A Counter Example For Neighbourhood Number Less Than Edge Covering Number Of a Graph

Surekha Ravi shankar Bhat, Ravi shankar Bhat*, Smitha Ganesh Bhat and Sayinath Udupa Nagara Vinayaka

***Abstract*—The open neighbourhood $N(v)$ of a vertex $v \in V$, is the set of all vertices adjacent to v. Then $N[v] = N(v) \cup \{v\}$ is called the enclave of v. We say that a vertex $v \in V$, n-covers an edge $x \in X$ if $x \in \langle N[v] \rangle$, the subgraph induced by the set $N[v]$. The n-covering number $\rho_n(G)$ introduced by Sampathkumar and Neeralagi [18] is the minimum number of vertices needed to n-cover all the edges of G. In this paper one of the results proved in [18] is disproved by exhibiting an infinite class of graphs as counter example. Further, an expression for number of triangles in any graph is established. In addition, the properties of clique regular graphs has been studied.**

***Index Terms*- n-coverings, clique number, independence number, matching number and edge covering number.**

I. Introduction

Any graph $G = (V, X)$ considered in this paper is finite, simple and undirected with $|V| = p$ and $|X| = q$. For standard notations and terminologies, see Berge [1] and West [20]. A property P of sets of vertices is said to be hereditary (superhereditary) if whenever S has the property P so does every proper subset (super set) of S. A set S is a vertex cover of G if at least one vertex of every edge is in S while S is independent if no two vertices in S are adjacent. The lower (upper) vertex covering number $\beta(G)(\Lambda(G))$ is the minimum (maximum) order of a minimal vertex cover while the upper (lower) independence number $\alpha(G)(i(G))$ is defined as the maximum (minimum) order of a maximal independent set (see [6], [13]). The edge analogue of above parameters are similarly defined. The edge covering number $\beta_1(G)$ is the minimum number of edges required to cover all the vertices of G. Finally, the matching number $\alpha_1(G)$ is the maximum number of independent edges of G. The independence property being hereditary in nature, while vertex covering property is superhereditary, the above parameters are related by classical theorem now known as Gallai's theorem, stated as for any graph $G, \alpha(G) + \beta(G) = i(G) + \Lambda(G) = \alpha_1(G) + \beta_1(G) = p$. The generalization of this result using hereditary properties is found in [8]. A set S is a dominating set if every vertex in $V - S$ is adjacent to a vertex in S. The lower (upper) domination number $\gamma(G)(\Gamma(G))$ is the minimum (maximum) number of vertices in a minimal dominating set. Let $N(v) = \{u \in V | u$ is adjacent to $v\}$ be the open neighbourhood of v and then $N[v] = N(v) \cup \{v\}$ called closed neighbourhood or enclave of v. Another graph invariant called neighbourhood number introduced and studied by Sampathkumar and Neeralagi [18] and subsequently attracted several researchers, for example [4], [19], [9], [11], [12], [7]. A set $S \subseteq V$ is said to be a neighbourhood set of G if $G = U_{v \in S} \langle N[v] \rangle$. For an isolate free graph we redefine neighbourhood number as *n-covering number*. We say that a vertex $v \in V$ n-covers an edge $x \in X$ if $x \in \langle N[v] \rangle$, the subgraph induced by the set $N[v]$. A set S is said to be a *n-covering* (or neighbourhood set) of G if the vertices of S n-cover all the edges of G. The *n-covering number* $\rho_n(G)$ is the minimum order of a n-covering of G.

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Surekha R. Bhat is a Professor at Department of Mathematics, Milagres College, Kallianpur, Udupi, Karnataka, India – 574111 e-mail: surekharbhat@gmail.com

R. S. Bhat is a Professor at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India – 576104 e-mail: rs.bhat@manipal.edu. * **Corresponding Author**

Smitha G. Bhat is an Assistant Professor at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India – 576104 e-mail: smitha.holla@manipal.edu

Sayinath Udupa N. V. is an Assistant Professor at Department of Mathematics, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India – 576104 e-mail: sayinath.udupa@gmail.com

Example 1.

For the Hajo's graph shown in the Fig. 1, the n-covering number $\rho_n(G) = 2$ and the corresponding ρ_n- set is $\{v_1, v_2\}$. The edge covering number $\beta_1(G) = 4$ and the corresponding β_1- set is $\{x_1, x_2, x_3, x_4\}$. Hence $\rho_n(G) = 2 < 4 = \beta_1(G)$.

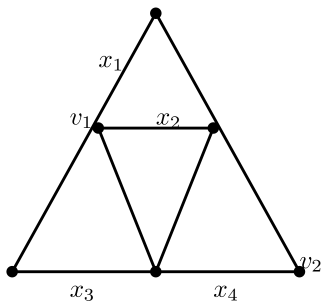

Fig. 1. Hajo's Graph

II. A COUNTER EXAMPLE FOR $\rho_n(G) \leq \beta_1(G)$

Sampathkumar and Neeralagi [18] proved that neighbourhood number is always less than edge covering number. That is $\rho_n(G) \leq \beta_1(G)$. But we observed that this result is not always true. We disprove this result by giving a counter example. For example, if G is a Petersen graph, then $\rho_n(G) = 6 > 5 = \beta_1(G)$. In fact, we construct an infinite family of graphs for which the difference between $\rho_n(G)$ and $\beta_1(G)$ can be made arbitrarily large. Let $G_1 = (V_1, X_1)$ and $G_2 = (V_2, X_2)$ be any two graphs. Then by merging two vertices $u \in V_1$ and $v \in V_2$, we mean identifying $u = v$. For example, let G_1 be the star $K_{1,r}$ with the central vertex a and G_2 be the star $K_{1,s}$ with the central vertex b. Then by merging the vertices a with b (equivalently, identifying the vertices $a = b$) we get the new graph $G = K_{1,r+s}$.

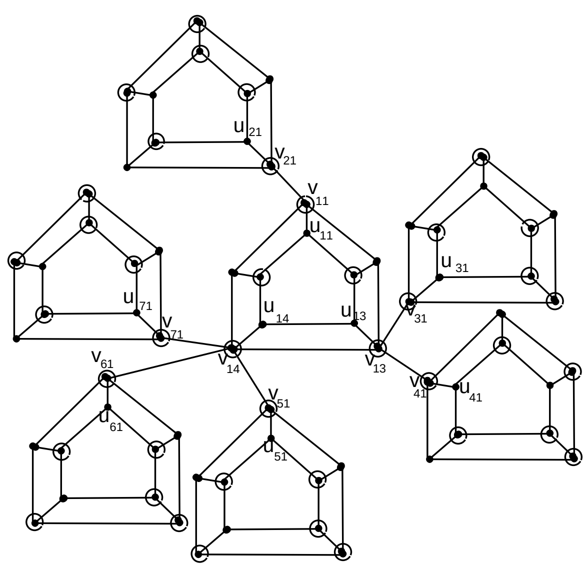

Fig. 2. The Graph $G(7,2)$ for $k = 7$ and $n = 2$ with $\rho_n(G) - \beta_1(G) = 7 = k$.

Proposition 1. *For any positive integer k, there exists an infinite class of graphs for which* $\rho_n(G) - \beta_1(G) = k$

Proof: We construct an infinite class of triangle free graphs $G(k,n), n \geq 2$ with the following three steps.
Step 1: Consider Two copies of any odd cycle C_{2n+1} for $n \geq 2$. Join i^{th} vertex of first copy of C_{2n+1} (inner cycle) with the i^{th} vertex of second copy of C_{2n+1} (outer cycle). We denote the obtained graph as $H = C_{2n+1} \oplus C_{2n+1}$. Now obtain k copies of the graph H. Let u_{ij} and v_{ij} for $1 \leq i \leq k; j = 1, 2, ...2n+1$ respectively be the labels of the vertices of inner and outer odd cycles C_{2n+1} in the i^{th} copy H_i of H. Then u_{ij} for $1 \leq i \leq k; j = 2, 4, ...2n, 2n+1$ and v_{ij} for $1 \leq i \leq k; j = 1, 3, 5, ...2n-1, 2n$ are called pivotal vertices.
Step 2: Choose $n+1$ integers $t_1, t_2, ...t_{n+1}$ such that $\sum_{r=1}^{n+1} t_r = k-1$. Form the stars $K_{1,t_1}, K_{1,t_2}, ...$ $...K_{1,t_{n+1}}$. Let c_r denote the central vertices of the star $K_{1,t_r}, 1 \leq r \leq n+1$.
Step 3: Merge each central vertex c_r with the pivotal vertex $v_{1(2r-1)}, 1 \leq r \leq n$ respectively and c_{n+1} with $v_{1(2n)}$. The obtained graph has $k-1$ pendant vertices say, $w_1, w_2.....w_{k-1}$. Finally, merge each pendant vertex w_l, with the pivotal vertex $v_{l+1,1}$ $1 \leq l \leq k-1$ to obtain the graph $G(k,n)$. The graph $G(7,2)$ is shown in the Fig. 2.
Now one can verify that the set of pivotal vertices $\{u_{ij} | 1 \leq i \leq k; j = 2, 4, ...2n, 2n+1\} \cup \{v_{ij} | 1 \leq i \leq k; j = 1, 3, ...2n-1, 2n\}$ is a ρ_n set of $G(k,n)$ of order $k(2n+2)$. Further, the set of edges $\{u_{ij}v_{ij}\} | 1 \leq i \leq k$; $j = 1, 2, 3, ...2n+1$ is a β_1 set of order $k(2n+1)$. Therefore $\rho_n(G(k,n)) - \beta_1(G(k,n)) = k(2n+2) - k(2n+1) = k$. For the graph $G(7,2)$ in Fig. 2, the encircled vertices $(6 \times 7 = 42)$ form a ρ_n-set and the set of edges $(5 \times 7 = 35)$ that join inner odd cycle with outer odd cycle in each copy of $H = C_5 \oplus C_5$ forms a β_1-set. Therefore $\rho_n(G(7,2)) - \beta_1(G(7,2)) = 42 - 35 = 7 = k$. ∎

We find a lower bound for matching number $\alpha_1(G)$ in terms of edges.

Proposition 2. *For any graph G with odd number of vertices and* $\alpha_1(G) = k$,

$$\frac{-1 + \sqrt{1 + 8q}}{4} \leq \alpha_1(G)$$

and equality holds if and only if G is a complete graph with odd number of vertices.

Proof: Let $\alpha_1(G) = k$. Since $\alpha_1(G) \leq \lfloor \frac{p}{2} \rfloor$, we have $p = 2k+1$ or $2k$. Then $q \leq^{2k+1} C_2 = k(2k+1)$. Hence $2k^2 + k - q \geq 0$. Solving this quadratic equation for k we get the desired result.
If G is a complete graph with odd number of vertices, say $p = 2k+1$, then one can easily verify that the

bound is attained. Conversely, suppose that $\alpha_1(G) = k = \frac{-1+\sqrt{1+8q}}{4}$. On simplifying this expression, we get $q = k(2k+1) =^{2k+1} C_2$. Hence we conclude that G is a complete graph with $p = 2k+1$ vertices.

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The bound for a graph with even number of vertices is similar, but in this case the lower bound is little improved and we have the following

Corollary 1. *For any graph G with even number of vertices and $\alpha_1(G) = k$,*

$$\frac{1+\sqrt{1+8q}}{4} \leq \alpha_1(G)$$

and equality holds if and only if G is a complete graph with even number of vertices.

III. Number of triangles in a graph

The *strength* of a vertex $s(v)$, $v \in V$ is defined as the number of edges in $\langle N[v] \rangle$. Then *maximum strength* of $G, \Delta_s(G) = max_{v\in V} s(v)$. Similarly *minimum strength* of $G, \delta_s(G)$ is defined. The *signature* $s_n(v)$ of a vertex $v \in V$ is the number of edges in $\langle N(v) \rangle$. Equivalently, $s_n(v)$ is the number of triangles containing v. Therefore if v is a vertex not in any triangle, then $s_n(v) = 0$. It is well known that sum of degrees of all vertices is twice the number of edges. We now show that the sum of signatures of all vertices is equal to thrice the number of triangles and hence obtain an expression for number of triangles in any graph.

Proposition 3. *For any graph G with t triangles,*

$$t = \frac{\sum_{v\in V} s_n(v)}{3} \tag{1}$$

and

$$\sum_{v\in V} s(v) = 2q + 3t \tag{2}$$

Further, $\sum_{v\in V} s(v)$ is odd $(even)$ if and only if number of triangles in G is odd $(even)$.

Proof: Since every triangle is counted three times in counting $s_n(v)$ of all the vertices, we have $\sum_{v\in V} s_n(v) = 3t$ proving the desired result (1). We note that for any vertex $v \in V$, the strength $s(v) = d(v) + s_n(v)$. Therefore $\sum_{v\in V} s(v) = \sum_{v\in V} d(v) + s_n(v) = 2q + 3t$.

The last part of the proposition follows from the fact that $\sum_{v\in V} s_n(v) = 3t$ is odd or even according as number of triangles in G is odd or even.

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For example, for the graph G in Fig. 3, number of triangles in $G = t = 5, q = 15$ and $\sum_{v\in V} s_n(v) = 15 = 3 \times 5 = 3t$. Again, $\sum_{v\in V} s(v) = 45 = 2\times 15 + 3\times 5 = 2q + 3t$ which are in agreement with the results (1) and (2).

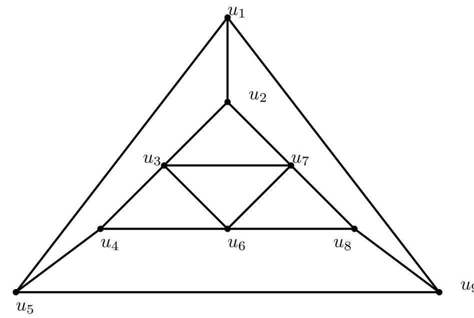

Fig. 3. A graph G with 5 triangles

Let $\Delta_{s_n}(G), \delta_{s_n}(G)$ denote the maximum and minimum signature of G respectively. If G is a graph in which every edge is in a triangle, then we can estimate the number of triangles.

Corollary 2. *For any graph G, in which every edge is in a triangle,*

$$\frac{p\delta_{s_n}}{3} \leq t \leq \frac{p\Delta_{s_n}}{3}$$

Further, the bound is sharp.

Proof: The result follows from the fact that $p\delta_{s_n} \leq \sum_{v\in V} s_n(v) = 3t \leq p\Delta_{s_n}$.

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A graph G is said to be *k-signature regular* if $s_n(v) = k$ for every $v \in V$. Any signature regular graph need not be regular. But if every edge is contained in a triangle, then every signature regular graph is also regular. For example, the graph obtained by removing the edges joining antipodal vertices from K_6 is 4-signature regular and 4-regular graph. Any signature regular graph attains both upper and lower bounds in the Corollary 2

We now obtain bounds for n-covering number of G in terms of maximum strength $\Delta_s(G)$.

Proposition 4. *For any graph G,*

$$\frac{q}{\Delta_s} \leq \rho_n(G) \leq q - \Delta_s + 1 \tag{3}$$

Further, these bounds are sharp.

Proof: Since a vertex can n-cover at most $\Delta_s(G)$ edges, to n-cover all the edges, we need at least $\frac{q}{\Delta_s(G)}$ vertices. Hence $\frac{q}{\Delta_s(G)} \leq \rho_n(G)$ which yields the desired lower bound. Let v be a vertex of maximum strength Δ_s. Then v n-covers all the edges in $\langle N[v] \rangle$. Let S be the set of vertices of order at most $|X - \langle N[v] \rangle| = q - \Delta_s$, formed by choosing one end of every edge in $X - \langle N[v] \rangle$. Then S, n-covers all the edges in $X - \langle N[v] \rangle$. Hence $S \cup \{v\}$ is a n-covering of G. Therefore $\rho_n(G) \leq |S \cup \{v\}| \leq q - \Delta_s + 1$.

For any complete bipartite graph, $G = K_{m,n}, m > n, \Delta_s = m, q = mn$, and $\rho_n(G) = n = \frac{mn}{m} = \frac{q}{\Delta_s}$.

Hence the lower bound in (3) is attained for any complete bipartite graph. A wounded spider is a tree obtained by subdividing at most $n-1$ edges of a star $K_{1,n}$. Any wounded spider attains the upper bound in (3). Further, any complete graph K_n attains both upper and lower bounds in (3).

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For any triangle free graph the maximum degree coincides with maximum strength of G. Hence the following corollary is immediate.

Corollary 3. *For any triangle free graph G with maximum degree Δ,*

$$\frac{q}{\Delta} \leq \rho_n(G) \leq q - \Delta + 1.$$

The proof of Proposition 4, suggests a similar lower bound for another parameter called open full domination number introduced by Brigham *et al*. [4]. A vertex v in a graph G openly dominates the subgraph $\langle N(v) \rangle$ induced by the (open) neighbourhood $N(v)$. A set S of vertices in G is a full open dominating set if every edge of G belongs to $\langle N(v) \rangle$ for some $v \in S$. The order of minimum full open dominating set is the full open domination number $\gamma_{FO}(G)$. A graph G has a full open dominating set if and only if G contains no isolated vertices and every edge of G lies on a triangle in G. We get a lower bound for full open domination number of a graph in terms of maximum signature $\Delta_{s_n}(G)$.

Proposition 5. *For any graph G in which every edge lies in a triangle,*

$$\frac{q}{\Delta_{s_n}(G)} \leq \gamma_{FO}(G) \tag{4}$$

Proof: Since a vertex v can openly dominate at most $\Delta_{s_n}(G)$ edges, to dominate all the edges of G we need at least $\frac{q}{\Delta_{s_n}}$ vertices, Hence $\gamma_{FO}(G) \geq \frac{q}{\Delta_{s_n}}$. ■

IV. New class of graphs

Definition 1. ***Well covered graphs*** *In 1970 Plummer [15] has introduced the concept of well covered graphs and further studied in [2], [16], [17]. A graph G is well covered if every maximal independent set is of same order. In other words G is well covered if and only if $i(G) = \alpha(G)$.*

The definition of well covered graphs motivated to define another special class of graphs called clique regular graphs. A complete subgraph is called a clique of G. The clique induced by the set S is maximal if the $S \cup \{v\}\}$ is not a clique of G for any $v \in V$. Let $K(G)$ denote the set of all maximal cliques of G and $|K(G)| = k$ denote the number of maximal cliques in G. Then clique number $\omega(G)$ is the order of maximum clique of G. Similarly minimum clique number $\vartheta(G)$ is the minimum order of a maximal clique of G.

Definition 2. ***Clique Regular Graphs*** *A graph G is clique regular if every clique is of same order. Thus G is k- clique regular graph if $\omega(G) = \vartheta(G) = k$.*

For example, friendship graph F_n is a graph in which n triangles have a vertex in common. Clearly, F_n is a 3-clique regular graph. Every cycle is 2-clique regular. A wind mill graph $Wd(n,k)$ is a graph in which k copies of complete graph K_n have a vertex in common. In fact $wd(3,k) = F_k$. Clearly, $Wd(n,k)$ is n-clique regular graph. The corona product $K_m \cdot K_n$ is obtained as follows. Take m copies of K_n. Then ith vertex of K_m is adjacent to every vertex of ith copy of K_n for $1 \leq i \leq m$. The corona product $K_3 \cdot K_2$ is 3-clique regular graph. In general, the corona product $K_m \cdot K_{m-1}$ is m-clique regular graph.

The next theorem establishes the relationship between clique regular graphs and well covered graphs.

Proposition 6. *A graph G is well covered if and only if $\overline{G}$ is clique regular*

Proof: A graph G is well covered

$\Longleftrightarrow i(G) = \alpha(G)$

$\Longleftrightarrow$ every maximal independent set is of same order in G

$\Longleftrightarrow$ every maximal clique is of same order in $\overline{G}$

$\Longleftrightarrow \omega(\overline{G}) = \vartheta(\overline{G})$

$\Longleftrightarrow \overline{G}$ is clique regular

■

From the above theorem it is immediate that the following graphs are well covered. For example, every cycle is 2-clique regular, hence the complement of every cycle is well covered. For a similar reason the complement of Petersen graph, complement of any triangle free graph are well covered. Further, complement of any windmill graph $Wd(n,k)$ and complement of any corona $K_m \cdot K_{m-1}$ are well covered graphs.

A graph is self complementary if and only if $\overline{G}$ the complement of G is isomorphic to G. That is $G \equiv \overline{G}$. We now define another important class of graphs called self well covered graphs. A graph G is said to be self well covered if both G and $\overline{G}$ are well covered. Similarly, a graph G is self clique regular if both G and $\overline{G}$ are clique regular. It is immediate that every self complementary well covered graphs are self well covered graphs. Again every self complementary clique regular graphs are self clique regular. But there are other graphs which are self well covered. The next result characterizes self well covered graphs.

Proposition 7. *Every self clique regular graph is self well covered.*

Proof: A graph G is self well covered
$\Longleftrightarrow i(G) = \alpha(G)$ both in G and $\overline{G}$
$\Longleftrightarrow$ every maximal independent set is of same order in G and $\overline{G}$
$\Longleftrightarrow$ every maximal clique is of same order both in G and $\overline{G}$
$\Longleftrightarrow \omega(G) = \vartheta(G)$both in G and $\overline{G}$
$\Longleftrightarrow$ both G and $\overline{G}$ are clique regular

■

For example, Every complete garph K_n is self well covered. The path P_4 and cycle C_5 are self well covered. The complete $k-$partite graph $K_{n_1,n_2...}$ $_{...n_k}$ is a graph in which the vertex set $V(G)$ is partetioned in to k independent sets with $|V_1| = n_1, |V_2| = n_2, ...$ $, ..., |V_k| = n_k$ and every pair of vertices in k sets are adjacent. In particular, $K_{2,2,...}$ $_{,...,2}$ is called cocktail party graph which is a typical example of self well covered graph.

A minmax relation is a theorem stating equality between the answers to a minimization problem and a maximization problem. The Konig- Egervary [10] theorem is such a relation for matching and vertex covering number which states that if G is a bipartite graph then the covering number $\beta(G)$ = the matching number $\alpha_1(G)$. We extend the Konig - Egervary theorem to self clique regular graphs as follows in which all the four min-max numbers are equal.

Proposition 8. *A graph G is self complementary and self clique regular graph if and only if* $i(G) = \alpha(G) = \omega(G) = \vartheta(G)$

Proof: From Proposition IV.II , we have a graph G is self clique regular $\Longleftrightarrow i(G) = \alpha(G)$ and $\omega(G) = \vartheta(G)$. Now it remains to show that $\alpha(G) = \omega(G)$. Since independent sets and complete sets exchange their properties on complementation for any graph we have $\alpha(G) = \omega(\overline{G})$. As G is self complementary $G \equiv \overline{G}$. Hence $\alpha(G) = \omega(\overline{G}) = \omega(G)$. ■

Definition 3. ***Clique paths and clique cycles.***

Similar to n-covering number another concept called line clique covering number is defined by Choudam [5] as the minimum number cliques that cover all the vertices of a graph. Parthasarathy and Choudam[14] studied the same extending it to product graphs. Surekha R Bhat [3] observed that a block behaves like an edge of a graph with multiple vertices. This concept led to define block walks and block paths in a graph. On similar lines we define clique paths, clique complete graph and generalized clique stars. It is well known that a clique graph $K_G(G)$ is a graph with vertex set as set of all maximal cliques of G and any two vertices in $K_G(G)$ are adjacent if any two cliques have a vertex in common.

A graph G is called a $clique\ path$, if $K_G(G)$ is a path. A graph G is called a $clique\ cycle$, if $K_G(G)$ is a cycle. A graph G is said to be $Clique-Complete$, if $K_G(G)$ is complete. A graph G is a $Clique-Star$, if $K_G(G)$ is a clique complete graph. A generalized star denoted $S(n, k)$ is a windmill graph in which each K_n has $n-1$ vertices in common.

A clique path, two types of clique cycles and a generalized star are shown in Fig. 4.

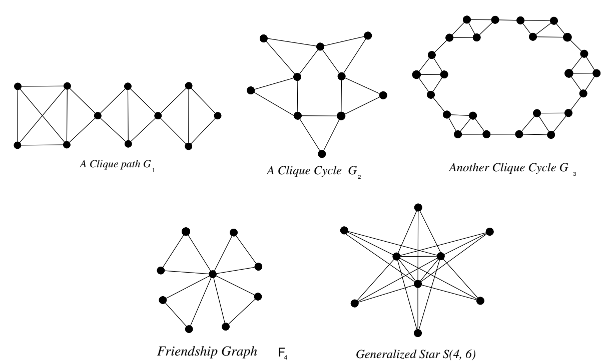

Fig. 4. A Clique Path G_1, A Clique Cycle G_2, Another Clique Cycle G_3, Friendship Graph F_4, Generalized Star $S(4, 6)$

A vertex $v \in V$ is said to be unicliqual if v lies in only one clique, otherwise v is called polycliqual vertex. A clique k is called a monoclique if k contains at least one uniclique vertex. For the clique cycle G_2 in Fig. 4, every clique is a monoclique whereas for the another clique cycle G_3 in Fig. 4, no clique is a monoclique. Friendship graph and Hajo's graph are examples of clique complete graphs. The clique cycle G_2 and friendship graph are 3-clique regular graphs.

V. Clique-Vertex degree, Polycliqual Vertex degree and Clique-Clique degree

Here we introduce some new clique-degree concepts.

Definition 4. *The* cv-degree *(*clique vertex degree*) of a clique l, $d_{cv}(l)$ is the number of vertices in clique l. The* polycliqual vertex-degree *of a clique l, $d_{pc}(l)$ is the number of polycliqual vertices in the clique l. Let V_{pc} denote the set of all polycliqual vertices of G and $|V_{pc}| = P_c$. The vc-degree (vertex clique degree) of a vertex $v \in V$ is the number of cliques incident on v.*

Let $\Delta_{cv}(G)$ and $\delta_{cv}(G)$ denote the maximum and minimum cv-degrees of G respectively. Then $\Delta_{pc}(G)$ and $\delta_{pc}(G)$, are defined similarly. We observe that

$\Delta_{cv}(G) = \omega(G)$ and $\delta_{cv}(G) = \vartheta(G)$. A clique-polycliqual vertex graph CPV(G) is a bipartite graph with partition V_1=set of all cliques of G and V_2=set of all polycliqual vertices of G and any two elements $l \in V_1$ and $u \in V_2$ are adjacent if the clique l is incident on the polycliqual vertex u in G.

Proposition 9. *For any graph G with k cliques and p_c polycliqual vertices,*

$$p_c + k - 1 \leq \sum_{u \in V_{pc}(G)} d_{vc}(u) = \sum_{l \in K(G)} d_{pc}(l) \leq k\Delta_{vc}$$

Further, $\sum_{u \in V_{pc}(G)} d_{vc}(u) = p_c + k - 1$ if and only if $CPV(G)$ is a tree.
$\sum_{u \in V_{pc}(G)} d_{vc}(u) = k\Delta_{vc}$ if and only if $CPV(G)$ is a complete bipartite graph.

Proof: First we note that for any $u \in V_{pc}, d_{vc}(u)$ is equal to $d(u), u \in V_2$ in $CPV(G)$. Therefore $\sum_{u \in V_{pc}(G)} d_{vc}(u) = \sum_{u \in V_2} d(u) = q(CPV)(G)$. Similarly, for any $l \in K(G)$, $d_{pc}(l)$ is equal to $d(l), l \in V_1$ in CPV(G). Therefore $\sum_{l \in K(G)} d_{pc}(l) = q(CPV)(G)$. Since $CPV(G)$ is a bipartite graph with $P_c + k$ vertices, $P_c + k - 1 \leq q(CPV)(G) \leq k\Delta_{vc}$. This proves the result.
It is evident that if if CPV(G) is a tree then it has $p_c + k - 1$ edges and if CPV(G) is a complete bipartite graph then it has $\Delta_{vc}k$ edges. This completes the proof. ■

Note 1. 1) *If G is a clique complete graph in which every polycliqual vertex is in every clique then $\sum_{u \in V_{pc}(G)} d_{vc}(u) = k\Delta_{vc}$.*
2) *If G is a block graph, then $\sum_{u \in V_{pc}(G)} d_{vc}(u) = p_c + k - 1$.*

Note 2. *In Hajo's graph every polycliqual vertex is not in every clique. Thus upper bound in above theorem is not attained for Hajo's graph.*

The next proposition is similar to above result and hence we state the result and omit the proof.

Proposition 10. *For any graph G with k cliques and p_c polycliqual vertices,*

$$p_c + k - 1 \leq \sum_{u \in P_C(G)} d_{vc}(u) = \sum_{l \in K(G)} d_{p_c}(l) \leq k\Delta_{p_c};$$

$\sum_{l \in K(G)} d_{p_c}(l) = p_c + k - 1$ if and only if $CPV(G)$ is a tree;
$\sum_{l \in K(G)} d_{p_c}(l) = k\Delta_{p_c}$ if and only if $CPV(G)$ is a complete bipartite graph.

Extending the above result taking all the vertices instead of polycliqual vertices, we get the following result. Clique vertex graph $CV(G)$ is a bipartite graph with bipartetion V_1 as set of all cliques of G and V_2 as set of all vertices of G. Any two elements $l \in V_1, u \in V_2$ are adjacent if they are incident.

Proposition 11. *For any graph G with k cliques,*

$$p + k - 1 \leq \sum_{u \in V(G)} d_{vc}(u) = \sum_{l \in K(G)} d_{cv}(l) \leq \omega k;$$

$\sum_{l \in K(G)} d_{cv}(l) = \omega k$ if and only if CV(G) is a complete bipartite graph;
$\sum_{l \in K(G)} d_{cv}(l) = p + k - 1$ if and only if CV(G) is a tree.

Proof: First we note that for any $u \in V, d_{vc}(u)$ is equal to $d(u), u \in V_2$ in $CV(G)$. Therefore $\sum_{u \in V(G)} d_{vc}(u) = \sum_{u \in V_2} d(u) = q(CV)(G)$. Similarly, for any $l \in K(G)$, $d_{cv}(l)$ is equal to $d(l), l \in V_1$ in CV(G). Therefore $\sum_{l \in K(G)} d_{cv}(l) = \sum_{l \in V_1} d(l) = q(CV)(G)$. Since $CV(G)$ is a bipartite graph with $P+k$ vertices, $P + k - 1 \leq q(CPV)(G) \leq k\omega$. This proves the result.
It is evident that if if CV(G) is a tree then it has $p+k-1$ edges and if CPV(G) is a complete bipartite graph then it has ωk edges. This completes the proof. ■

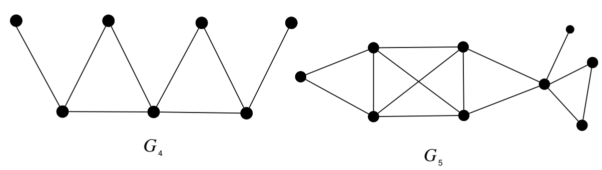

Fig. 5. Graph G_4 with 4 Cliques and Graph G_5 with 5 Cliques

Example 2. *For the graph G_4 of Fig. 5, number of cliques $k = 4$, $\sum_{l \in K(G)} d_{cv}(l) = 10$ and number of vertices $p = 7$. Thus $p+k-1 = 10 = \sum_{l \in K(G)} d_{cv}(l) =$ 10.*

For the graph G_5 of Fig. 5, number of cliques $k = 5$, $\sum_{l \in K(G)} d_{cv}(l) = 15$, $\omega = 4$ and number of vertices $p = 9$. Thus $p + k - 1 = 13 \leq \sum_{l \in K(G)} d_{cv}(l) = 15 \leq \omega k = 20$.

Corollary 4. *For any graph G,*

$$\frac{p-1}{\omega-1} \leq k$$

Proof: From Proposition 11, $p + k - 1 \leq \sum_{l \in K(G)} d_{cv}(l) \leq k\omega$. Thus

$$p + k - 1 \leq \omega k$$
$$p - 1 \leq k(\omega - 1)$$
$$\frac{p-1}{\omega-1} \leq k$$

■

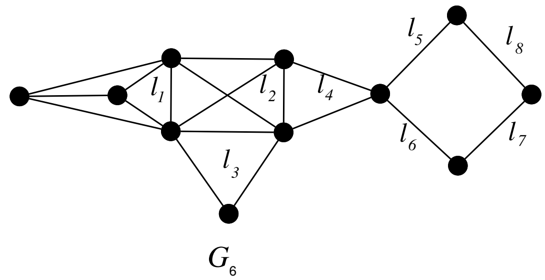

Fig. 6. Graph G_6 with Clique number equal to 4

Example 3. *For the graph G_6 of Fig. 6, number of cliques $k = 8$, $\omega = 4$ and number of vertices $p = 11$. Thus $\dfrac{p-1}{\omega-1} = \dfrac{10}{3} \leq k = 8$. One can easily check that for the Friendship graph F_n the equality holds in the above corollary.*

Corollary 5. *If G is a block graph, then*

$$\frac{p-1}{\omega-1} \leq k \leq \frac{p-1}{\vartheta-1}$$

VI. Conclusion

We have disproved the result $\rho_n(G) \leq \beta_1(G)$ by constructing an infinite class of graphs as counter example. New class of graphs such as clique regular graphs, clique complete graphs, clique paths , clique cycles and generalized stars are defined and studied.

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Surekha R. Bhat is a Professor in the department of Mathematics, Milagres College, Kallianpur, Udupi. She obtained Ph.D from Manipal university in 2013. Her research fields are Inverse domination in graphs, Cliques and block domination parameters in graphs

R. S. Bhat is a Professor in the department of Mathematics, Manipal institute of Technology, Manipal. He obtained Ph.D from National Institute of Technology, Surathkal in the year 2007. He guided two Ph.D students. His research fields are Domination in graphs, Energy of a graphs, Strong independent sets, strong covering sets, Cliques and Neighbourhood number.

Smitha G. Bhat is a Assistant Professor in the department of Mathematics, Manipal institute of Technology, Manipal. She obtained Ph.D from Manipal university in 2019. She obtained M.Sc in Mathematics from Mangalore University in the year 2007. Her research fields are Domination in graphs, Cliques and Neighbourhood number.

Sayinath Udupa N. V is Assistant Professor in the Department of Mathematics, Manipal Institute of Technology, Manipal. He obtained Ph.D from Manipal Academy of Higher Education in 2020. He obtained M.Sc in Applied Mathematics and Computing from Manipal Academy of Higher Education in the year 2012. He obtained B.Sc from Dr. NSAM first grade college in 2010. His research fields are Domination in graphs,Block domination parameters in graphs, Energy of graphs, Modular equations and Ramanujan's theta functions.