On the Dominant Local Metric Dimension of Corona Product Graphs

Reni Umilasari, Liliek Susilowati*, Slamin and Savari Prabhu

***Abstract*—A nontrivial connected graph T which one of the vertex is v, v is said to distinguish two vertex u, t if the distance between v and u is different from v to t, where $u, t \in V(T)$. Metric dimension is one topic in graph theory that uses the concept of distance. Combining the definition of the local metric dimension and dominating set, there is a new term, we called it dominant local metric dimension and symbolized as $Ddim_l(T)$. An ordered subset $W_l = \{w_1, w_2, \ldots, w_n\} \subseteq V(T)$ is called a dominant local resolving set of T if W_l is a local resolving set as well as a dominating set of T. The goal of this paper's research is to determine precise values of dominant local metric dimension for the corona product graphs. n copies of the graphs $P_1, P_2, ..., P_n$ of P are made to constructed the corona of any two graph T and P. After that, we link the i-th vertex of T to the vertices of P_i, where n is an order of graph T. T corona P is symbolized by $T \odot P$.**

***Index Terms*—dominating set, metric dimension, local resolving set, local metric dimension.**

I. Introduction

GRAPH theory is one of the theory in mathematics. In general, a graph can be described as a non-empty set with members referred to as vertices and an empty set with elements referred to as edges, which are an unordered pair of two different vertices. If an edge connects two vertices in a graph, they are said to be neighbors. The number of graph theory research topics keeps expanding. Dominating set and dominating number, graph coloring, graph labeling, and metric dimension are a few of the issues that have grown in popularity in the field of graph theory.

As early as 1850, the dominating set and dominating number were invented. Since European chess players are obsessed with finding solutions to the "dominating" problem queens, this hypothesis first emerged. On that game, the number of queens is determined by the "dominating set," which allows each queen to attack or dominate every position with a single move. In graph theory, queens are represented as vertices, and the paths queens take to travel between the chessboard's boxes are referred to as edges. Early in the 1960s, dominating set was introduced as formal theory. After that it was extensive use of both the dominating set and the dominating number. In a variety of applications, such as figuring out where to put how many cameras position of the supervisor in a building's hallway as well as the quantity of traffic police officers stationed at city corners to ensure that every street can be thoroughly monitored. Dominating set of a graph H usually symbolized as S. In formal definition, it is a subset of $V(H)$ which connected all vertices $V(H)\backslash S$ to S. We can write that if j is any vertex of $V(H)\backslash S$ then $d(j, x) = 1$, where $x \in S$. Minimum number of j is called the dominating number of graph H and symbolized $\gamma(H)$. As a result, the dominating set and the dominant number are tightly related.

Some researchers can develop the topics in real life like the benefit of metric dimension in real life, one of them is written by Khuller, et al., which was inspired by the movement of robots in two-dimensional Euclid space (R^2), Khuller, et al. attempted to relate the metric dimension concept to the navigation of a robot, they refer to a motion field as a graph. The vertex in the graph is the place where the robot stops or does activities, while the edges are the paths where the robot walks [1]. Another application of this topic in the graph is dominating set which can be used to determine the placement of ATM in some locations [2]. Several researchers have studied and developed metric dimensions in the last decades. They developed the concept of metric dimensions. For example, the fractional metric dimension [3] and also the characterization written by Arumugam and Varughese [4], and a few writers combined the combination of the notions of group in algebra and metric dimension, such as research conducted by Bazak which focuses on finding the Zero-Divisor Graph for the Ring Zn [5]. We refer to [6], [7], [8], and [9] for more information related to metric dimension and local metric dimension.

Some research in dominating set and metric dimensions conducted by Foucaud, et al. generated the formulas and metric dimension complexity and location domination on intervals and permutations graphs [10], and Susilowati, et al. figured out the dominant metric dimension of some specific graphs [11]. Then, the dominant local metric dimension is formulatted by Umilasari, et al. They mention if Wl is a local resolving set and a dominating set of T, then it is referred to as a dominant local resolving set, where T is connected graph and an ordered set $W_l = w_1, w_2, \ldots, w_n V(T)$. The dominant local basis is the dominant local resolving set with the smallest cardinality. The dominant local metric dimension, also known as the number of vertices in the dominant local basis of G, is denoted by $Ddim_l(G)$ [12]. Certain characteristics of the dominant local resolving set, lower and upper bound of the dominant local metric dimension, and the major finding of the paper give some exact value of the dominant local metric dimension for certain classes of graphs. In this paper, we look deeper into the concept by observing the value of the dominant local metric dimension of product graphs, especially in corona product of graphs.

Manuscript received January 4, 2022; revised October 21, 2022. This work was supported by DRPM, KEMENRISTEK of Indonesia, Dcree 8/E1/KPT/2021, Contract No.4/E1/KP.PTNBH/2021 and 460/UN3.15/PT/2021, year 2021.

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n copies of the graphs $P_1, P_2, ..., P_n$ of P are made to constructed the corona of any two graph T and P. After that, we link the i-th vertex of T to the vertices of P_i, where n is an order of graph T. T corona P is symbolized by $T \odot P$ [13]. Throughout this part, we speak of P_i as an i-th copy of P connected to i-th vertex of P in $T \odot P$ for every $i \in 1, 2, \ldots, n$. Before presenting the main result of this research, we give some theorems about $\gamma(T)$ and $dim(T)$

Theorem 1.1 [14] The dominating number of some graphs are given below:

a. Let $T = P_m$ or $R = C_n$ with $m \geq 2$ and $n \geq 3$, then $\gamma(T) = \lceil \frac{|V(G)|}{3} \rceil$
b. Let $T = K_m$ or $T = K_{1,n-1}$. If $m, \geq 1, n \geq 2 \rightarrow \gamma(T) = 1$.
c. Let $T = K_{m,n}$ with $m, n \geq 3$, then $\gamma(T) = 2$.

Theorem 1.2 [15] Let H be a nontrivial connected graph with $|V(H)| = n$.

- $dim_l(H) = n - 1 \Leftrightarrow H = K_n$
- $dim_l(H) = n \Leftrightarrow H$ is bipartite

II. THE CHARACTERISTIC OF $Ddim_l(G)$ AND THE EXCAT VALUE FOR SPECIAL GRAPHS

This section shows some results relating to the characteristic of the local resolving set and $Ddim_l$ of some graphs in which results have been presented by Umilasari et al. [16].

Lemma 2.1 Given a connected graph H. If there is $S \subseteq V(H)$, then for every set H containing a local resolving set is a local resolving set.

Lemma 2.2 Given a connected graph H. If there is $W_l \subseteq V(H)$, $\forall v_i, v_j \in W_l \Rightarrow r(v_i|W_l) \neq r(v_j|W_l)$.

Lemma 2.3 Given a connected graph H with the order j, then $max\{\gamma(H), dim_l(H)\} \leq Ddim_l(H) \leq min\{\gamma(H) + dim_l(H), j - 1\}$.

Theorem 2.4

a. If $k \geq 2$, then $Ddim_l(P_k) = \gamma(P_k)$.
b. If $k \geq 4$, then $Ddim_l(C_k) = \gamma(C_k)$.
c. $Ddim_l(H) = 1 \Leftrightarrow H \cong S_k$.
d. $Ddim_l(H) = k - 1 \Leftrightarrow H \cong K_j$, $j \geq 2$.
e. If $p \geq 2$, $q \geq 2$, then $Ddim_l(K_{p,q}) = \gamma(K_{p,q})$.

III. MAIN RESULT

This section presents the value of $Ddim_l(G \odot H)$ where G is any graphs and H is special graph. First of all, we show a Lemma as the property of a local resolving set, then we describe the proof of $Ddim_l(G \odot P_m), Ddim_l(G \odot C_n), Ddim_l(G \odot K_n), Ddim_l(G \odot S_n)$ and $Ddim_l(G \odot K_{m,n})$.

Lemma 3.1 Given a connected graph G. If there is no local dominant resolving set with cardinality p, then $\forall S \subseteq V(G)$ and $|S| < p$ is not a local dominant resolving set.

Proof Suppose that there is $S \subseteq V(G)$ with $|S| < p$ as a local dominant resolving set, so for every $uv \in E(G)$ we get $r(u|S) \neq r(v|S)$. Then, we can find a set $T \subseteq V(G) - S$, in case $|S \cup T| = p$, such that $S \cup T$ is a local dominant resolving set too. Consequently, there is a contradiction between the first and the final statement. □

The next lemma shows that $Ddim_l(K_1 + P_n)$ has relation with the $Ddim_l(P_n)$.

Lemma 3.2 If P_n is a path graph, with $|V(P_n)| = n \geq 5$, then

$$Ddim_l(K_1 + P_n) = Ddim_l(P_n).$$

Proof. Let $V(K_1) = \{u\}$ and $V(P_n) = \{v_i | 1 \leq i \leq n\}$ with $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n - 1\}$. We give labels of $K_1 + P_n$ is $V(K_1 + P_n) = \{u, v_i | 1 \leq i \leq n\}$, while for the edge we write $E(K_1 + P_n) = \{uv_i, | 1 \leq i \leq n\}$ $\cup \{v_i v_{i+1} | 1 \leq i \leq n - 1\}$. Based on Theorem 2.4, we know that $Ddim_l(P_n) = \lceil \frac{n}{3} \rceil$. To determine $Ddim_l(K_1 + P_n)$, it divides into two cases as follows.

a. $n \equiv 0 (mod\ 3)$
Put $W_l = \{v_2, v_5, v_8, v_{11}, \ldots, v_{3i-1}\}$ so that $|W_l| = \lceil \frac{n}{3} \rceil = Ddim_l(P_n)$. Based on Lemma 2.2 $\forall v_i, v_j \in W_l$ we get $r(v_i|W_l) \neq r(v_j|W_l)$ with $i \neq j$. Since $\forall u, v_i \in V(K_1 + P_n) \backslash W_l$ we have gotten:

$$r(u|W_l) = (\underbrace{1, 1, 1, \ldots, 1}_{\lceil \frac{n}{3} \rceil - tuple})$$

$$r(v_i|W_l) = \begin{cases} (1, 2, 2, 2, \ldots, 2, 2, 2), \\ \text{for } i = 1, 3 \\ (2, 2, 2, \underbrace{1}_{\lceil \frac{i}{3} \rceil^{th}}, 2, 2, \ldots, 2, 2, 2), \\ \text{for } i \equiv 0, 1 (mod\ 3), 3 < i < n - 2 \\ (2, 2, 2, \ldots, 2, 2, 2, 1), \\ \text{for } i = n - 2, n \end{cases}$$

Furthermore, every two adjacent vertices have different representations toward W_l. Since P_n is a path with $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n - 1\}$, then every v_{3i-1} is adjacent to v_{3i-2} and v_{3i} is adjacent to $v_{3(i+1)-2}$. Beside that, u_1 is adjacent to every vertex in W_l. Therefore, W_l is a local dominant resolving set with lowest cardinality. Choose any $S \subseteq V(K_1 + P_n)$ with $|S| < |W_l|, |S| = |W_l| - 1$. Then it will be shown 2 cases for S.

i. S does not contain u, then $S \subseteq V(P_n)$. Based on Theorem 1.1, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. Since $S < \lceil \frac{n}{3} \rceil$, then S is not a dominating set of $K_1 + P_n$.
ii. S contains u, then the vertex set of P_n which are also the elements of S consist of $\lceil \frac{n}{3} \rceil - 2$ elements. Hence, there exists $v_f, v_{f+5} \in S$, and $v_{f+1}, v_{f+2}, v_{f+3}, v_{f+4} \notin S$. Therefore, $r(v_{f+2}|S) = r(v_{f+3}|S)$. Consequently, S is not a local resolving set of $K_1 + P_n$.

Considering the two scenarios described above, S is not a local dominant resolving set of $K_1 + P_n$. By Lemma 3.1, it means W_l is a local dominant basis of $K_1 + P_n$ for $n \equiv 0 (mod\ 3)$.

b. $n \not\equiv 0 (mod\ 3)$
Put $W_l = \{v_2, v_5, v_8, v_{11}, \ldots, v_{3i-1}\}$ so that $|W_l| = \lceil \frac{n}{3} \rceil = Ddim_l(P_n)$. Based on Lemma 2.2 $\forall v_i, v_j \in W_l$ we get are $r(v_i|W_l) \neq r(v_j|W_l)$ with $i \neq j$. Then, $\forall u, v_i \in V(K_1 + P_n)$ we get:

$$r(u|W_l) = (\underbrace{1, 1, 1, \ldots, 1}_{\lceil \frac{n}{3} \rceil - tuple})$$

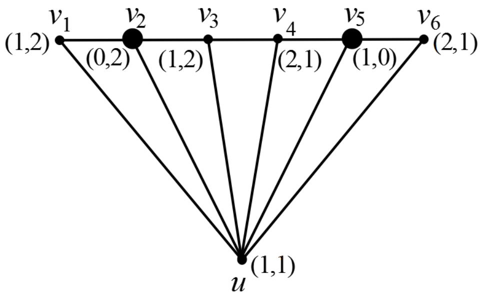

Fig. 1. $K_1 + P_6$ has dominant local metric dimension equals two.

$$r(v_i|W_l) = \begin{cases} (1,2,2,2,\ldots,2,2,2), \\ \text{for } i = 1,3 \\ (2,2,2,\underbrace{1}_{\lceil \frac{i}{3} \rceil^{th}},2,2,\ldots,2,2,2), \\ \text{for } i \equiv 1,2(mod\ 3), 3 < i < n-2 \\ (2,2,2,\ldots,2,2,2,1), \\ \text{for } i \equiv 2(mod\ 3), i = n-1 \\ (2,2,2,\ldots,2,2,1,1), \\ \text{for } i \equiv 1(mod\ 3), i = n-1 \end{cases}$$

Furthermore, every two adjacent vertices have different representations toward W_l. Since P_n is a path graph with $E(P_n) = \{v_i v_{i+1} | 1 \leq i \leq n-1\}$, then every v_{3i-1} is adjacent to v_{3i-2} and v_{3i} is adjacent to $v_{3(i+1)-2}$. Beside that, u_1 is adjacent to every vertex in W_l. Therefore, W_l is a dominating set of $K_1 + P_n$. Thus, W_l is a local dominant resolving set with lowest cardinality. Take any $S \subseteq V(K_1 + P_n)$ with $|S| < |W_l|, |S| = |W_l| - 1$. Then there are two cases for S.

i. S does not contain u, then $S \subseteq V(P_n)$. Based on Theorem 1.1, $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. Since $S < \lceil \frac{n}{3} \rceil$, then S is not a dominating set of $K_1 + P_n$.

ii. S contains u, then the vertex set of P_n which are also the elements of S consist of $\lceil \frac{n}{3} \rceil - 2$ elements. Hence, there exists $v_f, v_{f+5} \in S$, and $v_{f+1}, v_{f+2}, v_{f+3}, v_{f+4} \notin S$. Therefore, $r(v_{f+2}|S) = r(v_{f+3}|S)$. Consequently, S is not a local resolving set of $K_1 + P_n$.

Considering the two scenarios described above,, S is not a local dominant resolving set of $K_1 + P_n$. By Lemma 3.1, it can be said that W_l is a local dominant basis of $K_1 + P_n, n \not\cong 0(mod\ 3)$.

From the 2 conditions above, it has been proven that $W_l = \lceil \frac{n}{3} \rceil = Ddim_l(P_n)$ is a local dominant basis of $K_1 + P_n$, for $n \geq 5$. Then, it can be concluded that $Ddim_l(K_1 + P_n) = Ddim_l(P_n)$ for $n \geq 5$. □

Figure 1 shows that the bigger vertices form the dominant local basis of $K_1 + P_6$. Next, the $Ddim_l$ of a connected graph operated by corona product to path graph is presented below.

Theorem 3.3 Given a connected graph G, $|V(G)| = m \geq 2$.If P_n is a path, with $n \geq 4$, then

$$Ddim_l(G \odot P_n) = |V(G)| \times Ddim_l(P_n).$$

Proof. Let $V(G) = \{u_i | 1 \leq i \leq m\}$, $V(P_n) = \{v_j | 1 \leq j \leq n\}$ and $E(P_n) = \{v_j v_{j+1} | 1 \leq j \leq n-1\}$. The i-th copy of P_n with $1 \leq i \leq m$ is called $(P_n)_i$ with $V((P_n)_i) = \{v_{ij} | 1 \leq j \leq n\}$. We give the vertex label of $G \odot P_n$ is $V(G \odot P_n) = \{v_{0i} | 1 \leq i \leq m, u_i \in V(G)\} \cup \{v_{ij} | 1 \leq i \leq m, 1 \leq j \leq n, v_i \in V(P_n)\}$, and $E(G \odot P_n) = \{v_{0i} v_{0j} | u_i u_j \in E(P_m)\} \cup \{v_{ij} v_{i(j+1)} | v_j v_{(j+1)} \in E(P_n)\} \cup \{v_{0i} v_{ij} | u_i \in V(G), v_j \in V(P_n)\}$. To show the $Ddim_l(P_n)$, we divided the number of n into 2 cases.

1. For $n = 4$
We can demonstrate that $B_i = \{v_{i2}, v_{i3}\}$ is a dominant local resolving set of $K_1 + (P_4)_i$. $\forall 1 \leq i \leq m, |B_i| = |B| = 2$. Choose $W_l = \cup_{i=1}^{m} B_i$ so $|W_l| = |V(G)| \times 2$. Without remove of generality, it can be observed that the representation of every vertex $V(P_4)_i \backslash B_i$ to B_i is different.
$$r(v_{i1}|Bi) = (1,2)$$
$$r(v_{i4}|Bi) = (2,1)$$
$$r(v_{0i}|Bi) = (1,1)$$
Because every two adjacent vertices in $V(P_4)_i$ have a distinct representation with regard to B_i then it is established that B_i is a local resolving set of $(P_4)_i$. Here, since $W_l \subseteq V(G \odot P_4)$ and $B_i \subseteq W_l$, by Lemma 2.1 then W_l is a local resolving set of $G \odot P_4$. Because u_i and v_{i1} is adjacent with v_{i2} and v_{i4} is adjacent with v_{i3} so that W_l is a dominating set of $G \odot P_4$. Thus, W_l is a dominant local metric dimension of $G \odot P_4$. Next, we choose any $S \subseteq V(G \odot P_4)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$, then there exists i such that S consist of maximally $|B_i| - 1$ elements of $V(P_4)_i$. Since B_i is a dominant local basis of $(P_4)_i$ then there exist two vertex elements of $V((P_4)_i)$ which have the same representation to S or there exists a vertex of $V((P_4)_i)$ which is not adjacent with any vertex in S, so that S is not a local resolving set or S is not a dominating set of $G \odot P_4$. Look at Lemma 3.1 then $W_l = \cup_{i=1}^{m} B_i$ is the dominant local basis of $G \odot P_4$. Hence, $Ddim_l(G \odot P_4) = |V(G)| \times Ddim_l(P_4)$ for $n = 4$.
2. For $n \geq 5$
Let B be a local dominant basis of P_n, B_i is a local dominant basis of $(P_n)_i$, hence for every $i = 1,2,3,\ldots,m, |B_i| = |B|$. Choose $W_l = \cup_{i=1}^{m} B_i$, by Lemma 3.2 since B_i is a local dominant basis of $K_1 + (P_n)_i$ then W_l is a local dominant resolving set of $G \odot P_n$. Then, it will be demonstrated that W_l is a local dominant resolving set with lowest cardinality. Take any $S \subseteq V(G \odot P_n)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$, then there exists i such that S comprise maximally $|B_i| - 1$ elements of $K_1 + (P_n)_i$. Since B_i is a local dominant basis of $K_1 + (P_n)_i$ then there exist two vertices in $K_1 + (P_n)_i$ have the same representation, so that S is neither a local resolving set nor dominating set of $G \odot P_n$. Based on Lemma 3.1 then $W_l = \cup_{i=1}^{m} B_i$ is a local dominant basis of $G \odot P_n$. Since $|B_i| = Ddim_l(K_1 + (P_n)_i)$ and by Lemma 3.2 we know that $Ddim_l(K_1 + P_n) = Ddim_l(P_n)$. Therefore, it has been proven that for $n \geq 5$ $Ddim_l(G \odot P_n) = |V(G)| \times Ddim_l(P_n)$.

From the explanation above in poin (1) and (2), it is proven that for $n \geq 4$, $Ddim_l(G \odot P_n) = |V(G)| \times Ddim_l(P_n)$. □

Figure 2 shows the example of $C_3 \odot P_6$ has dominant local metric dimension equals six. The dominant local basis is shown by bigger vertices. Following that, we demonstrate

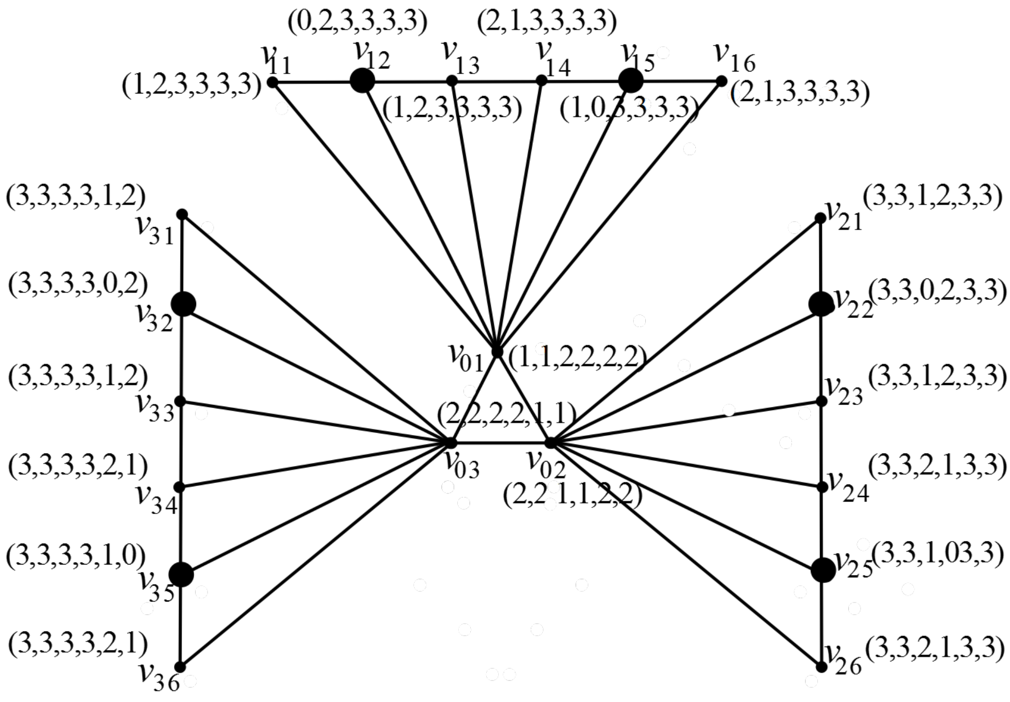

Fig. 2. $C_3 \odot P_6$ has dominant local metric dimension equals six.

the dominant local metric dimension of $G \odot C_m$. Before it, we would show a Lemma related to the proof.

Lemma 3.4 If C_m is a cycle and $|V(C_m)| = m \geq 6$, then

$$Ddim_l(K_1 + C_m) = Ddim_l(C_m).$$

Proof. Let $V(K_1) = \{u_1\}$, $V(C_m) = \{v_i | i = 1, 2, 3, \ldots, m\}$ and $E(C_m) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, m - 1\} \cup \{v_1 v_m\}$.We give the vertex labels of $K_1 + C_m$, that is $V(K_1 + C_m) = \{u_1, v_i | 1 \leq i \leq m\}$, while $E(K_1 + C_m) = \{u_1 v_i, |1 \leq i \leq m\} \cup \{v_i v_{i+1} | 1 \leq i \leq m-1\} \cup \{v_1 v_m\}$. To ascertain $Ddim_l(K_1 + C_m$, it divides into 2 parts.

a For $m \equiv 0 (mod\ 3)$

Put $W_l = \{v_1, v_4, v_7, v_{10}, \ldots, v_{3i-2}\}$, then $|W_l| = \lceil \frac{m}{3} \rceil = Ddim_l(C_m)$. Based on Lemma 2.2 for every $v_i, v_j \in W_l$ we get $r(v_i|W_l) \neq r(v_j|W_l)$ for $i \neq j$. Next, since for every u_1 and $v_i \in V(K_1+C_m) \backslash W_l$ we get:

$$r(u_1|W_l) = (\underbrace{1, 1, 1, \ldots, 1}_{\lceil \frac{m}{3} \rceil - tuple})$$

$$r(v_i|W_l) = \begin{cases} (1, 2, 2, 2, \ldots, 2, 2, 2), \\ \text{for } i = 2, m \\ (2, 2, 2, \underbrace{1}_{\lfloor \frac{i}{3} \rfloor^{th}}, 2, 2, \ldots, 2, 2, 2), \\ \text{for } 2 < i < m-3 \\ (2, 2, 2, \ldots, 2, 2, 2, 1), \\ \text{for } i = m-1, m-3 \end{cases}$$

Furthermore, every two adjacent vertices have a different representation to W_l then W_l is a local resolving set of $K_1 + C_m$. Since C_m is a cycle with $E(C_m) = \{v_i v_{i+1} | i = 1, 2, 3, \ldots, m-1\} \cup \{v_1 v_m\}$, then for every vertex v_{3i-1} is adjacent to v_{3i-2} and v_{3i} is adjacent to $v_{3(i+1)-2}$. Besides that, the vertex u_1 is adjacent to the elements of W_l. Therefore, W_l is dominant local resolving set of $K_1 + C_m$. Next, we choose any $S \subseteq V(K_1 + C_m)$ with $|S| < |W_l|$, let $|S| = |W_l| - 1$. Then, there are 2 conditions of S.

i S does not contain u_1, then $S \subseteq V(C_m)$. Based on Theorem 1.1, $\gamma(C_m) = \lceil \frac{m}{3} \rceil$. Because of $S < \lceil \frac{m}{3} \rceil$, then S is not dominating set of $K_1 + C_m$.

ii S contains u_1, then the elements of $V(C_m)$ which are also the element of S consist of $\lceil \frac{m}{3} \rceil - 2$ elements. Hence, there exists $v_f, v_{f+5} \in S$ and $v_{f+1}, v_{f+2}, v_{f+3}, v_{f+4} \notin S$. Consequently, $r(v_{f+2}|S) = r(v_{f+3}|S)$. Therefore, S is not a local resolving set of $K_1 + C_m$.

Considering the two scenarios described above, S is not a dominant local resolving set of $K_1 + C_m$. By Lemma 3.1, it can be concluded that W_l is a local dominant basis of $K_1 + C_m, m \equiv 0(mod\ 3)$.

b For $m \not\equiv 0(mod\ 3)$

Put $W_l = \{v_1, v_4, v_7, v_10, \ldots, v_{3i-2}, v_m\}$ then $|W_l| = \lceil \frac{m}{3} \rceil = Ddim_l(C_m)$. Based on Lemma 2.2 for every $v_i, v_j \in W_l$ we get $r(v_i|W_l) r(v_j|W_l)$ for $i \neq j$. Next, since for every u_1 and $v_i \in V(K_1 + C_m) \backslash W_l$ we get:

$$r(u_1|W_l) = (\underbrace{1, 1, 1, \ldots, 1}_{\lceil \frac{m}{3} \rceil - tuple})$$

$$r(v_i|W_l) = \begin{cases} (1, 2, 2, 2, \ldots, 2, 2, 2), \\ \text{for } i = 2 \\ (2, 2, 2, \underbrace{1}_{\lfloor \frac{i}{3} \rfloor^{th}}, 2, 2, \ldots, 2, 2, 2), \\ \text{for } 2 < i < m-2 \\ (2, 2, 2, \ldots, 2, 2, 2, 1), \\ \text{for } i = m-1 \end{cases}$$

Furthermore, every two adjacent vertices have a different representation to W_l then W_l is a local resolving set of $K_1 + C_m$. Since C_m is a cycle with $E(C_m) = \{v_i v_{i+1} | i = 1, 2, 3, \dot{;} m-1\} \cup \{v_1 v_m\}$, then for every

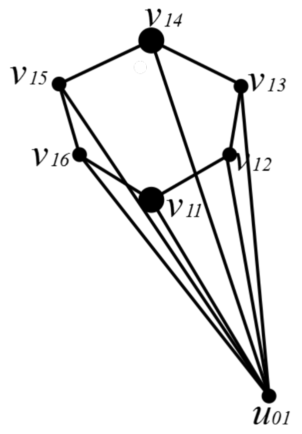

Fig. 3. $K_1 \odot C_6$ has dominant local metric dimension equals two.

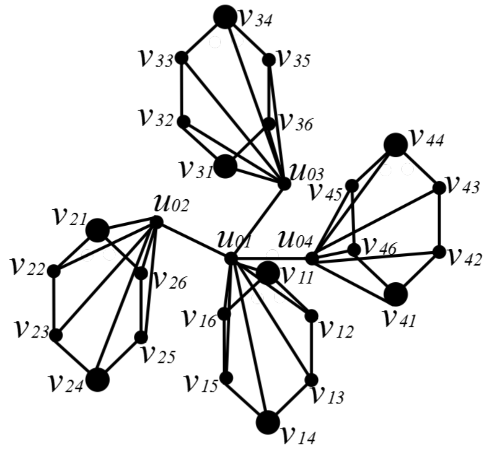

Fig. 4. $S_4 \odot C_6$ has dominant local metric dimension equals eight.

vertex v_{3i-1} is adjacent to v_{3i-2} and v_{3i} is adjacent to $v_{3(i+1)-2}$. Besides that, the vertex u_1 is adjacent to the elements of W_l. Therefore, W_l is dominating set of $K_1 + C_m$ and W_l is dominant local resolving set of $K_1 + C_m$. Next, it is shown that W_l is a dominant local resolving set with minimum cardinality. Choose any $S \subseteq V(K_1 + C_m)$ with $|S| < |W_l|$, $|S| = |W_l| - 1$. Then, there are two cases of S.

i. S does not contain u_1, then $S \subseteq V(C_m)$. Based on Theorem 1.1, $\gamma(C_m) = \lceil \frac{m}{3} \rceil$. Because of $S < \lceil \frac{m}{3} \rceil$, then S is not dominating set of $K_1 + C_m$.
ii. S contains u_1, then the elements of $V(C_m)$ which are also the element of S consist of $\lceil \frac{m}{3} \rceil - 2$ elements. Hence, there exists $v_x, v_{x+5} \in S$ and $v_{x+1}, v_{x+2}, v_{x+3}, v_{x+4} \notin S$. Consequently, $r(v_{x+2}|S) = r(v_{x+3}|S)$. Therefore, S is not a local resolving set of $K_1 + C_m$.

Considering the two scenarios described above, S is not a dominant local dominant resolving set of $K_1 + C_m$. By Lemma 3.1, it can be concluded that W_l is a local dominant basis of $K_1 + C_m, m \equiv 0(mod\ 3)$.

From the two possibilities in poin (a) and (b), it has been proven that $W_l = \lceil \frac{m}{3} \rceil$ is a local dominant basis of $K_1 + C_m$, for $m \geq 6$. Based on Theorem 2.4, we know that $Ddim_l(C_m) = \lceil \frac{m}{3} \rceil$. Then, it can be concluded that for $m \geq 6$, $Ddim_l(K_1 + C_m) = Ddim_l(C_m)$. □

Figure 3 gives the example that $Ddim_l(K_1 + C_6) = Ddim_l(C_6) = 2$. While Figure 4 shows that $Ddim_l(S_4 \odot C_6) = 8$.

Theorem 3.5 Given a connected graph G, $|V(G)| = n \geq 2$. If C_m is a cycle with $m \geq 6$, then

$$Ddim_l(G \odot C_m) = |V(G)| \times Ddim_l(C_m).$$

Proof. Let $V(G) = \{u_i | 1 \leq i \leq n\}$, $V(C_m) = \{v_j | 1 \leq j \leq m\}$ and $E(C_m) = \{v_j v_{j+1} | 1 \leq j \leq m-1\} \cup \{v_m v_1\}$. The i-th copy of C_m with $1 \leq i \leq n$ is called $(C_m)_i$ with $V(C_m)_i) = \{v_{ij} | 1 \leq j \leq m\}$. We give the labels of $G \odot C_m$ by $V(G \odot C_m) = \{v_{0i} | 1 \leq i \leq n, u_i \in V(G)\}$ $\cup \{v_{ij} | 1 \leq i \leq n, 1 \leq j \leq m, v_i \in V(C_m)\}$, and $E(G \odot C_m) = E(G) \cup_{i=1}^{n} E(C_m)_i \cup \{u_i v_{ij} | u_i \in V(G), v_{ij} \in V(C_m)_i\}$. Let B as a local dominant basis of $K_1 + C_m$, B_i is a local dominant basis of $K_1 + (C_m)_i$ so that for every $i = 1, 2, 3, \ldots, n, |B_i| = |B|$. Select $W_l = \cup_{i=1}^{n} B_i$, based on Lemma 3.2 since B_i is a local dominant basis of $K_1 + (C_m)_i$ then W_l is a local dominant resolving set of $G \odot C_m$. Next, we take any $S \subseteq V(G \odot C_m)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$, then $\exists i \ni |S| \leq |B_i| - 1$. Since B_i is a local dominant basis of $K_1 + (C_m)_i$ then there exist two vertices in $K_1 + (C_m)_i$ have the same representation or there exists a vertex in $K_1 + (C_m)_i$ that is not adjacent to any vertex in S, so that S is not a local resolving set or S is not a dominating set of $G \odot C_m$. Based on Lemma 3.1 then $W_l = \cup_{i=1}^{m} B_i$ is a local dominant basis of $G \odot C_m$. Since B_i is a local dominant basis of $K_1 + (C_m)_i$ with $|B_i| = Ddim_l(K_1 + (C_m)_i)$ and by Lemma 3.3 we know that $Ddim_l(K_1 + C_m) = Ddim_l(C_m)$. Therefore, it has been proven that for $m \geq 6$ $Ddim_l(G \odot C_m) = |V(G)| \times Ddim_l(C_m)$. □

The following theorems explain $Ddim_l(G \odot K_n), Ddim_l(G \odot S_n), Ddim_l(G \odot K_{m,n})$.

Theorem 3.6 Given a connected graph G, $|V(G)| = m \geq 2$. If K_n is a complete graph with $n \geq 2$, then

$$Ddim_l(G \odot K_n) = |V(G)| \times Ddim_l(K_n)$$

Proof. Let $V(G) = \{u_i | 1 \leq i \leq m\}$, $V(K_n) = \{v_j | 1 \leq j \leq n\}$ and $E(K_n) = \{v_j v_k | 1 \leq j, k \leq n, j \neq k\}$. The i-th copy of K_n with $1 \leq i \leq m$ is called $(K_n)_i$ whose the vertex and edge label are $V((K_n)_i) = \{v_{ij} | 1 \leq j \leq n\}$ and $E(K_n)_i) = \{v_{ij} v_{ik} | v_j v_k \in E(K_n)\}$ for every $1 \leq i \leq m$. While $G \odot K_n$ has $V(G \odot K_n) = \{v_{0i} | 1 \leq i \leq m\} \cup \{v_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G \odot K_n) = \{v_{0i} v_{0j} | u_i u_j \in E(G)\} \cup \{v_{ij} v_{ik} | v_j v_k \in E(K_n)\} \cup \{v_{0i} v_{ij} | u_i \in V(G), v_j \in V(K_n)\}$. Suppose B be a local dominant basis of K_n, B_i for $(K_n)_i$, thus $\forall 1 \leq i \leq m, |B_i| = |B|$. Put $W_l = \cup_{i=}^{m} B_i$, with $B_i = \{v_{ij} | 1 \leq j \leq n-1\}$ for every $1 \leq i \leq m$, then $|W_l| = m(n-1)$. Derived from Lemma 2.2, select two adjacent vertices in $V(G \odot K_n) \backslash W_l$. In every case, every two adjacent vertices have different representations toward W_l.

Take any $x, y \in V(G \odot K_n)$ with $xy \in E(G \odot K_n)$, then there exist three cases:

i. For $x, y = v_{0i}, v_{0j} \in V(G \odot K_n) \backslash W_l$ with $i \neq j$. Since G is a connected graph, $d(v_{0j}, v) = d(v_{0j}, v_{0i}) + d(v_{0i}, v)$ for every $v \in B_i$ so that $d(v, v_{0i}) \neq d(v, v_{0j})$ caused $r(v_{0i}|B_i) \neq r(v_{0j}|B_i)$. Since of $B_i \subseteq W_l$ then $r(v_{0i}|W_l) \neq r(v_{0j}|W_l)$.
ii. For $x, y = v_{ij}, v_{ik} \in V(G \odot K_n) \backslash W_l$ with $j \neq k$, $v_{ij} v_{ik} \in E(K_n)_i$ for $i = 1, 2, \ldots, m$. Since B_i is a local

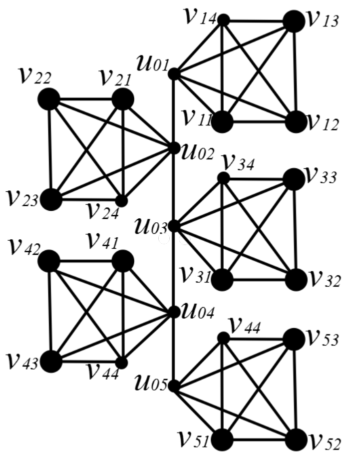

Fig. 5. $P_5 \odot K_4$ has dominant local metric dimension equals fifteen.

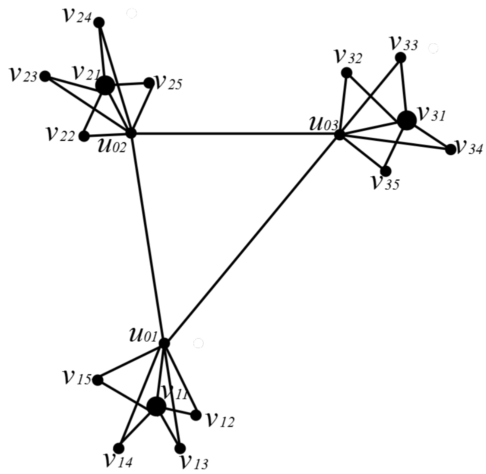

Fig. 6. $C_3 \odot S_5$ has dominant local metric dimension equals three.

basis of $(K_n)_i$ and $r(v_{ij}|B_i) \neq r(v_{ik}|B_i)$. Because $B_i \subseteq W_l$, then $r(v_{ij}|W_l) \neq r(v_{ik}|W_l)$.

iii. For $x, y = v_{0i}v_{ij} \in V(G \odot K_n)\backslash W_l$. There exist two possibilities.

1) v_{0i} with $v_{ij} \notin B_i$.
If $i \neq k, d(v_{0i}, v_{0k}) \neq d(v_{ij}, v_{0k}) \rightarrow \forall v \in B_i$ causes $d(v, v_{0i}) \neq d(v, v_{ij})$. So, $B_i \subseteq W_l$ make $r(v_{0i}|W_l) \neq r(v_{ij}|W_l)$.

2) v_{0i} with $v_{ij} \in B_i$.
Since $v_{ij} \in B_i$ then there exists a zero element in $r(v_{ij}|B_i)$. Besides that, $d(v_{0i}, v_{ij}) = 1$ and $v_{0i} \notin B_i$ so, there are no zero elements in $r(v_{0i}|B_i)$. Consequently, $r(v_{ij}|B_i) \notin r(v_{0i}|B_i)$. Then, $B_i \subseteq W_l$ implies $r(v_{0i}|W_l) \neq r(v_{ij}|W_l)$.

Considering the two scenarios described above, $W_l = \cup_{i=1}^{m} B_i$ is a local resolving set of $G \odot K_n$. Then, $\forall v_{0i} \in V(G), 1 \leq i \leq m$ $d(v_{0i}, v_{ij} = 1$ where $v_{ij} \in W_l$ and $\forall v_{in} \in V((K_n)_i), 1 \leq i \leq m$ $d(v_{in}, v_{ij} = 1$ where $v_{ij} \in W_l$. Then, W_l is a dominating set. So that, $W_l = \cup_{i=1}^{m} B_i$ is a local dominant resolving set of $G \odot K_n$. Then, that $W_l = \cup_{i=1}^{m} B_i$ is a local dominant resolving set with smallest cardinality. Put any $S \subseteq V(G \odot K_n)$ with $|S| < |W_l|$. Let $|S| = |W_l| - 1$, then $\exists i \ni |S| \leq |B_i| - 1$ elements of $(K_n)_i$. Since B_i is a local dominant basis of $(K_n)_i$ then there exist 2 vertices in $(K_n)_i$ have same representation toward S, it means S is not a local dominant resolving set of $G \odot K_n$. Looking back to the Lemma 3.1 is known that $W_l = \cup_{i=1}^{m} B_i$ is a local dominant basis of $G \odot K_n$ with $|B_i| = Ddim_l((K_n)_i)$, hence it has been proven that $Ddim_l(G \odot K_n) = |W_l| = |V(G)| \times Ddim_l(K_n)$. □

Figure 5 gives an example of $G \odot K_n$. It is $P_5 \odot K_4$ whose has dominant local metric dimension equals fifteen. The vertices that are printed larger are elements of the dominant local basis of $P_5 \odot K_4$.

We can demonstrate the following theorems by using similar techniques to demonstrate that $Ddim_l$ of a connected graph G with a star and a complete bipartite graph is as stated.

Theorem 3.7 Given a connected graph G, $|V(G)| = m \geq 2$. If S_n is a star with $n \geq 3$, then

$$Ddim_l(G \odot S_n) = |V(G)| \times Ddim_l(S_n).$$

Proof. Since the steps are similar to the Theorem 3.5, we only show the dominant local basis of $G \odot S_n$. Let $V(G) = \{u_i 1 \leq i \leq m\}$, the vertex set of star is $V(S_n) = \{v\} \cup \{v_i|1 \leq i \leq n-1\}$, $E(S_n) = \{vv_i|1 \leq i \leq n-1\}$. For $G \odot S_n$, $V(G \odot S_n) = \{u_{0i}|u_i \in V(G), 1 \leq i \leq m\} \cup \{v_i|v \in V(S_n), 1 \leq i \leq m\} \cup \{v_{ij}|v_j \in V(S_n), 1 \leq i \leq m, 1 \leq j \leq n-1\}$ and $E(G \odot S_n) = \{u_{0i}u_{0k}|u_i u_k \in E(G); 1 \leq i, k \leq m, i \neq k\} \cup \{u_{0i}v_i|1 \leq i \leq m\} \cup \{v_i v_{ij}|1 \leq i \leq m, 1 \leq j \leq n-1\}$. Select $W_l = \cup_{i=1}^{m} B_i$, with $B_i = \{v_i\}$ for every $1 \leq i \leq m$. Therefore, $Ddim_l(G \odot S_n) = |W_l| = m \times Ddim_l(S_n) = |V(G)| \times Ddim_l(S_n)$ for $n \leq 2$. □

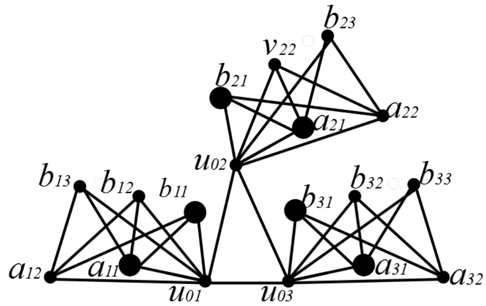

Fig. 7. $C_3 \odot K_{3,2}$ has dominant local metric dimension equals six.

Theorem 3.8 Given a connected graph G, $|V(G)| = p \geq 2$. If $K_{(m,n)}$ is a complete bipartite graph with $m, n \geq 2$, then

$$Ddim_l(G \odot K_{(m,n)}) = |V(G)| \times Ddim_l(K_{m,n})$$

Proof. It can be proven similarly to the two theorems before. Let $V(G) = \{u_k|1 \leq k \leq p\}$, $V(K_{m,n}) = \{a_i|1 \leq i \leq m\}$ $\cup\{b_j|1 \leq j \leq n\}$, and $E(K_{m,n}) = \{a_i b_j|1 \leq i \leq m; 1 \leq j \leq n\}$. We give the vertex label of $G \odot K_{m,n}$ is $V(G \odot K_{m,n}) = V(G) \cup_{k=1}^{p} V((K_{m,n})_k)$ for the edge $E(G \odot K_{m,n}) = E(G) \cup_{k=1}^{p} E((K_{m,n})_k) \cup \{u_k a_{ki}, u_k b_{kj}|u_k \in V(G); a_{ki}, b_{kj} \in V((K_{m,n})_k)\}$. Choose $W_l = \cup_{k=1}^{p} B_k$, with $B_k = \{a_{k1}, b_{k1}\}$ for every $1 \leq k \leq p$, then we get $Ddim_l(G \odot K_{m,n}) = |W_l| = |V(G)| \times Ddim_l(K_{m,n})$. □

IV. Conclusion

In this study, we identified a local resolving property as well as $Ddim_l(G \odot P_m), Ddim_l(G \odot C_n), Ddim_l(G \odot K_n), Ddim_l(G \odot S_n)$ and $Ddim_l(G \odot K_{m,n})$. The presented results still lead to several open questions, such as how about $Ddim_l(G \odot H)$ for G and H are any two connected graphs.

This research can also be expanded to another operation of graphs.

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