The Relationships among the Kinds of Concept Lattices Based on the Dual Intersectable Context

Ting Qian*, Erkang Qin, Cheng Wei, Wanting Zhao, and Xiaoli He

***Abstract*—The three-way object oriented concept lattice is extend researches of rough concept analysis by combining three-way decision. In this paper, we investigate the relationships between the kinds of concept lattices and the three-way object oriented concept lattice bassd on the dual intersectable context. Firstly, the relationships between the object oriented concept lattice and the three-way object oriented concept lattice are studied. Some conclusions are obtained. (1) They are isomorphic based on the dual context; (2) They are not isomorphic based on the dual intersectable context. And the the isomorphic relation are ausschied in the dual intersectable complement contex. In addition, the relationship between the concept lattice and the three-way object oriented concept lattice are studied. The isomorphic relationship between them is proved in the dual intersectable context.**

***Index Terms*—formal context, concept lattice, three-way decision, object oriented lattice, dual intersectable context.**

I. Introduction

BASED on the practical application of lattice theory in the mathematical order theory, formal concept analysis (FCA) was given ([1], [2]). It studies the hierarchical structures which is induced by a binary relation between the objects and attributes. With the development of data methods, FCA has become an effective mathematical method for data processing and knowledge discovery [3-10]. Furthermore, to adapt to various requirements of data analysis, the AFS conceptual analysis [11], the variable threshold conceptual analysis [12], the real conceptual analysis [13], the power conceptual analysis [14], the approximate conceptual analysis [15] and the closed label conceptual analysis [16] and so on came into being.

Recently, three-way decision [17] grows rapidly. And combining with it, three-way concept analysis (3WCA) was firstly given by Qi et al. in 2014 ([18], [19]). Later, 3WCA has become the hot research topic. Three-way conceptual approach for cognitive memory functionalities are introduced by Shivhare and Cherukuri [20]. Conflict analysis of 3WCA under one-vote is proposed by Zhi et al. [21]. In 2020, the method of constructing three-way concept lattice was proposed by the composite of classical lattices in [22]. The three-way rough semiconcept is proposed by Mao and Cheng [23]. Later, Wei et al. extend 3WCA to formal decision contexts [24]. Inspired by 3WCA, Xin et al. used attribute correlation degree to study the three-way intuitionistic fuzzy formal concept lattice in 2021 [25].

Combing FCA with rough set, the property oriented concept lattice (L_p) and the object oriented concept lattice (L_o) which were proposed by Düntsch and Gediga [26] and Yao [27] respectively, are two important concept lattices. From the perspective of three-way decision, the idea locally completely possessed are shown. Based on it, three-way object (property) oriented concept lattices(OEOL(OEPL)) were proposed by Wei and Qian [28]. Then, more details are introduced in [29].

However, in the process of research, we find that the three-way concept lattices are much more complex than the concept lattice. In addition, FCA is relatively mature after nearly 40 years of development. Therefore, it is a good opinion to deal with the problem of three-way concept lattices by using the related methods of concept lattices. Based on it, Qian et al. investigated the isomorphic relationship of kinds of concept lattices based on the dual intersectable context(DIC), and meanwhile proposed the construction methods of three-way concept lattices [30].

Compared with concept lattice, the three-way object oriented concept is as complicated as three-way concept lattice. Thus, the same problem occurs in OEOL. That is, (1) Can we use the existing FCA and L_o to study OEOL directly. (2) Whether the relationship between concept lattice and three-way concept lattice, especially isomorphism, can be obtained by describing the characteristics of formal context. On the basis of literature [30], there are clear research ideas to these two questions. Therefore, The relationships among the kinds of concept lattices will be studied based on the dual intersectable context in this paper.

The follow-up arrangement is as follows. Section II reviewes some basic notions related to FCA and 3WCA. Section III investigates the relations between L_o and OEOL of the DC and the DIC, respectively. Several related and examples are given. Finally, Section IV drawn all of conclusions. theorems

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II. Preliminaries

Some notation is fixed as follows firstly. Let $|T| < \infty$ and $T \neq \emptyset$. The power set of T is $\mathcal{P}(T)$, and $\mathcal{DP}(T) = \mathcal{P}(T) \times \mathcal{P}(T)$. Some basic set-theoretic operators on $\mathcal{DP}(T)$ can be proposed based on set operators. For any $(X,Y),(Z,W) \in \mathcal{DP}(T)$, we define

$$(X,Y) \cap (Z,W) = (X \cap Z, Y \cap W),$$
$$(X,Y) \cup (Z,W) = (X \cup Z, Y \cup W),$$
$$(X,Y)^c = (X^c, Y^c),$$

TABLE I
A FORMAL CONTEXT (G, M, I)

G	a	b	c	d	e
1	1	0	1	1	1
2	1	0	1	0	0
3	0	1	0	0	1
4	1	0	0	0	0
5	1	1	0	0	1

$$(X, Y) \subseteq (Z, W) \Leftrightarrow X \subseteq Y \text{ and } Z \subseteq W.$$

A. Concept lattices

There are some definitions recalled in FCA.

Definition 2.1 [2] Suppose G, M and I be three sets, where $I \subseteq G \times M$, (G, M, I) is called as a formal context. The elements of G are called the objects, so G is called the object set. The elements of M are called the attributes, so M is called the attribut set. If $(g, m) \in I$(or gIm), then we read it as "the object g has the attribute m".

Given a formal context (G, M, I), $X \subseteq G$ and $A \subseteq M$, a pair of dual operators is defined.

$$X^* = \{m \in M | (g, m) \in I \text{ for all } g \in X\},$$
$$A^* = \{g \in G | (g, m) \in I \text{ for all } m \in A\}.$$

Then (G, M, I) is called canonical if and only if there are not $g \in G$ and $m \in M$ to make $\{g\} \times M \subseteq I$, and $G \times \{m\} \subseteq I$.

If $X^* = A$ and $A^* = X$, then (X, A),X and A are called a concept, the extent and the intent. The set of all concepts forms a complete lattice with the following partial order $\leq$. And it is called the concept lattice which is denoted by $L(G, M, I)$ (Simply mark it as CL). For any $(X_1, A_1), (X_2, A_2) \in L$, if $X_1 \subseteq X_2$, $(X_1, A_1) \leq (X_2, A_2)$ is defined. We can prove $\leq$ is a partial order relation.

They are also proved that $((X_1 \cap X_2, (A_1 \cup A_2)^{**})$ and $((X_1 \cup X_2)^{**}, A_1 \cap A_2)$ are infimum and supermum of (X_1, A_1) and (X_2, A_2), respectively.

Example 2.1 Table 1 is a formal context, $G = \{1, 2, 3, 4, 5\}$ and $M = \{a, b, c, d, e\}$. Figure 1 shows CL. For simplicity, listing elements represents sets except G, M and $\emptyset$. The sets in other examples are similarly marked.

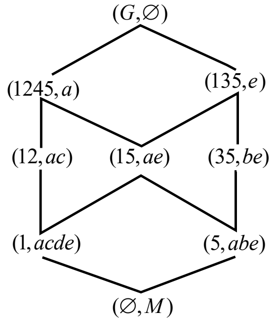

Fig. 1. CL of Table I

B. The relevant definitions of L_o

Similar to concept lattice, Yao proposed the definition of L_o based on rough set theory [27].

Definition 2.2 [26], [27] Let (G, M, I) be a formal context, $X \subseteq G$ and $A \subseteq M$, $\natural(\Diamond) : \mathcal{P}(G) \to \mathcal{P}(M)$ and $\Diamond(\natural) : \mathcal{P}(M) \to \mathcal{P}(G)$ are defined, as follows:

$$X^{\natural} = \{m \in M | m^* \subseteq X\},$$
$$X^{\Diamond} = \{m \in M | m^* \cap X \neq \emptyset\},$$
$$A^{\Diamond} = \{g \in G | g^* \cap A \neq \emptyset\},$$
$$A^{\natural} = \{g \in G | g^* \subseteq A\}.$$

If $X^{\natural} = A$ and $A^{\Diamond} = X$, then (X, A) is called an object oriented concept. And X and A are resepectively called the extent and the intent of (X, A). If $X_1 \subseteq X_2$, then $(X_1, A_1) \leq (X_2, A_2)$ is defined. We can prove $\leq$ is a partial order relation. The family of all object oriented concepts forms a complete lattice with the above relation. We call it the object oriented concept lattice and denote it by $L_o(G, M, I)$ (Simply mark it as L_o).

They are also proved that $((X_1 \cap X_2)^{\natural\Diamond}, A_1 \cap A_2)$ and $(X_1 \cup X_2, (A_1 \cup A_2)^{\Diamond\natural})$ are infimum and supermum of (X_1, A_1) and (X_2, A_2), respectively.

Example 2.2 Figure 2 is L_o of Table 1.

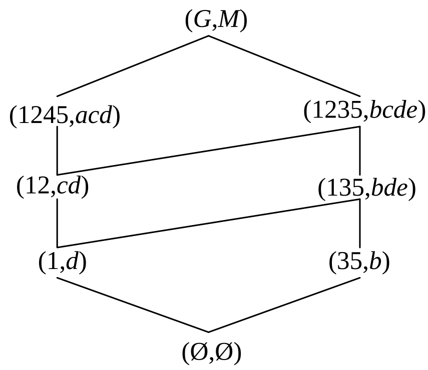

Fig. 2. L_o of Table I

C. The relevant definitions of OEOL

The related contents of OEOL are proposed by Wei and Qian [28], [29].

Definition 2.3 [18], [19] Let (G, M, I) be a formal context. For $X \subseteq G, A \subseteq M$ and $I^c = (G \times M) \setminus I$, define another pair of operators, $\bar{*} : \mathcal{P}(G) \to \mathcal{P}(M)$ and $\bar{*} : \mathcal{P}(M) \to \mathcal{P}(G)$, as follows. For $X \subseteq G$ and $A \subseteq M$,

$X^{\bar{*}} = \{m \in M | \forall x \in X(\neg(xIm))\} = \{m \in M | \forall x \in X(xI^c m)\}$,

$A^{\bar{*}} = \{g \in G | \forall a \in A(\neg(gIa))\} = \{g \in G | \forall a \in A(gI^c a)\}$.

We give a following pair of new operators based on "locally" meaning. .

Definition 2.4 [28], [29] Let (G, M, I) be a formal context. For $X \subseteq G$ and $A \subseteq M$, define $\bar{\natural}(\bar{\Diamond}) : \mathcal{P}(G) \to \mathcal{P}(M)$ and $\bar{\Diamond}(\bar{\natural}) : \mathcal{P}(M) \to \mathcal{P}(G)$:

$$X^{\bar{\natural}} = \{m \in M | m^{\bar{*}} \subseteq X\},$$
$$X^{\bar{\Diamond}} = \{m \in M | m^{\bar{*}} \cap X \neq \emptyset\},$$
$$A^{\bar{\Diamond}} = \{g \in G | g^{\bar{*}} \cap A \neq \emptyset\},$$
$$A^{\bar{\natural}} = \{g \in G | g^{\bar{*}} \subseteq A\}.$$

Combing the operators $\natural$, $\Diamond$, $\bar{\natural}$ and $\bar{\Diamond}$ together, the following pair of new operators is obtained in [28], [29].

Definition 2.5 [28], [29] Let (G, M, I) be a formal context. For $X \subseteq G$ and $A, B \subseteq M$, define $\triangleright : \mathcal{P}(G) \to$

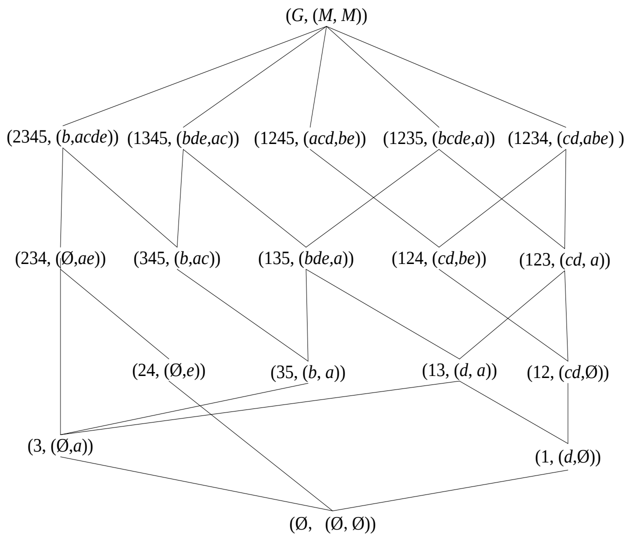

Fig. 3. OEOL of Table I

$\mathcal{DP}(M)$ and $\lhd : \mathcal{DP}(M) \to \mathcal{P}(G)$ by $X^{\rhd} = (X^{\natural}, X^{\overline{\natural}})$ and $(A,B)^{\lhd} = A^{\Diamond} \cup B^{\overline{\Diamond}}$. We call them a pair of three-way object oriented operators which abbreviate as OEO-operators.

Definition 2.6 [28], [29] Let (G,M,I) be a formal context, $X \subseteq G$ and $A, B \subseteq M$. If $X = (A,B)^{\lhd}$ and $X^{\rhd} = (A,B)$, then $(X,(A,B))$ is called the OEO-concept, X is called the extent and (A,B) is called the intent of $(X,(A,B))$. We use $OEOL(G,M,I)$ to express a set of all OEO-concepts. $\leq$ between $(X,(A,B))$ and $(Y,(C,D))$ is a binary relation defined by: $(X,(A,B)) \leq (Y,(C,D)) \Leftrightarrow X \subseteq Y \Leftrightarrow (A,B) \subseteq (C,D)$. We can prove $OEOL(G,M,I)$ with $\leq$ is a complete lattice called the three-way object oriented concept lattice (Simply mark it as OEOL).

Example 2.3 Figure 3 is OEOL of Table 1.

III. THE RELATIONSHIPS

In the section, we are going to consider the relationships among kinds of CL and OEOL based on some special contexts. To briefly describe the relationship, we give the following notation. $A \cong B$: A and B is isomorphic. $A \hat{\cong} B$: A and B is anti isomorphic. Simply mark $L(G,M,I^c)$ and $L_o(G,M,I^c)$ as NCL and NL_o, respectively.

A. The relationships between L_o and OEOL based on the DC

In order to facilitate research, we recalled the definition of the DC.

Definition 3.1 [30] Let (G,M,I) be a formal context, $a \in M$. If there is $b \in M$ for $a \in M$ which satisfies $a^* \cup b^* = G$ and $a^* \cap b^* = \emptyset$, then b is called a dual attribute of a.

Obviously, if b is called a dual attribute of a, then a is called a dual attribute of b. So, we called a and b as dual attributes.

Example 3.1 Table 2 is a formal context, $G = \{1,2,3\}$ and $M = \{a,b,c,d\}$. According to Definition 3.1, a and d are dual attributes.

Definition 3.2 [30] Suppose (G,M,I) be a formal context. For any $a \in M$, if there is a dual attribute of a, then (G,M,I) is called a dual context which abbreviated as DC.

TABLE II
A FORMAL CONTEXT (G,M,I)

G	a	b	c	d
1	1	1	0	0
2	0	0	1	1
3	1	0	1	0

Example 3.2 According to Definition 3.2, Table 2 is obviously a DC.

Theorem 3.1 If (G,M,I) is a clarified context and DC, then (1)$|M|$ is even number; (2)$L_o \cong NL_o$.

Proof (1) Because (G,M,I) is a DC, each attribute has dual attributes. Furthermore, each attribute corresponds to its dual attribute one by one, as a result of clarified. So $|M|$ is even number. (2) $\alpha : G \to G$ $\alpha(g) = g$, $\beta : M \to M$ $\alpha(m) = \hat{m}$, where a and m are dual attributes. So $(G,M,I) \cong (G,M,I^c)$. Thus $L_o \cong NL_o$.

Theorem 3.2 Let (G,M,I) is a clarified context and DC. Then $L_o \cong$ OEOL.

Proof Suppose $\alpha : L_o \to$ OEOL, for any $(X,A) \in L_o$, $\alpha(X,A) = (X,(A,X^{\overline{\natural}}))$. Now we just have to prove that it is an isomorphic mapping.

Firstly, we show that it makes sense. That is, we will prove $(X,(A,X^{\overline{\natural}})) \in$ OEOL. We will only prove that $X = A^{\Diamond} \cup X^{\overline{\natural}\overline{\Diamond}}$ by Definintion 2.5. In fact, since $(X,A) \in L_o$, we get $X = A^{\Diamond}$. And we obtain $X^{\overline{\natural}\overline{\Diamond}} \subseteq X$ by the properties of $\natural$ and $\Diamond$. So we obtain $(X,(A,X^{\overline{\natural}})) \in$ OEOL.

Secondly, we will prove α is a surjective. For any $(X,(A,B)) \in$ OEOL, we will only prove that $(X,A) \in L_o$. We can easily obtain $A = X^{\natural}$ by Definintion 2.5. Then we will only prove $X = A^{\Diamond}$ by Definintion 2.2. In fact, $B^{\overline{\Diamond}} = \hat{B}^{\Diamond}$. So, $X = A^{\Diamond} \cup B^{\overline{\Diamond}} = A^{\Diamond} \cup \hat{B}^{\Diamond} = (A \cup \hat{B})^{\Diamond}$. And then, we get X is the extent of the object oriented concep. That is, $X^{\natural\Diamond} = X$. Thus, $X = A^{\Diamond}$.

In addition, $(X,A) \neq (Y,B)$ if and only if $X \neq Y$. $X \neq Y$ if and only if $(X,(A,X^{\overline{\natural}})) \neq (Y,(B,Y^{\overline{\natural}}))$. $(X,(A,X^{\overline{\natural}})) \neq (Y,(B,Y^{\overline{\natural}}))$ if and only if $\alpha(X,A) \neq \alpha(Y,B)$. So $(X,A) \neq (Y,B)$ if and only if $\alpha(X,A) \neq \alpha(Y,B)$. Thus, α is a injective.

Finally, we show that it is an order isomorphism. In fact, $(X,A) \leq (Y,B)$ if and only if $X \leq Y$. $X \leq Y$ if and only if $(X,(A,X^{\overline{\natural}})) \leq (Y,(B,Y^{\overline{\natural}}))$. $(X,(A,X^{\overline{\natural}})) \leq (Y,(B,Y^{\overline{\natural}}))$ if and ony if $\alpha(X,A) \leq \alpha(Y,B)$. So, $(X,A) \leq (Y,B)$ if and only if $\alpha(X,A) \leq \alpha(Y,B)$.

From what has been discussed above, $L_o \cong$ OEOL.

Theorem 3.3 Suppose (G,M,I) be a DC, then $CL \hat{\cong} L_o$.

Proof Suppose $\varphi : CL \to L_o$, for any $(X,A) \in CL$, $\varphi(X,A) = (X^c, \hat{A})$. Now we just have to prove that it is an anti isomorphic mapping.

Firstly, we show that it makes sense. That is, we will prove $(X^c, \hat{A}) \in L_o$. We will only prove that $X^c = \hat{A}^{\Diamond}$ and $X^{c\natural} = \hat{A}$ by Definition 2.2. In fact, since $(X,A) \in CL$, $X^{c\natural} = \{m \in M | m^* \subseteq X^c\} = \{m \in M | X \cap m^{*c}\} = \{m \in M | X \cap m^{\overline{*}} = \emptyset\} = \{m \in M | X \cap m^{*c} = \emptyset\} = \{m \in M | gI^c m \text{ for all } g \in X\} = X^{\overline{*}}$. In DC, $X^{\overline{*}} = \hat{A}$, that is $X^{c\natural} = \hat{A}$. And then, $\hat{A}^{\Diamond} = \{g \in G | g^* \cap \hat{A} \neq \emptyset\}$, so $\hat{A}^{\Diamond c} = \{g \in G | g^{\overline{*}} \cap \hat{A} = \emptyset\} = \{g \in G | \hat{A} \subseteq g^{\overline{*}}\} = \hat{A}^{\overline{*}}$. In dual context, $A^* = \hat{A}^{\overline{*}}$, so $A^{*c} = \hat{A}^{\Diamond}$. Since $A^{*c} = X^c$, we get $X^c = \hat{A}^{\Diamond}$. Then we obtain $(X^c, \hat{A}) \in L_o$.

Secondly, we will prove φ is a surjective. For any $(X, A) \in L_o$, we will only prove that $(X^c, \hat{A}) \in CL$. We get $A = X^{\natural} = \{m \in M | m^* \subseteq X\}, X = A^{\Diamond}$ by Definintion 2.2. In fact, $X^{c*} = \{m \in M | X^c \subseteq m^*\} = \{m \in M | m^{*c} \subseteq X^c\} = \{m \in M | m^{\overline{*}} \subseteq X\} = \{\hat{m} \in M | \hat{m}^* \subseteq X\} = \hat{A}$. In addition, $\hat{A}^* = A^{\overline{*}}$ in the DC. So $\hat{A}^* = A^{\overline{*}} = \{g | A \subseteq g^{\overline{*}}\}$. And $A^{\Diamond c} = \{g | A \cap g^* \neq \emptyset\}^c = \{g | A \cap g^* = \emptyset\} = \{g | A \subseteq g^{\overline{*}}\}$. Thus, $\hat{A}^* = A^{\Diamond c} = X^c$.

In addition, $(X, A) \neq (Y, B)$ equivalent to $X \neq Y$. $X \neq Y$ equivalent to $X^c \neq Y^c$. $X^c \neq Y^c \iff (X^c, \hat{A}) \neq (Y^c, \hat{B})$. So $\varphi(X, A) \leq \varphi(Y, B)$. Thus, φ is a injective.

Finally, we show that it is an anti isomorphism. In fact, $(X, A) \leq (Y, B)$ equivalent to $X \leq Y$. $X \neq Y$ equivalent to $X^c \neq Y^c$. $X^c \neq Y^c \Longleftrightarrow (X^c, \hat{A}) \geq (Y^c, \hat{B})$. So $\varphi(X, A) \geq \varphi(Y, B)$.

From what has been discussed above, $CL \hat{\cong} L_o$.

Theorem 3.4 Suppose (G, M, I) be a DC, then $NL \hat{\cong}$ OEOL.

Proof Suppose $\beta : NL \to$ OEOL be a map, for any $(X, A) \in NL$, $\beta(X, A) = (X^c, (\hat{A}, X^{c\overline{\natural}}))$. In fact, we can have $\beta = \alpha \circ \varphi$ by Theorem 3.2 and Theorem 3.3. Through the properties of composite mapping, we can obtain β is an anti isomorphic mapping.

B. The relationships between L_o and OEOL based on the DIC

In order to facilitate research, we recalled the definition of the dual intersection context firstly.

Definition 3.3 [30] Suppose (G, M, I) be a formal context, $a \in M$. If $G \setminus a^* = \cap a_j^*$ where $a_j \in M$, then a is called a dual intersection attribute.

Definition 3.4 [30] (G, M, I) is called a attribute-induced dual intersection context when every $a \in M$ is a dual intersection attribute. Let's simply write it down as DIC.

By comparing above definitions, we note that DIC is the promotion of DC.

Theorem 3.5 Suppose (G, M, I) be a DIC, then the extent set L_o includeds in the extent set of NL_o.

Proof Taking $(X, A) \in L_o$, we have $X^{c\natural} = \{m \in M | m^* \subseteq X^c\} = \{m \in M | X \cap m^{*c}\} = \{m \in M | X \cap m^{\overline{*}} = \emptyset\} = \{m \in M | X \cap m^{*c} = \emptyset\} = \{m \in M | g I^c m$ for all $g \in X\} = X^{\overline{*}}$. Thus, $A = X^{\natural} = X^{c\overline{*}}$. Similarly, $X^c = A^{\Diamond c} = \{g | A \cap g^* \neq \emptyset\}^c = \{g | A \cap g^* = \emptyset\} = \{g | A \subseteq g^{\overline{*}}\} = A^{\overline{*}}$. Thus, $A = X^{c\overline{*}}$ and $X^c = A^{\overline{*}}$.

Due to the dual intersection property of (G, M, I), for any $m \in M$, we have $G \setminus m^* = \cap {m_j}^*$, that is $m^{\overline{*}} = \cap {m_j}^*$. For the above X^c, $X^c = \bigcap m^{\overline{*}}$, where $m \in A$. So, $X^c = \bigcap \cap {m_j}^*$. Thus, we get $X^{c**} = X^c$ is the extent of the formal concep by the properties of $*$.

In addition, $X^{c*} = \{m \in M | X^c \subseteq m^*\} = \{m \in M | m^{\overline{*}} \subseteq X\} = X^{\overline{\natural}}$. And for any $B \subseteq M$, $B^{\overline{\Diamond} c} = \{g | B \cap g^{\overline{*}} \neq \emptyset\}^c = \{g | B \cap g^{\overline{*}} = \emptyset\} = \{g | B \subseteq g^*\} = B^*$. Replace B with X^{c*}, we obtain $X^{c*\overline{\Diamond} c} = X^{c**}$. Since $X^{c**} = X^c$ and $X^{c*} = X^{\overline{\natural}}$, we get $X^{\overline{\natural}\overline{\Diamond}} = X$ Thus, X is the extent of NL_o.

Therefore, the extent set L_o includeds in the extent set of NL_o by arbitrariness of X.

TABLE III
A Formal Context (G, M, I)

G	a	b	c	d	e
1	1	1	0	1	0
2	0	1	1	0	1
3	1	0	1	0	1

In fact, L_o and NL_o are not isomorphic, when (G, M, I) is a DIC. Let's illustrate it.

Example 3.3 Table 3 is a formal context and Table 4 is its complement context, where the object set $G = \{1, 2, 3\}$, the attribute set $M = \{a, b, c, d, e\}$. I and I^c shown in tables. We can compute L_o, NL_o and OEOL which are shown in Figure 4, Figure 5 and Figure 6 respectively. Firstly we can easily verify that (G, M, I) is a DIC. But, we observed that (1)L_o and NL_o are not isomorphic: (2)NL_o and OEOL are not isomorphic in the DIC.

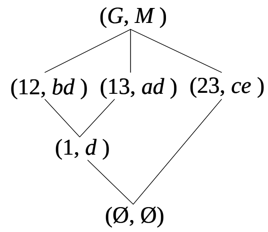

Fig. 4. L_o of Table III

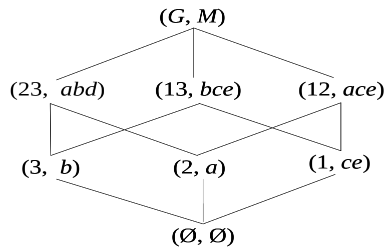

Fig. 5. NL_o of Table III

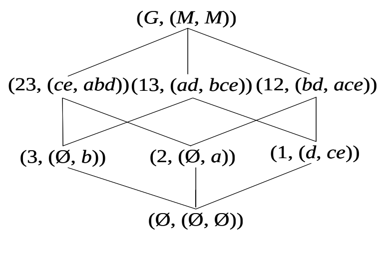

Fig. 6. OEOL of Table III

Theorem 3.6 Suppose (G, M, I^c) be a DIC, then $L_o \cong$ OEOL.

Proof For any X is the extent of OEOL, we might as well set $(X, (A, B)) \in$ OEOL, so $A = X^{\natural}, B = X^{\overline{\natural}}$ and $X = A^{\Diamond} \cup B^{\overline{\Diamond}}$ by Definintion 2.5. According to the condition and Theorem 3.5, we can obtain $L_{oE}(G, M, I^c)$ is a subset of $L_{oE}(G, M, I)$. And then $B^{\overline{\Diamond}} \in L_{oE}(G, M, I)$. Thus, $X = A^{\Diamond} \cup B^{\overline{\Diamond}} \in L_{oE}(G, M, I)$. Therefore, $OEOL_E(G, M, I)$ is a subset of $L_{oE}(G, M, I)$.

In addition, for any $X \in L_{oE}(G,M,I)$, obviously $(X,(X^{\natural}, X^{\overline{\natural}})) \in$ OEOL by the properties of $\natural$ and $\Diamond$. That is, $L_{oE}(G,M,I)$ is a subset of $OEOL_E(G,M,I)$.

To sum up, $L_{oE}(G,M,I)$ equals $OEOL_E(G,M,I)$. Define $\psi : L_o \to$ OEOL, for any $(X, X^{\natural}) \in L_o$, $\psi(X, X^{\natural}) = (X,(A,B))$. Obviously, ψ is an isomorphic map.

Theorem 3.7 Suppose (G,M,I) be a DIC, then $CL \hat{\cong} NL_o$.

Proof We construct $\gamma : CL \to NL_o$, $\gamma(X,A) = (X^c, A)$, where $(X,A) \in CL$. And prove it an isomorphic mapping.

Firstly, $(X,A) \in CL$, so $X^{c\overline{\natural}} = \{m \in M | m^{\overline{*}} \subseteq X^c\} = \{m \in M | X \cap m^*\} = \{m \in M | X \cap m^* = \emptyset\} = \{m \in M | X \cap m^* = \emptyset\} = \{m \in M | gIm$ for all $g \in X\} = X^* = A$. Secondly, $A^{\Diamond c} = \{g | A \cap g^{\overline{*}} \neq \emptyset\}^c = \{g | A \cap g^{\overline{*}} = \emptyset\} = \{g | A \subseteq g^*\} = A^* = X$. That is, $A^{\overline{\Diamond}} = X^c$. Thus, $(X^c, A) \in NL_o$. Similarily, if $(X,A) \in L_o$, then $(X^c, A) \in NL$. To sum up, φ is a surjection from CL to NL_o.

In addition, if $(X,A) \neq (Y,B)$,then $X^c \neq Y^c$. So $(X^c, A) \neq (Y^c, B)$, that is $\gamma(X,A) \leq \gamma(Y,B)$. Thus, γ is a injective.

Finally, we show that it is an anti isomorphism. In fact, $(X,A) \leq (Y,B)$ is equivalent to $X \leq Y$. $X \leq Y$ equivalent to $X^c \geq Y^c$. And we have $(X^c, A) \geq (Y^c, B) \iff \gamma(X,A) \geq \gamma(Y,B)$. Thus, $(X,A) \leq (Y,B)$ is equivalent to $\gamma(X,A) \geq \gamma(Y,B)$.

From what has been discussed above, $CL \hat{\cong} NL_o$.

Theorem 3.8 Suppose (G,M,I) be a DIC, then $CL \hat{\cong}$ OEOL.

Proof Let $\eta = \psi \circ \gamma|_{L_E(G,M,I)}$, ψ and γ come from Theorem 3.6 and Theorem 3.7, respectively. Obviously, it is an anti isomorphic mapping through the properties of composite mapping.

Combined with literature [30], the relations of concept lattices are shown in the Figure 7 and Figure 8 below based on the DC and the DIC, respectively.

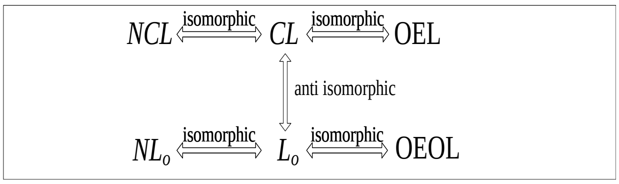

Fig. 7. the relationships based on the DC

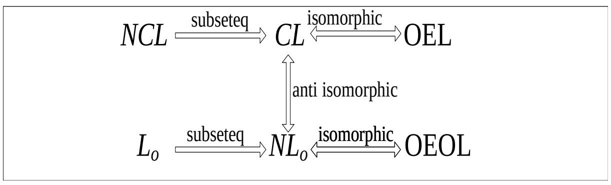

Fig. 8. the relationships based on the DIC

IV. Conclusion

In the paper, we study the relations between L_o and OEOL based on the DC and the DIC, respectively. Firstly, we prove that L_o and OEOL are isomorphic based on the DC but they are not isomorphic based on the DIC. In addition, the anti isomorphic relationship between CL and OEOL are both obtained based on the DC and the DIC, respectively. Finally, the isomorphic relation between L_o and OEOL is obtained when its complement context is a DIC. In the future, the relationship between L_p and OEPL based on the DC and the DIC will be studied, respectively. The necessary and sufficient conditions for their isomorphism will be discussed.

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