Analysis of Casson Flow Through Parallel and Uniformly Porous Walls of Different Permeability

Devaki B, *Sampath Kumar V S and Nityananda P Pai

Abstract—**The present study deals with a steady two-dimensional Casson in-compressible fluid flow in a channel due to non-uniform suction or injection through its porous walls. The constitutive equations are reduced to a nonlinear ordinary differential equation using similarity transformation with appropriate boundary conditions. Now, the modeled equation is solved using Homotopy Perturbation method (HPM) and by an effective finite difference method (FDM). The results obtained for velocity, skin friction at the upper and lower walls are displayed in the form of figures and tables. It is interesting to note that the results are in good agreement.**

Index Terms—**Cassson fluid, porous channel, Navier-Stokes equations, homotopy perturbation method, finite difference method.**

I. Introduction

THE non-Newtonian fluids have wide applications in everyday life, and it is well known that the fluids that appear in industrial and engineering processes are mostly non-Newtonian fluids. The mechanics of non-Newtonian fluid flows present a special challenge to engineers, physicists, and mathematicians. The properties of such fluids cannot be explored by simple Navier Stokes equations. The Navier Stokes theory is inadequate for such fluids and there is no single constitutive equation that exhibits all properties of such non-Newtonian fluids. In the process, several non-Newtonian fluid models have been proposed([1], [2], [3], [4], [5]). Amongst these Casson fluid satisfactorily describes the properties of many polymers over a wide range of shear stress.

In 1957, N. Casson [6] developed the Casson fluid model. We can observe in the literature that the Casson fluid model is sometimes stated to fit rheological data better than general viscoplastic models for many materials. Casson fluid model exhibits yield stress. If the shear stress is less than the yield stress applied to the fluid, it behaves like a solid whereas if the shear stress is greater than the yield stress is applied, it starts to move. An approximate Casson fluid model for tube flow of blood is studied by W. P. Walawander *et al.* [7]. Abdul-Sattar *et al.* [8] in their paper explain the effect of squeezing flow of a Casson fluid between parallel plates on magnetic fields.

Porous media are encountered in many natural as well as man-made systems. The laminar flow in a channel with porous walls was studied by Berman [9]. Further investigation was done by Yuan [10]. Later, White *et al.* [11] studied the laminar flow in a uniformly porous channel. The porous medium is characterized by its permeability which is a measure of the flow conductivity in the porous medium. Laminar flow between two parallel porous walls with variable permeability is studied by various authors ([12], [13], [14]). Further, Hafeez Y *et al.* [15], Attia *et al.* [16] have studied the effect of injection and suction in Casson fluid.

Non-linear phenomena play a vital role in Fluid Mechanics, Quantum Physics, Magnetohydrodynamics, etc. There are various analytical and numerical methods to find its exact or approximate solution ([17], [18]). In this paper, we have used a powerful semi-analytical method, known as Homotopy Perturbation Method (HPM). The HPM was first introduced by J. H. He in 1998 ([19], [20]). To solve different types of differential and integral equations many researchers have used HPM ([21], [22], [23], [24], [25], [26]). This method is a combination of traditional Perturbation method and homotopy in topology. In HPM the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. This method continuously deforms a tough problem into a number of simple problems, easy to solve. The major drawback of the traditional perturbation method is the over-dependence on the existence of small parameters. This condition is overstrict and greatly effects the application of the Perturbation method because most of the nonlinear problems do not even contain the so-called small parameter. The HPM doesn't depend upon a small parameter involved in the problem. The approximation obtained by HPM is uniformly valid not only for small parameters but also for very large parameters.

II. Formulation of the problem

The steady incompressible Casson fluid flow along a two-dimensional channel with uniformly porous walls with different velocities at the walls is considered. Let x and z coordinate axes are taken parallel and perpendicular to the channel walls, respectively (Figure 1). Let u and v are velocity components in x and z directions, respectively.

At the wall $z = 0$, the velocity components $v = V_1$ and $v = V_2$ at the wall $z = h$, where h is the channel width. The Reynolds number $R_1 = (V_1 h)/\mu$ is defined for the cases $|V_1| \geq |V_2|$ and $R_2 = (V_2 h)/\mu$ for $|V_2| \geq |V_1|$ where μ is the viscosity. By assuming the flow to be steady, laminar and incompressible, and by taking the velocity components to be $u = u(x, z)$ and $v = v(x, z)$, the momentum and continuity equations reduce to

Manuscript received May 19, 2022; revised October 24, 2022.

Devaki B is a Research Scholar in Mathematics Department, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (email: devaki.badekkila@learner.manipal.edu)
Sampath Kumar V S is an Assistant Professor(Selection grade) of Mathematics Department, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (email-sampath.kvs@manipal.edu)
Nityananda P Pai is a Professor in Mathematics Department, Manipal Institute of Technology, Manipal Academy of Higher Education, Manipal, Karnataka, India, 576104 (email-nithyanand.pai@manipal.edu)
*Corresponding author- Sampath Kumar V S.

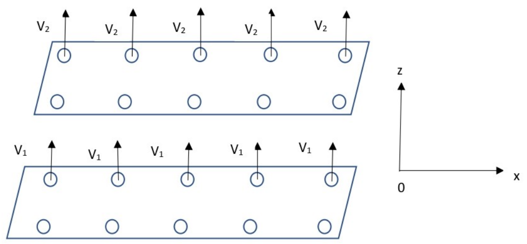

Fig. 1: Schematic diagram of the flow

$$u\frac{\partial u}{\partial x}+\frac{v}{h}\frac{\partial u}{\partial \lambda}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(1+\frac{1}{\gamma}\right)\left(2\frac{\partial^2 u}{\partial x^2}+\frac{1}{h^2}\frac{\partial^2 u}{\partial \lambda^2}+\frac{1}{h}\frac{\partial^2 v}{\partial x\partial \lambda}\right) \quad (1)$$

$$u\frac{\partial v}{\partial x}+\frac{v}{h}\frac{\partial v}{\partial \lambda}=-\frac{1}{h\rho}\frac{\partial p}{\partial \lambda}+\nu\left(1+\frac{1}{\gamma}\right)\left(\frac{\partial^2 v}{\partial x^2}+\frac{2}{h^2}\frac{\partial^2 v}{\partial \lambda^2}+\frac{1}{h}\frac{\partial^2 u}{\partial x\partial \lambda}\right) \quad (2)$$

and

$$\frac{\partial u}{\partial x}+\frac{1}{h}\frac{\partial v}{\partial \lambda}=0, \quad (3)$$

respectively, where ρ is the fluid density, p is the pressure , ν is the kinematic viscosity, γ is the Casson parameter and λ is the non-dimensional variable given by $\lambda=\frac{z}{h}$. The flow of fluid is through a two-dimensional channel having fluid sucked or injected with constant velocities V_1 and V_2 through its porous walls at $z=0$ and $z=h$, respectively. The boundary conditions to be satisfied by the flow are

$$u(x,0)=0, u(x,h)=0 \quad (4)$$

$$v(x,0)=V_1, v(x,h)=V_2 \quad (5)$$

For suction or injection flow, the problems to be solved for the case $|V_2|\geq|V_1|$ can be reduced to the case $|V_1|\leq|V_2|$. But for mixed flow, the case $|V_1|\geq|V_2|$ and $|V_2|\geq|V_1|$ are two different problems and need to be solved separately. It is assumed that for this two-dimensional incompressible flow, there exists a stream function of the form

$$\xi(x,\lambda)=\left[\frac{hU(0)}{\alpha_2}-V_2x\right]f(\lambda) \quad (6)$$

where $\alpha_2=1-\frac{V_1}{V_2}$, for the case $|V_1|\geq|V_2|$.
The expressions for the velocity components are

$$u(x,h)=\left[\frac{V(0)}{\alpha_2}-\frac{V_2x}{h}\right]f'(\lambda) \quad (7)$$

$$v(\lambda)=V_2f(\lambda). \quad (8)$$

Similarly, for the case $|V_2|\geq|V_1|$ there is a stream function of the form

$$\xi(x,\lambda)=\left[\frac{hU(0)}{\alpha_1}-V_1x\right]f(\lambda) \quad (9)$$

where $\alpha_1=\frac{V_2}{V_1}-1$.
The above choice of stream function and velocity components reduce (1) and (2) to scalar equation.

$$\left(1+\frac{1}{\gamma}\right)f'''+R_2(f'^2-ff'')=K_2 \quad (10)$$

where K_2 is a constant and f is a function of λ.
The boundary conditions are

$$f(0)=1-\alpha_2, f'(0)=0 \quad (11)$$

$$f(1)=1, f'(1)=0 \quad (12)$$

where $\alpha_2=1-\frac{V_1}{V_2}$.
Similarly, for the case $|V_1|\geq|V_2|$, the reduced equation is

$$\left(1+\frac{1}{\gamma}\right)f'''+R_1(f'^2-ff'')=K_1. \quad (13)$$

The boundary conditions are

$$f(0)=1, f'(0)=0 \quad (14)$$

$$f(1)=1+\alpha_1, f'(1)=0 \quad (15)$$

where $\alpha_1=\frac{V_2}{V_1}-1$.
The boundary conditions (11) and (14) imply the suction case α_2 and α_1, must lie in the range $1\leq\alpha_2\leq2$ and $-1\geq\alpha_1\geq-2$, whereas for the mixed cases they are in the range $0\leq\alpha_2\leq1$ and $0\geq\alpha_1\geq-1$.

Differential equations of the type (10) and (13) are usually solved by direct integration which frequently involves more than one integration process, because of the two-point nature of the boundary conditions. Moreover, to confirm the validity of numerical results they are to be solved using other possible available methods. Thus, the use of a series solution provides an effective approach.

III. Method of Solution

A. Homotopy Perturbation Method (HPM)

To describe the HPM solution [27] for non-linear differential equation, we consider,

$$D[f(\eta)]-f_1(\eta)=0 \quad (16)$$

where D denotes the operator, $f(\eta)$ is unknown function, η denote the independent variable and f_1 is known function. D can be written as

$$D=L+N \quad (17)$$

where L is a simple linear part, N is remaining part of the equation (16).

The proper choice of L,N form the governing equation, one can get the homotopy equation as follows

$$H(\phi(n,q;q))=(1-q)[L(\phi,q)-L(v_0(\eta))] +q[D(\phi,q)-f_1(\eta)]=0 \quad (18)$$

where q is the embedding parameter which varies from 0 to 1 and $v_0(\eta)$ is the initial guess to the equation (16). So we assume the solution of equation (18) as follows

$$\phi(n,q)=\sum_{n=0}^{\infty}q^nf_n(\eta) \quad (19)$$

The solution to the considered problem is equation (19) at $q=1$. For the problem considered here, the first three terms in the solution are

For mixed injection:

$$f_0(\eta)=1+3\alpha_1\eta^2-2\alpha_1\eta^3 \quad (20)$$

$$f_1(\eta) = \frac{1}{70(1+\gamma)}\big[-35\alpha_1 R\gamma - 16\alpha_1^2 R\eta^2\gamma + 70\alpha_1 R\eta^3\gamma + 27\alpha_1^2 R\eta^3\gamma - 35\alpha_1 R\eta^4\gamma - 21\alpha_1^2 R\eta^5\gamma + 14\alpha_1^2 R\eta^6\gamma - 4\alpha_1^2 R\eta^7\gamma\big] \quad (21)$$

$$f_2(\eta) = \frac{R^2\gamma^2}{646800(1+\gamma)^2}\big[32340\alpha_1\eta^2 + 13860\alpha_1^2\eta^2 - 761\alpha_1^3\eta^2 - 129360\alpha_1\eta^3 - 70840\alpha_1^2\eta^3 - 5858\alpha_1^3\eta^3 + 161700\alpha_1\eta^4 + 62370\alpha_1^2\eta^4 - 64680\alpha_1^2\eta^5 + 29568\alpha_1^3\eta^5 - 11580\alpha_1^2\eta^6 - 34804\alpha_1^3\eta^6 + 64680\alpha_1^2\eta^7 + 14256\alpha_1^3\eta^7 - 16170\alpha_1^2\eta^8 - 3465\alpha_1^3\eta^8 + 3080\alpha_1^3\eta^9 - 2464\alpha_1^3\eta^{10} + 448\alpha_1^3\eta^{11}\big] \quad (22)$$

For mixed suction:

$$f_0(\eta) = 1 - \alpha_2 + 3\alpha_2\eta^2 - 2\alpha_2\eta^3 \quad (23)$$

$$f_1(\eta) = \frac{1}{70(1+\gamma)}\big[-35\alpha_2 R\eta^2\gamma + 19\alpha_2^2 R\eta^2\gamma + 70\alpha_2 R\eta^3\gamma - 43\alpha_2^2 R\eta^3\gamma - 35\alpha_2 R\eta^4\gamma - 21\alpha_2^2 R\eta^5\gamma + 14\alpha_2^2 R\eta^6\gamma - 4\alpha_2 R\eta^7\gamma\big] \quad (24)$$

$$f_2(\eta) = \frac{R^2\gamma^2}{646800(1+\gamma)^2}\big[32340\alpha_2\eta^2 - 50820\alpha_2^2\eta^2 + 17719\alpha_2^3\eta^2 - 129360\alpha_2\eta^3 + 187880\alpha_2^2\eta^3 - 64378\alpha_2^3\eta^3 - 161700\alpha_2\eta^4 - 261030\alpha_2^3\eta^4 + 64680\alpha_2\eta^5 + 194040\alpha_2^2\eta^5 - 99792\alpha_2^3\eta^5 - 118580\alpha_2^2\eta^6 + 83776\alpha_2^3\eta^6 + 64680\alpha_2^2\eta^7 - 50424\alpha_2^3\eta^7 - 16170\alpha_2^2\eta^8 + 12705\alpha_2^3\eta^8 + 3080\alpha_2^3\eta^9 - 2464\alpha_2^3\eta^{10} + 448\alpha_2^3\eta^{11}\big] \quad (25)$$

B. *Finite Difference Method*

Finite Difference Method (FDM) is one of the oldest methods used to solve differential equations that are difficult or impossible to solve analytically. The finite difference method is applied directly to the differential form of the governing equations. The principle is to employ a Taylor series expansion for the discretization of the derivatives of the flow variables.

IV. Results and Discussion

A new type of series solution is presented for Casson fluid flow through parallel and uniformly porous walls of different permeability. The behavior of Casson parameter and Reynolds number on velocity profile is described in this section. For this purpose Figure(2-13) shows the velocity profiles for various values of Reynolds number and Casson parameter for different permeability factors are displayed.

The behaviour of flow for different values of Casson fluid parameter is shown in Figures (2-13). All the velocity profiles show that, velocity increases for η values and reaches to a maximum and thereafter decreases. It is clear from the figure that, flow pattern changes at one point in the interval $[0, 1]$ for different Reynolds numbers and γ values. In the interval $0 \leq \eta \leq 0.5$, velocity increases and it decreases in $0.5 \leq \eta \leq 1$. The same behaviour is observed even when permeability factor values is increased. Figures (14-16) represent the velocity profile for mixed case with different values of Reynolds number for $\gamma = 0.1, 0.2, 0.3$ respectively. Figure 17 and Figure 19 represent the velocity profile with variation of Casson parameter for $R_1 = 20$ and $R_1 = -20$ respectively. The same behaviour is observed even when permeability factor values is increased (Figure 18).

The effect of different permeability factors on skin friction at the walls for different values of Reynolds number and Casson parameter are represented in the Tables(1-3). Also to find the accuracy of the HPM solution, the values of the skin friction at the walls are compared with pure numerical values. In this method we have generated higher order terms upto 25 using Mathematica.

The results of $f''(1)$, $f''(0)$ for fixed values of permeability factor and different values of R_1, γ are shown in Table (1-3). The values for $f''(1)$, $f''(0)$ are monotonically increasing with the values of R_1 and $\gamma = 0.1, 0.2, 0.3$ (Table 1). But, Table (2) depicts that the vlues of $f''(1)$ decreasing with values of R_1 and γ. For positive values of α_1 and α_2 where as reverse trend is observed for $f''(0)$. It is important to know the convergence of series solutions obtained. So, we have verified the results described in Table (1-3), and Figures (2-13) by pure numerical method (FDM) and both the methods give results which are in good agreement.

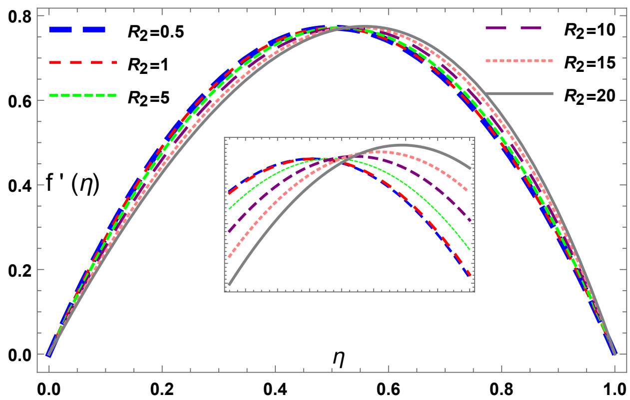

Fig. 2: Velocity profiles with $\alpha_2 = 0.51425$, for different values of Reynolds numbers R_2 and $\nu = 0.1$

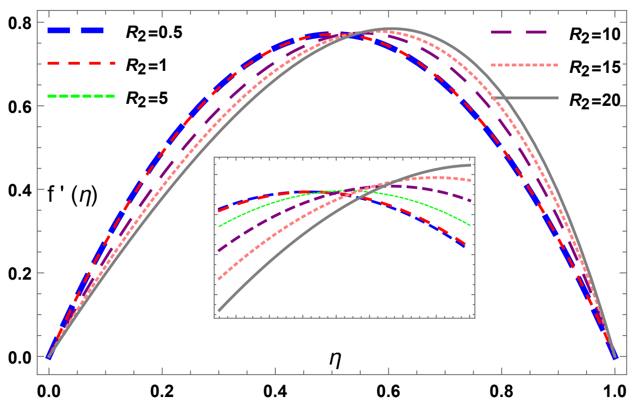

Fig. 3: Velocity profiles with $\alpha_2 = 0.51425$, for different values of Reynolds numbers R_2 and $\nu = 0.2$

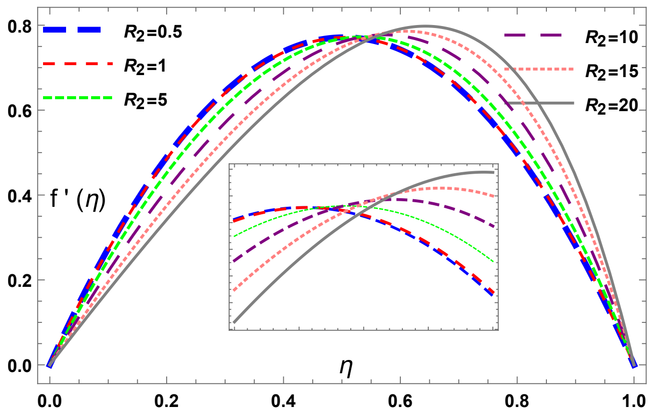

Fig. 4: Velocity profiles with $\alpha_2 = 0.51425$, for different values of Reynolds numbers R_2 and $\gamma = 0.3$

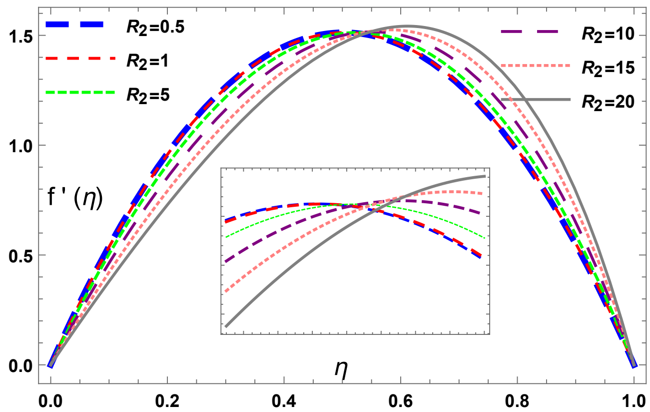

Fig. 7: Velocity profiles with $\alpha_2 = 1.00863$, for different values of Reynolds numbers R_2 and $\gamma = 0.3$

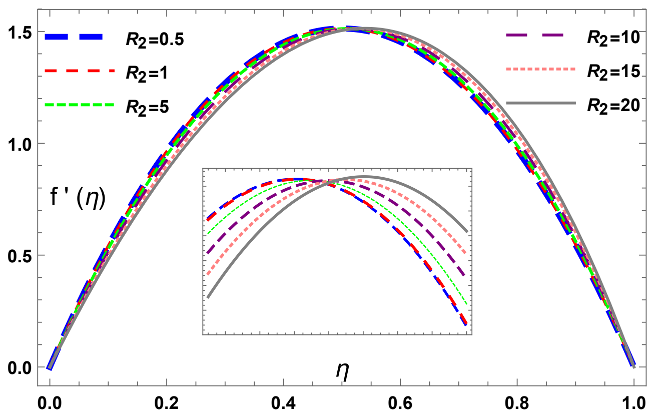

Fig. 5: Velocity profiles with $\alpha_2 = 1.00863$, for different values of Reynolds numbers R_2 and $\gamma = 0.1$

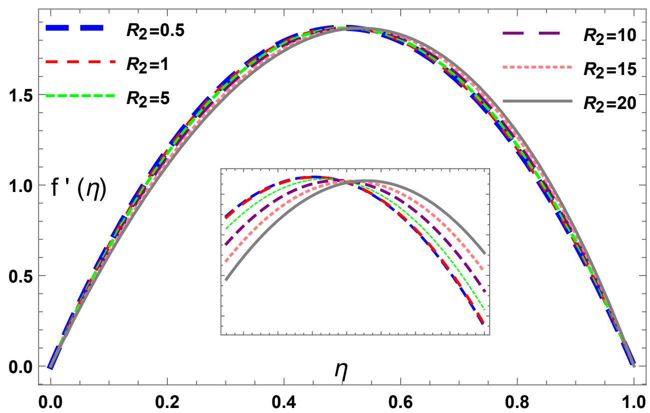

Fig. 8: Velocity profiles with $\alpha_2 = 1.24575$, for different values of Reynolds numbers R_2 and $\gamma = 0.1$

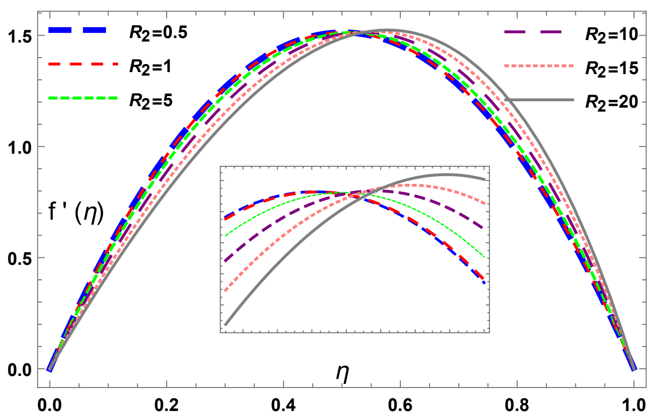

Fig. 6: Velocity profiles with $\alpha_2 = 1.00863$, for different values of Reynolds numbers R_2 and $\gamma = 0.2$

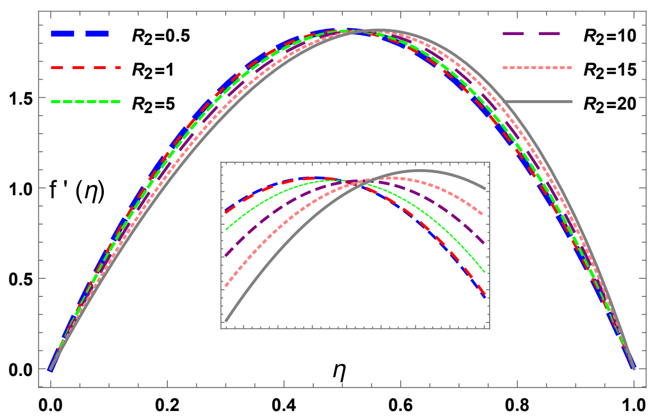

Fig. 9: Velocity profiles with $\alpha_2 = 1.24575$, for different values of Reynolds numbers R_2 and $\gamma = 0.2$

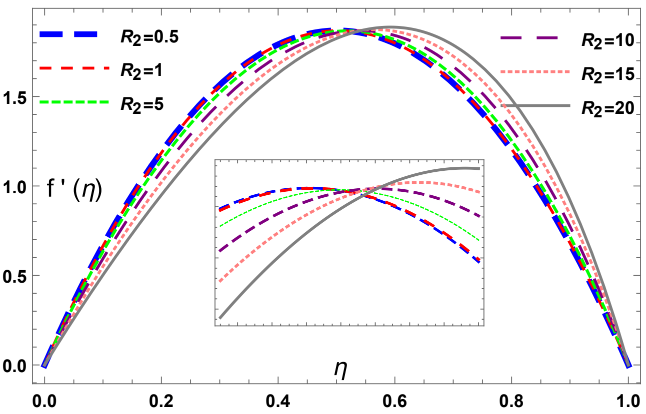

Fig. 10: Velocity profiles with $\alpha_2 = 1.24575$, for different values of Reynolds numbers R_2 and $\gamma = 0.3$

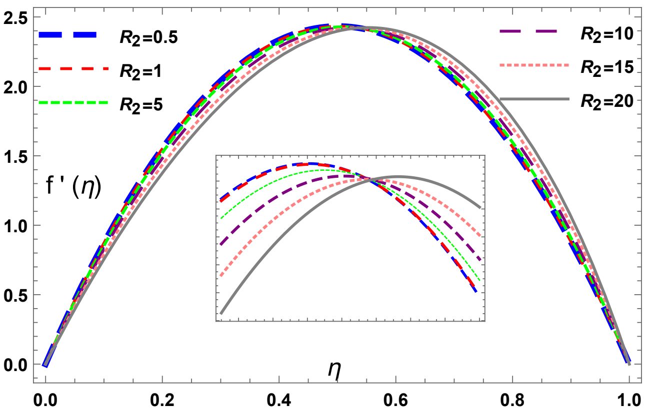

Fig. 13: Velocity profiles with $\alpha_2 = 1.62204$, for different values of Reynolds numbers R_2 and $\gamma = 0.3$

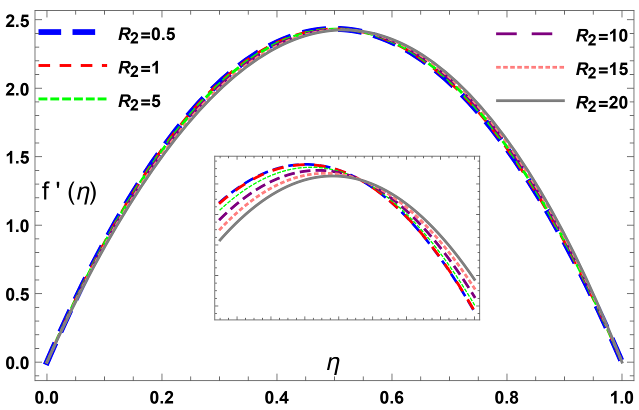

Fig. 11: Velocity profiles with $\alpha_2 = 1.62204$, for different values of Reynolds numbers R_2 and $\gamma = 0.1$

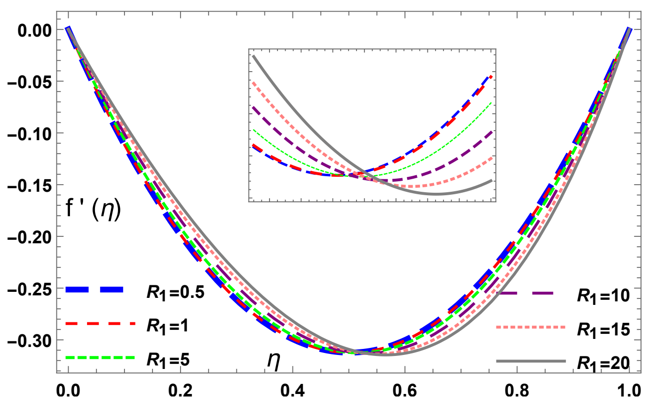

Fig. 14: Velocity profiles with $\alpha_1 = -0.20820$, for different values of Reynolds numbers R_1 and $\gamma = 0.1$

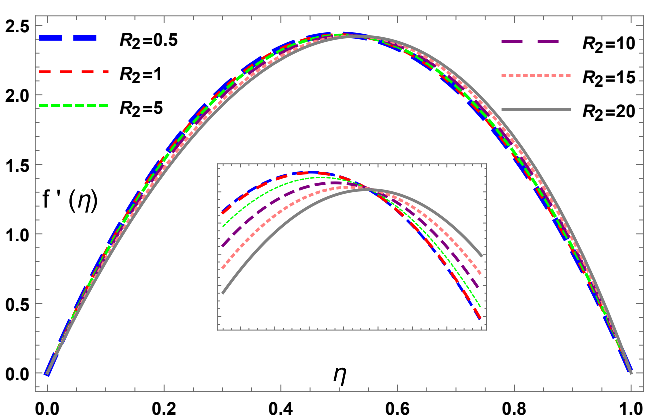

Fig. 12: Velocity profiles with $\alpha_2 = 1.62204$, for different values of Reynolds numbers R_2 and $\gamma = 0.2$

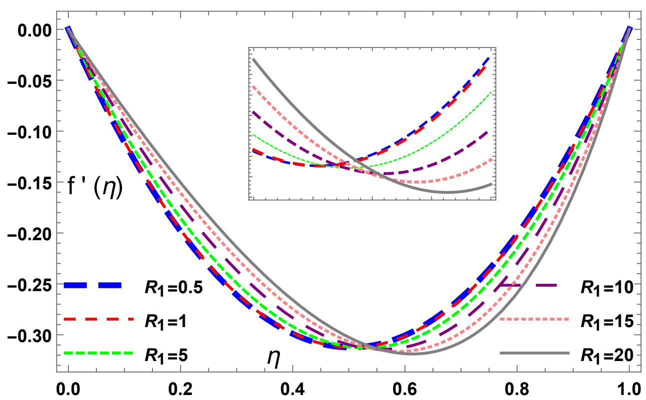

Fig. 15: Velocity profiles with $\alpha_1 = -0.20820$, for different values of Reynolds numbers R_1 and $\gamma = 0.2$

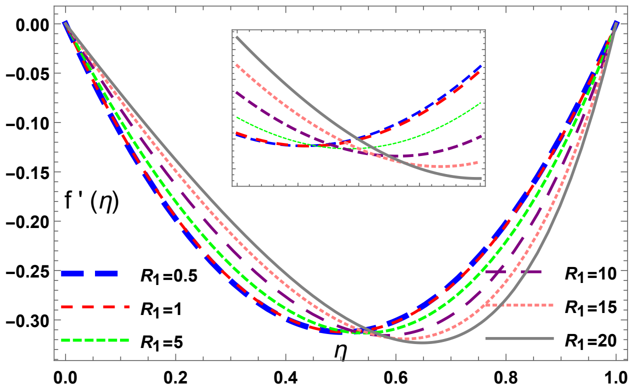

Fig.16.Velocity profile with $\alpha_1 = -0.20820$ for different Values of Reynolds number R_1 and $\gamma = 0.3$

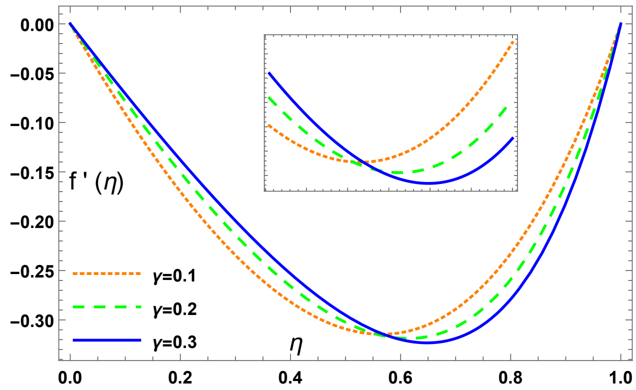

Fig. 17.Velocity profile with $\alpha_1 = -0.20820$, $R_1 = 20$ and for different values of γ

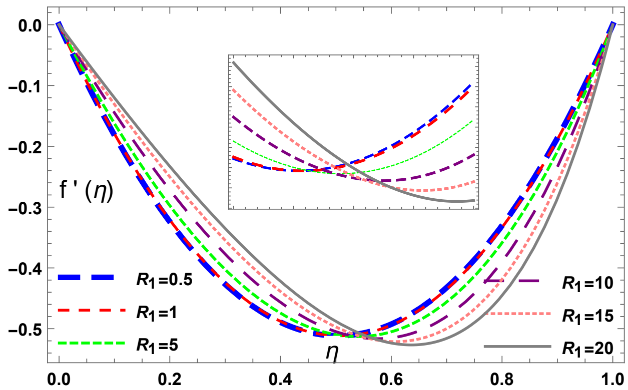

Fig.18.Velocity profile with $\alpha_1 = -0.33410$ for different Values of Reynolds number R_1 and $\gamma = 0.3$

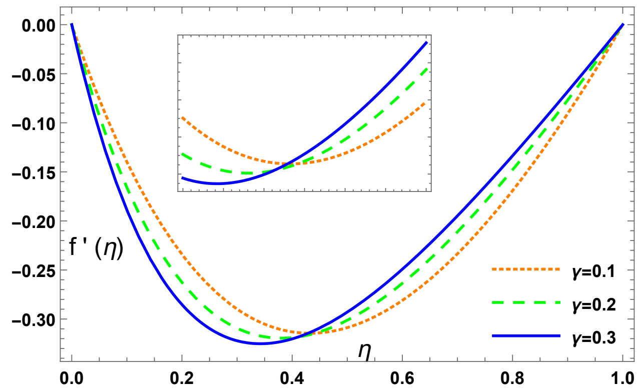

Fig. 19.Velocity profile with $\alpha_1 = -0.20820$, $R_1 = -20$ and for different values of γ

TABLE I: Results for physical parameters (mixed injection)

R_1	γ	$\alpha_1 = -0.20820$				$\alpha_1 = -0.33410$			
		$f''(1)$ (HPM)	FDM	$f''(0)$(HPM)	FDM	$f''(1)$ (HPM)	FDM	$f''(0)$(HPM)	FDM
-20	0.1	0.96814	0.96634	-1.65479	-1.64697	1.56836	1.56935	-2.61595	2.59718
-15		1.02463	1.02893	-1.54163	-1.53190	1.66583	1.65741	-2.44482	-2.45337
-10		1.09342	1.09625	-1.43624	-1.42679	1.77106	1.77158	-2.28591	-2.28842
-5		1.16827	1.16213	-1.33876	-1.34669	1.88402	1.88350	-2.13922	-2.14269
-1		1.23253	1.22619	-1.26648	-1.27020	1.97988	1.96303	-2.03057	-2.03590
1		1.26611	1.22619	-1.23223	-1.27020	2.02961	2.00984	-1.9791	-1.99645
5		1.33611	1.33404	-1.16745	-1.17477	2.13257	2.11665	-1.88176	-1.89582
10		1.42884	1.43431	-1.09332	-1.09079	2.26763	2.26631	-1.77029	-1.77317
15		1.52714	1.52961	-1.02651	-1.02431	2.40939	2.40498	-1.66965	-1.67138
20		1.63069	1.62212	-.966637	-0.97139	2.55739	2.75450	-1.57922	-1.47454
-20	0.2	0.79288	0.79331	-2.08485	-2.08120	1.28680	1.29100	-3.24718	-3.28364
-15		0.87840	0.87791	-1.83865	-1.83134	1.42432	1.42330	-2.87304	-2.89630
-10		0.98209	0.98080	-1.61622	-1.61791	1.58654	1.60556	-2.53766	-2.55121
-5		1.10547	1.10359	-1.41945	-1.41818	1.77456	1.78866	-2.24271	-2.26056
-1		1.21883	1.22293	-1.28112	-1.27667	1.94360	1.96220	-2.03606	-2.04239
1		1.28038	1.27610	-1.21833	-1.22227	2.03419	2.05046	-1.94232	-1.96062
5		1.41299	1.40826	-1.10516	-1.10652	2.22699	2.24673	-1.77334	-1.78463
10		1.59561	1.59933	-0.98585	-0.97794	2.48824	2.51713	-1.59471	-1.60337
15		1.79507	1.79607	-0.88885	-0.88830	2.76936	2.79034	-1.44864	-1.45705
20		2.00894	2.01115	-0.81114	-0.80706	3.06720	3.09245	-1.33062	-1.33763
-20	0.3	0.69196	0.68387	-2.50393	-2.51221	1.12840	1.12824	-3.92107	-3.39149
-15		0.78122	0.78601	-2.12468	-2.12830	1.27862	1.28418	-3.33333	-3.32697
-10		0.90065	0.89637	-1.78513	-1.78358	1.47199	1.47917	-2.81377	-2.79610
-5		1.05563	1.05830	-1.49201	-1.48465	1.71343	1.70395	-2.36995	-2.37551
-1		1.20736	1.19600	-1.29368	-1.30290	1.94244	1.93932	-2.07145	-2.06927
1		1.29259	1.28595	-1.20673	-1.21420	2.06866	2.07209	-1.94079	-1.93533
5		1.48110	1.48210	-1.05646	-1.05842	2.34316	2.34002	-1.71479	-1.71507
10		1.74767	1.75804	-0.90940	-0.90698	2.72320	2.72379	-1.49246	-1.49447
15		2.04296	2.03655	-0.80070	-0.80094	3.13672	3.15056	-1.32626	-1.31637
20		2.36022	2.34373	-0.72194	-0.72548	3.57529	3.56873	-1.20397	-1.19940

TABLE II: Results for physical parameters (large suction)

R_2	γ	$\alpha_2 = 1.91600$				$\alpha_2 = 1.94300$			
		$f''(1)$ (HPM)	FDM	$f''(0)$(HPM)	FDM	$f''(1)$ (HPM)	FDM	$f''(0)$(HPM)	FDM
-20	0.1	-11.11610	-11.11790	11.35830	11.35440	-11.30860	-11.11790	11.4749	11.35440
-15		-11.20200	-11.19920	11.39220	11.38810	-11.3881	-11.19920	11.51860	11.38810
-10		-11.29370	-11.28760	11.42650	11.42710	-11.47250	-11.28760	11.56380	11.42710
-5		-11.39150	-11.39530	11.46110	11.45550	-11.56240	-11.39530	11.61020	11.45550
-1		-11.47450	-11.47160	11.48900	11.48880	-11.63840	-11.47160	11.64830	11.48880
1		-11.51780	-11.51470	11.50300	11.49800	-11.67790	-11.51470	11.66770	11.49800
5		-11.60790	-11.59940	11.53090	11.53480	-11.7600	-11.59940	11.70700	11.53480
10		-11.7277	-11.72040	11.56560	11.55830	-11.86870	-11.72040	11.75700	11.55830
15		-11.8564	-11.84550	11.59990	11.60250	-11.98500	-11.84550	11.80810	11.60250
20		-11.99460	-11.99090	11.63330	11.63370	-12.10940	-11.99090	11.85990	11.63370
-20	0.2	-10.86530	-10.86650	11.25010	11.24620	-11.07520	-10.86650	11.33880	11.24620
-15		-10.99690	-10.99150	11.30870	11.30480	-11.19790	-10.99150	11.41180	11.30480
-10		-11.14410	-11.14260	11.36960	11.36600	-11.33450	-11.14260	11.48930	11.36600
-5		-11.30950	-11.30500	11.43220	11.43150	-11.48710	-11.30500	11.57140	11.43150
-1		-11.45690	-11.45090	11.48320	11.48000	-11.62220	-11.45090	11.64030	11.48000
1		-11.53610	-11.54020	11.50880	11.50180	-11.69460	-11.54020	11.67580	11.50180
5		-11.70720	-11.71000	11.55990	11.55700	-11.85010	-11.71000	11.74860	11.55700
10		-11.94740	-11.94850	11.62230	11.61790	-12.06700	-11.94850	11.84260	11.61790
15		-12.22220	-12.21400	11.68080	11.68450	-12.31290	-12.21400	11.93860	11.68450
20		-12.53870	-12.53390	11.73200	11.73600	-12.59330	-12.53390	12.03450	11.73600
-20	0.3	-10.68890	-10.68610	11.16500	11.16250	-10.90950	-10.68610	11.23530	11.16250
-15		-10.84630	-10.84230	11.24140	11.23970	-11.05740	-10.84230	11.32800	11.23970
-10		-11.02940	-11.03200	11.32260	11.31670	-11.22810	-11.03200	11.42930	11.31670
-5		-11.24360	-11.24440	11.40790	11.39960	-11.42640	-11.42640	11.53930	-11.39996
-1		-11.44210	-11.44640	11.47830	11.47320	-11.60870	-11.44640	11.63360	11.47320
1		-11.55180	-11.55070	11.51370	11.51370	-11.70890	-11.55070	11.68270	11.51370
5		-11.79590	-11.79050	11.58420	11.58430	-11.93040	-11.93050	11.78440	11.58430
10		-12.15540	-12.14870	11.66780	11.66970	-12.25330	-12.14870	11.91630	11.66970
15		-12.59160	-12.58720	11.73890	11.73760	-12.63990	-12.58720	12.04910	11.73760
20		-13.12900	-13.11720	11.78290	11.78280	-13.10920	-13.11720	12.17350	11.78280

TABLE III: Results of physical parameters (large injection)

R_2	γ	$\alpha_2 = 1.61940$				$\alpha_2 = 1.92780$			
		$f''(1)$ (HPM)	FDM	$f''(0)$(HPM)	FDM	$f''(1)$ (HPM)	FDM	$f''(0)$(HPM)	FDM
-20	0.1	-9.06681	-9.05923	10.02100	10.02140	-11.20010	-11.19710	11.40940	11.40370
-15		-9.20967	-9.20571	9.95365	9.95343	-11.28320	-11.28160	11.44750	11.44360
-10		-9.36470	-9.36523	9.88078	9.87137	-11.37180	-11.36960	11.48650	11.48090
-5		-9.53314	-9.54989	9.80189	9.78864	-11.46610	-11.46560	11.52630	11.51900
-1		-9.67849	-9.67263	9.73405	9.73193	-11.54610	-11.54830	11.55870	11.55490
1		-9.75496	-9.75531	9.69846	9.69577	-11.58770	-11.58590	11.57500	11.57080
5		-9.91602	-9.91062	9.62368	9.62350	-11.67440	-11.66860	11.60780	11.60310
10		-10.13370	-10.13120	9.52301	9.52091	-11.7894	-11.78800	11.64920	11.64300
15		-10.37140	-10.36260	9.41358	9.41540	-11.91270	-11.91100	11.69070	11.68180
20		-10.63130	-10.62870	9.29444	9.29178	-12.04500	-12.03610	11.73200	11.73290
-20	0.2	-8.66488	-8.66646	10.21110	10.20850	-10.95680	-10.95710	11.28100	11.28640
-15		-8.87291	-8.87024	10.11270	10.11131	-11.08460	-11.08190	11.35390	11.34720
-10		-9.11315	-9.11246	9.99914	9.99491	-11.22720	-11.22800	11.42200	11.41950
-5		-9.39180	-9.39113	9.86807	9.86384	-11.38710	-11.38340	11.49310	11.48960
-1		-9.64739	-9.64761	9.74855	9.74614	-11.52910	-11.52060	11.55190	11.55040
1		-9.78761	-9.79631	9.68328	9.67552	-11.60540	-11.59790	11.58180	11.58130
5		-10.09610	-10.09090	9.54038	9.54228	-11.76970	-11.76850	11.64230	11.63780
10		-10.54210	-10.54340	9.33530	9.33096	-11.99980	-11.98930	11.71830	11.72020
15		-11.06820	-11.05310	9.09513	9.09126	-12.26210	-12.25900	11.79300	11.78970
20		-11.69140	-11.69200	8.81191	8.80489	-12.56300	-12.56670	11.86340	11.85770
-20	0.3	-8.39621	-8.39641	10.33800	10.33300	-10.78510	-10.78520	11.19580	11.18880
-15		-8.63540	-8.63216	10.22510	10.22370	-10.93840	-10.93770	11.27930	11.27210
-10		-8.92529	-8.92094	10.08790	10.08850	-11.11610	-11.11580	11.36930	11.37060
-5		-9.27964	-9.27579	9.92074	9.91962	-11.32340	-11.31870	11.46540	11.45990
-1		-9.62142	-9.63075	9.76066	9.75335	-11.51490	-11.51150	11.54620	11.54250
1		-9.81560	-9.80878	9.67027	9.66772	-11.62050	-11.61770	11.58760	11.58530
5		-10.25910	-10.25800	9.46523	9.46357	-11.85480	-11.84620	11.67150	11.66570
10		-10.93880	-10.93480	9.15404	9.15369	-12.19840	-12.19410	11.77600	11.77540
15		-11.79720	-11.79230	8.76388	8.76096	-12.61320	-12.60540	11.87360	11.87550
20		-12.89120	-12.87880	8.26588	8.27100	-13.12120	-13.10660	11.95200	11.94810

V. Conclusions

The current analysis is focused on Casson fluid flow between parallel and porous walls with different porosity at the walls. From careful observations of the results lead to the following conclusions

1) for a given α_2 and varying R_2 (increasing) the maximum value of the velocity of the fluid shifts towards the axis whereas for smaller values of R_2 it is parabolic.
2) velocity of the fluid decrease as there is increase in the Reynolds number till it reaches peak, then the relation get revesed in the other half.
3) as the Casson parameter value increases, we can observe that the maximum value of $f'(\eta)$ is shifting to its right side.
4) for $\gamma = 0.1$ the velocity curves almost coincides with each other representing their same characteristics.
5) The skin friction $f''(1)$ at the wall increses for different Casson parameters in the range $-20 < R_1 < 20$ But for the same case $f''(0)$ shows reverse trend (for mixed injection)
6) For the case of suction $f''(1)$ and $f''(0)$ both increases in magnitude for different Cason parameters in the range $-20 < R_2 < 20$.
7) The physical parameter $f''(1)$, increse in magnitude in the domain $-20 < R_2 < 20$, where as $f''(0)$ decreses for different value of γ, the Casson parameter in the case of injection

Acknowledgment

The authors thank the management of the university (MAHE) for the constant encouragement to conduct research work.

References

[1] E. C. Bingham, *An investigation of the laws of the plastic flows*. U S Govt. Printing Office, 1917.
[2] A. W. Sisko, "The flow of lubricating greases," *Industrial and Engineering Chemistry Research*, vol. 50, no. 12, pp. 1789–1792, 1958.
[3] R. P. Chhabra, *Non-Newtonian Fluids: An Introduction*. Springer New York, 2010.
[4] B. Reyes, A. A. Howard, P. Perdikaris, and A. M. Tartakovsky, "Learning unknown physics of non-newtonian fluids," *Physical Review Fluids*, vol. 6, no. 7, pp. 1–8, 2021.
[5] W. Warjito, B. Budiarso, and D. Adanta, "Computational analysis of flow field on cross-flow hydro turbines," Engineering Letters, vol. 29, no. 1, pp. 87–94, 2021.
[6] N. Casson, *Rheology of Disperse Systems*. Pergamon Press, Oxford, 1959.
[7] W. P. Walawender, T. Y. Chen, and D. Cala, "An approximate casson fluid model for tube flow of blood," *Biorheology*, vol. 12, no. 2, pp. 111–119, 1975.
[8] A.-S. J. A. Al-Saif and A. M. Jasin, "A novel algorithm for studying the effects of squeezing flow of a casson fluid between parallel plates on magnetic field," *Journal of Applied Mathematics*, vol. 2019, no. 1, pp. 1–19, 2019.
[9] A. S. Berman, "Laminar flow in channels with porous walls," *Journal of Applied Physics*, vol. 24, no. 9, pp. 1232–1235, 1953.
[10] S. W. Yuan, "Further investigation of laminar flow in channels with porous walls," *Journal of Applied Physics*, vol. 27, no. 3, pp. 267–269, 1956.
[11] F. M. White, B. F. Barfield, and M. J. Goglia, "Laminar flow in a uniformly porous channel," *Journal of Applied Mechanics*, vol. 25, no. 4, pp. 613–617, 1958.
[12] R. M. Terril and G. M. Shrestha, "Laminar flow through parallel and uniform porous walls of different permeability," *Journal of Applied Mathematics and Physics (ZAMP)*, vol. 16, no. 4, pp. 470–482, 1965.
[13] S. Krishnambal, "Laminar flow between two parallel porous walls with variable permeability and wall slip," *American Society of Mechanical Engineers, Fluids Engineering Division (Publication) FED*, vol. 257, no. 1, pp. 1325–1331, 2002.
[14] A. M. Obalalu, O. A. Ajala, A. Taofeeq, F. A. Wahaab, O. Aliu, and L. L. Adebayo, "Natural convective non-newtonian casson fluid flow in a porous medium with slip and temperature jump boundary conditions," *Pet Coal*, vol. 62, no. 4, pp. 1532–1545, 2021.
[15] Y. Hafeez, "Flow of viscous fluid between two parallel porous plates with bottom injection and top suction," *Progress in physics*, vol. 10, no. 1, pp. 49–51, 2014.
[16] H. A. Attia, W. A. E. M. Ahmed, W. Abbas, and M. Abdeen, "Unsteady flow in a porous medium between parallel plates in the presence of uniform suction and injection with heat transfer," *International Journal of Civil Engineering*, vol. 12, no. 3, pp. 277–281, 2014.
[17] L. Hollis, E. Barnhill, N. Conlisk, L. Thomas-Seale, N. Roberts, P. Pankaj, and P. Hoskins, "Finite element analysis to compare the accuracy of the direct and mdev inversion algorithms in mr elastography," IAENG International Journal of Computer Science, vol. 43, no. 2, pp. 137–146, 2016.
[18] K. Para, S. Sungnul, S. Sirisubtawee, and S. Phongthanapanich, "A comparison of numerical solutions for advection-diffusion-reaction equations between finite volume and finite difference methods," Engineering Letters, vol. 30, no. 2, pp. 566–581, 2022.
[19] J.-H. He, "Recent development of the homotopy perturbation method," *Topological methods in nonlinear analysis*, vol. 31, no. 2, pp. 205–209, 2008.
[20] ——, "A note on the homotopy perturbation method," *Thermal Science*, vol. 14, no. 2, pp. 565–568, 2010.
[21] E. Babolian, A. Azizi, and J. Saeidian, "Some notes on using the homotopy perturbation method for solving time-dependent differential equations," *Mathematical and Computer Modelling*, vol. 50, no. 1, pp. 213–224, 2009.
[22] J. Saberi-Nadjafi and S. Zahmatkesh, "Homotopy perturbation method (hpm) for solving higher order boundary value problems (bvp)," *Applied Mathematical and Computational Sciences*, vol. 1, no. 2, pp. 199–224, 2010.
[23] S. Liao, "Notes on the homotopy analysis method: Some definitions and theorems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 983–997, 2009.
[24] N. M. Bujurke, V. Madalli, and B. G. Mulimani, "Long series analysis of laminar flow through parallel and uniformly porous walls of different permeability," *Comput. Methods Appl. Mech. Engrg*, vol. 160, no. 1, pp. 39–56, 1998.
[25] V. S. S. Kumar and N. P. Pai, "Suction and injection effect on flow between two plates with reference to casson fluid model," *Multidiscipline Modeling in Materials and Structures*, vol. 15, no. 3, pp. 559–574, 2019.
[26] A. Ansari, H. Temimi, M. Kinawi, and A. Siddiqui, "A note on certain perturbation methods for solving the problem of fully developed flow through a porous channel," IAENG International Journal of Applied Mathematics, vol. 40, no. 4, pp. 224–232, 2010.
[27] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3, pp. 257–262, 1999.