $\overline{\Phi^d}$ matches GSRN. ■

Theorem 2: An allocating rule ϕ on $\mathcal{MCM}$ matches GCLS, UEE, GSRN and CSY if and only if $\phi = \overline{\Phi^d}$.

Proof: Based on definition of $\overline{\Phi^d}$, Remark 2 and Lemmas 1, 4, 5, it is shown that $\overline{\Phi^d}$ matches GCLS, CSY, UEE and GSRN. The rest of proofs could be finished by Theorem 1 and Lemma 3. ■

IV. Discussion and Conclusions

Based on the notion of cross-disciplinary investigation among different areas, this article would like to use axiomatic outcomes to verify the accuracy and the plausibility of the utility allocating notion by inquiring "how is the notion defined", "why does one consider the notion", "is such notion exact" and "how economical is such notion"?

By Sections 2 and 3, it is shown that the major advantage of the CSSR is that the CSSR of a multi-choice model totally exists and to produce an exact affect for a specific member operating with a specific energetic grade that different from the general proposition with multi-choice models, which producing a type of entire affect for a specific member by picking the marginal contributions of this member among its all energetic grades. One would like to claim that the CSSR can produce ‘ "suitable outcome" exactly over resource allocating processes. In order to present how the CSSR could be used and to rise its meaning more visible, one would further discuss the interaction among game-theoretical axioms and resource allocating processes.

1) **Grade completeness**: Suitable resource-allocating processes should make complete usage of whole utility. That is, a suitable resource-allocating process should meet the grade completeness axiom.
2) **Scaled criterion for models**: members own its particular properties of activities. Interplaies among members are often generate from two-member interplaies followed by coalitional interplaies. That is, a suitable resource-allocating process should content the scaled criterion for models axiom.
3) **Unmixed excesses equality**: If any two members are equal unmixed excesses to whole circumstances after the operation of member grouping, the affects of these two members should be the same. Thereupon, a suitable resource-allocating process should content the unmixed excesses equality axiom.
4) **Grade synchronization**: Suitable resource-allocating procedures, in which each member should be applied with the proper energetic grade to approach the aim, rather than the mete (small or large) basing on the energetic grade, should accomplish the most adequate efficacy in accordance with the proportionality standard. That is, a suitable resource-allocating procedure should match the grade synchronization axiom.
5) **Consistency**: Suitable resource-allocating procedures should be inspected under an iterative continuous procedure, and should present consistent outcomes. A suitable resource-allocating procedure should hence content the consistency axiom.

As presented in Section 2, one could have that the framework of utility-allocating procedures could be generalized as a multi-choice consideration. By using Theorems 1 and 2, it is clear that the CSSR is the unique allocating rule simultaneously matching consistency, grade completeness, grade synchronization, scaled criterion for models and unmixed excesses equality. By the items 1–5, it is also easy to have that the axioms of consistency, grade completeness, grade synchronization, scaled criterion for models and unmixed excesses equality should be necessary qualifications under utility-allocating procedures. Thereupon, the CSSR might be adopted to be a suitable allocating notion under resource allocating processes.

The goal of this paper is to offer different analysis for utility-allocating procedures.

1) A generalized analogue of the PEANSC, the cumulative scaled-single rule, is generalized by simultaneously pondering the members, its energetic grades and scales.
2) To dissect the applied rationality and the mathematical accuracy of the cumulative scaled-single rule, two axiomatic results are provided.
3) By applying the axiomatic procedures to utility-allocating procedures, the applicability and the plausibility for the cumulative scaled-single rule have been further determined by applying some instances and constructions.

One could compare the outcomes of this paper with existing outcomes. Several major dissimilarities are as follows:

1) Under the context of traditional side-payment considerations, allocating rules have only pondered on non-partaking or partaking among all members. As stated above, however, it is equitable that every member should resort distince energetic grades. Thereupon, different from the core, the EANSC, the PEANSC and other allocating rules on traditional side-payment considerations, the cumulative scaled-single rule is generated to resolve utility-allocating notion by means of scales and multi-choice behavior simultaneously.
2) The cumulative scaled-single rule does not present in pre-existing researches. The axiomatic ideas is a multi-choice generalizations of associated outcomes of Moulin [11], Hart and Mas-Colell [3] and Maschler

and Owen[10].

3) By pondering real-world circumstances, the cumulative scaled-single rule is considered to arise associated affect of a specific member if it partakes with a specific energetic grade. Based on multi-choice considerations, pre-existing allocating rules have been considered to arise a kind of entire affect for a specific member by gatherling the marginal dedications of this member among its energetic grades. As stated in Introduction, Hwang and Liao [6], Liao [8] and Nouweland et al. [12] considered associated allocating rules to arise several types of entire affect for a specific member by gatherling associated dedications of the member among all its energetic grades.
   - Based on the core on traditional side-payment considerations, Hwang and Liao [6] proposed some core concepts by using *duplicate behavior among members and its energetic grades*. Differing from Hwang and Liao [6], this paper focuses on the rule of the PEANSC by both pondering the members, its energetic grades and scales. The other main discrimination is the fact that this paper presents the axioms of grade synchronization and unmixed equality symmetry to examine the cumulative scaled-single rule defined in this paper. The notion of scales and associated axioms of grade synchronization and unmixed equality symmetry do not present in Hwang and Liao [6].
   - Based on the EANSC on traditional side-payment considerations, Liao [8] proposed the *maximal EANSC* by using the *maximal pure affects* of members among its energetic grades. Differing from Liao [8], this paper focuses on the rule of the PEANSC by both pondering the members, its energetic grades and scales. The other main discrimination is the fact that this paper presents the axioms of grade synchronization and unmixed equality symmetry to examine the cumulative scaled-single rule defined in this paper. The notion of scales and associated axioms of grade synchronization and unmixed equality symmetry do not present in Liao [8].
   - Based on the Shapley value on traditional side-payment considerations, Nouweland et al. [12] introduced the *multi-choice Shapley value* by using the *replicated behavior* due to the members and its energetic grades. Differing from Nouweland et al. [12], this paper focuses on the rule of the PEANSC by both pondering the members, its energetic grades and scales. The other main discrimination is the fact that this paper presents the axioms of grade synchronization and unmixed equality symmetry to examine the cumulative scaled-single rule defined in this paper. The notion of scales and associated axioms of grade synchronization and unmixed equality symmetry do not present in Nouweland et al. [12].

4) By both focusing on the members and its energetic grades, Hwang and Li [5] introduced a generalized core under multi-choice non-side-payment considerations. Inspired by Hwang and Li [5], the cumulative scaled-single rule of this paper is considered by both pondering the members and its energetic grades under multi-choice side-payment considerations. One should also compare the works of this paper with the outcome of Hwang and Li [5]. There are some main differences:
   - The allocating rule due to Hwang and Liao [5] is depended on multi-choice non-side-payment consideration. The cumulative scaled-single rule of this paper is depended on multi-choice side-payment consideration. In addition, the notion of scales does not present in Hwang and Liao [5].
   - The allocating rule defined by Hwang and Liao [5] is a generalization of the core. The cumulative scaled-single rule of this paper is a generalization of the PEANSC.

One motivation is arisen from the results of this article as follows:

- Whether other allocating rules and its axiomatic characterizations could be adopted to generalize the most efficient suitable notions under utility-allocating situations.

To our knowledge, these issues are still open questions.

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