Solution for Open Questions in Yen (2021) Osler (2001) and Çalışkan (2020, 2022)

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Abstract—**The purpose of studying inventory models is to compensate for the uncertainty in the timing and quantity of demand and supply, which is even more significant in the post-epidemic era. To solve inventory models by algebraic methods, a recent paper provided a new approach to studying the problem that had been examined by five published papers. This paper also demonstrated that his new approach can solve another transit bus model, and yet, he raised an open question. The purpose of this paper is to offer a formative answer accordingly. Moreover, we solve an open question proposed by another paper to prove identity with two cubic roots without referring to its original cubic polynomial. Meanwhile, we discuss another problem in a related paper to show that comparisons among several approximated inventory models by their minimum values are not proper that should be executed on the original model. At last, we examine a recently published paper to point out questionable results, and then offer revisions. This paper will help researchers solve their inventory models with proper solution approaches.**

***Index Terms*—Economic ordering, Economic production, Algebraic method, Inventory model**

I. Introduction

Developing solution methods without referring to calculus can help practitioners with a limited mathematical background to accept models and their applications. Recently, with an intuitive point of view, Yen [1] published a paper in the International Journal of Applied Mathematics to present an algebraic approach to solving inventory models with two backorder costs. Yen's method is simpler than Grubbström and Erdem [2], Cárdenas-Barrón [3], Ronald et al. [4], Chang et al. [5], and Luo and Chou [6]. Moreover, Yen [1] tried to demonstrate that his approach can solve the transit bus model proposed by Chang and Schonfeld [7], but left an open question. The first purpose of this paper is to present a positive answer to the open question proposed by Yen [1]. Then, we show that our algebraic skills can solve the other open question proposed by Osler [8]. Meanwhile, we point out questionable comparisons in Çalışkan [9] to help researchers to select the best-approximated inventory model. Moreover, we study Çalışkan [10] to point out that he added an extra condition for the objective function to be a convex function to indicate that Çalışkan [10] needed other restrictions to solve the open question proposed by Lau et al. [11]. Moreover, Çalışkan [10] squared both sides of an inequality without checking why the left-hand side is non-negative which violated algebraic rules. Our paper will help researchers to realize transit bus models and inventory models.

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II. The Open Question Proposed by Yen [1]

Yen [1] considered the transit bus model developed by Chang and Schonfeld [7], with the following objective function,

$$C(r,h)=\frac{d_1}{rh}+d_2(r+d)+d_3h+d_4 \qquad (2.1)$$

where $d_1=BDTW$, $d_2=\frac{qxLTW}{4g}$, $d_3=qwzLTW$, and $d_4=qvLMTW$ are four abbreviations to abstractly express the presentation of Equation (2.1).

The goal of Equation (2.1) is to find the minimum solution for the route width r^*, the headway h^*, and the minimum cost $C(r^*,h^*)$.

In Chang and Schonfeld [7], they applied calculus to solve $\frac{\partial}{\partial r}c(r,h)=0$ and $\frac{\partial}{\partial h}c(r,h)=0$ to derive that

$$r^*=\left(\frac{d_1d_3}{d_2^2}\right)^{1/3}, \qquad (2.2)$$

and

$$h^*=\left(\frac{d_1d_3}{d_3^2}\right)^{1/3}. \qquad (2.3)$$

Yen [1] tried to use algebraic methods to solve the minimum problem of Equation (2.1), and then Yen [1] mentioned that based on Equations (2.2) and (2.3) he obtained that

$$r^*h^*=\left(\frac{d_1^{\,2}}{d_2d_3}\right)^{1/3}, \qquad (2.4)$$

such that he derived that r^*h^* is a constant and then he assumed the following condition

$$rh=C_0. \qquad (2.5)$$

Yen [1] changed Equation (2.1) with two variables h and r to a new system of variables h and C_0,

$$C(\frac{c_0}{h},h)=\frac{d_1}{c_0}+d_2(\frac{c_0}{h}+d)+d_3h+d_4, \qquad (2.6)$$

and then he rewrote Equation (2.6) as

$$C(\frac{c_0}{h},h)=\frac{d_1}{c_0}+d_2(\frac{c_0}{h}+d)+d_3h+d_4=$$

$$\left(\sqrt{\frac{d_2c_0}{h}}-\sqrt{d_3h}\right)^2+2\sqrt{d_2d_3c_0}+d_2d+d_4+\frac{d_1}{c_0}. \qquad (2.7)$$

From Equation (2.7), Yen [1] claimed that

$$h^*=\sqrt{d_2C_0/d_3}, \qquad (2.8)$$

and then he changed the optimal problem of Equation (2.1) from a two-variable problem of h, and C_0 into the following one-variable problem of C_0,

$$C(\frac{c_0}{h},h^*) = d_2 d + 2\sqrt{d_2 d_3 c_0} + d_4 + \frac{d_1}{c_0}. \quad (2.9)$$

Hence, Yen [1] overlooked the constant term of $d_2 d + d_4$ and then abstractly rewrote Equation (2.9) as follows, with $A_1 = d_1 > 0$, $B_1 = 2\sqrt{d_2 d_3} > 0$ and a new variable $x = \sqrt{C_0}$, then the minimum problem became

$$f(x) = \frac{A_1}{x^2} + B_1 x. \quad (2.10)$$

Yen [1] raised an open question to solve Equation (2.10) by algebraic methods.

III. Our algebraic method for the open question of Yen [1]

We assume the minimum solution of Equation (2.10) as x^*, and then assume the two sequences, (α_n) and (β_n) where

$$\alpha_n = x^* + (1/n), \quad (3.1)$$

and

$$\beta_n = x^* - (1/n), \quad (3.2)$$

for $n = 1,2,\ldots$. We know that when n goes to infinity, then α_n will go to x^*. Similarly, when n goes to infinity, then β_n will also go to x^*.

We derive that

$$f(\alpha_n) - f(x^*) \geq 0, \quad (3.3)$$

and

$$f(\beta_n) - f(x^*) \geq 0, \quad (3.4)$$

owing to that x^* is the minimum solution.

We plug Equation (2.10) into Equations (3.3) and (3.4) to imply that

$$\frac{A_1}{(\alpha_n x^*)^2}\left[(x^*)^2 - (\alpha_n)^2\right] + B_1(\alpha_n - x^*) \geq 0, \quad (3.5)$$

and

$$\frac{A_1}{(\alpha_n x^*)^2}\left[(x^*)^2 - (\beta_n)^2\right] + B_1(\beta_n - x^*) \geq 0. \quad (3.6)$$

We cancel out the common factor, $\alpha_n - x^*$ from Equation (3.5). Owing to $\alpha_n - x^* > 0$, the direction of inequality will not be altered, then we obtain that

$$\frac{-A_1}{(\alpha_n x^*)^2}\left[\alpha_n + x^*\right] + B_1 \geq 0. \quad (3.7)$$

When n goes to infinity, α_n goes to x^* to result in

$$\frac{B_1}{2A_1} \geq \frac{1}{(x^*)^3}. \quad (3.8)$$

On the other hand, we cancel out the common factor, $\beta_n - x^*$ from Equation (3.6). Owing to $\beta_n - x^* < 0$, the direction of inequality will be altered, then we obtain that

$$\frac{-A_1}{(\beta_n x^*)^2}\left[\beta_n + x^*\right] + B_1 \leq 0. \quad (3.9)$$

When n goes to infinity, β_n goes to x^* to result in

$$\frac{B_1}{2A_1} \leq \frac{1}{(x^*)^3}. \quad (3.10)$$

We combine the results of Equations (3.8) and (3.10) to derive that

$$\frac{B_1}{2A_1} = \frac{1}{(x^*)^3}, \quad (3.11)$$

and then we find that

$$x^* = (2A_1 / B_1)^{1/3}. \quad (3.12)$$

Therefore, we provide an algebraic method to solve the open question proposed by Yen [1].

IV. Application of our derivations

We recall that $(C_0)^* = (x^*)^2$, with $A_1 = d_1$, and $B_1 = 2\sqrt{d_1 d_2}$, and then apply our result of Equation (3.12) to imply that

$$(C_0)^* = (d_1^2 / d_2 d_3)^{1/3}. \quad (4.1)$$

Based on Equation (1.8), we obtain that

$$h^* = \sqrt{d_2 (C_0)^* / d_3} = (d_1 d_2 / d_3^2)^{1/3}, \quad (4.2)$$

We recall Equation (1.4) to derive that

$$r^* = (C_0)^* / h^* = (d_1 d_3 / d_2^2)^{1/3}. \quad (4.3)$$

Our results of Equations (4.2) and (4.3) are identical to that of Chang and Schonfeld [7] obtained by calculus as mentioned in Equations (2.2) and (2.3).

Therefore, our algebraic approach can solve the bus transit model proposed by Chang and Schonfeld [7].

V. A related problem

Here, we begin to discuss the second open question that will be solved by our algebraic methods. We will verify the following identity containing two cubic roots,

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = 1. \quad (5.1)$$

without referring to the following cubic equation

$$x^3 + 3x - 4 = 0. \quad (5.2)$$

We factor Equation (5.2) to imply that

$$x^3 + 3x - 4 = (x-1)(x^2 + x + 4). \quad (5.3)$$

Based on Equation (5.3), we know that $x^3 + 3x - 4 = 0$ has three roots as one real root,

$$x = 1, \quad (5.4)$$

with two complex number roots,

$$x = \frac{-1+\sqrt{15}i}{2}, \quad (5.5)$$

and

$$x = \frac{-1-\sqrt{15}i}{2}. \quad (5.6)$$

Motivated by Equation (5.1), we assume that

$$\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}} = y. \quad (5.7)$$

Our goal is to show that $y=1$.

There are infinite combinations for

$$a y^m + b y^n, \quad (5.8)$$

where a and b are integers; m, and n are positive numbers.

If we know Equation (5.2) in advance, then our goal is $y^3 + 3y$ such that we select $a = 1$, $m = 3$, $b = 3$, and $n = 1$, and then we evaluate $y^3 + 3y$ to find that

$$\begin{aligned} & y^3 + 3y \\ &= \left[\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}\right]^3 + 3\left[\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}\right] \\ &= 2+\sqrt{5} + 3\left(\sqrt[3]{2+\sqrt{5}}\right)\left(\sqrt[3]{2-\sqrt{5}}\right)\left[\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}\right] \\ &\quad + 2 - \sqrt{5} + 3\left[\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}\right]. \end{aligned} \quad (5.9)$$

Owing to

$$\left(\sqrt[3]{2+\sqrt{5}}\right)\left(\sqrt[3]{2-\sqrt{5}}\right)=\sqrt[3]{4-5}=-1, \tag{5.10}$$

We can further simplify the derivation of Equation (5.9) as

$$y^3+3y=4. \tag{5.11}$$

Based on Equation (5.11), it indicates that y is a solution of $y^3+3y-4=0$.

We rewrite Equation (5.7) as

$$y=\sqrt[3]{2+\sqrt{5}}-\sqrt[3]{\sqrt{5}-2}, \tag{5.12}$$

to indicate that $y=\sqrt[3]{2+\sqrt{5}}-\sqrt[3]{\sqrt{5}-2}$ is a real number. We refer to Equations (5.4-5.6) to find that

$$y=1 \tag{5.13}$$

that is,

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=1, \tag{5.14}$$

the desired result of Equation (5.1) appears. However, the above derivation, using Equation (5.2) as a motivation, the purpose of this open question is to derive Equation (5.1) without referring to Equation (5.2).

This open issue had been discussed in several papers, such as Sofo [12], Osler [8], and Osler [13]. Intended to enhance the completeness of the theories in the previous papers, this study attempts to prove the indeterminate mathematical problems left. In the next section, a detailed review of the three papers mentioned above will be provided.

VI. REVIEW OF SOFO [12], OSLER [8], AND OSLER [13]

Sofo [12] considered this open problem to assume that

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=I, \tag{6.1}$$

and then Sofo [12] cubed on both sides of Equation (6.1) to derive that

$$I^3+3I-4=0. \tag{6.2}$$

Owing to $I=1$ being the only real solution for Equation (6.2), Sofo [12] concluded that Equation (5.1) is proved. We point out that Sofo [12] had proved Equation (5.1), but Sofo [12] used Equation (6.2) which is identical to Equation (5.2).

Osler [8] mentioned that let us start with the expression

$$x=\sqrt[n]{a+\sqrt{b}}+\sqrt[n]{a-\sqrt{b}}, \tag{6.3}$$

and then raise both sides of Equation (6.3) to their *n*th power, and then the Cardan polynomial, $C_n(c,x)=2a$, of degree *n*, appeared, where

$$c=\sqrt[n]{a+\sqrt{b}}\sqrt[n]{a-\sqrt{b}}=\sqrt[n]{a^2-b}. \tag{6.4}$$

In the following, we will provide an example of the Cardan polynomial.

For example, $n=3$, then

$$x^3=\left(\sqrt[3]{a+\sqrt{b}}+\sqrt[3]{a-\sqrt{b}}\right)^3,$$

$$=\sum_{k=0}^{3}\frac{3!}{k!(3-k)!}\left(\sqrt[3]{a+\sqrt{b}}\right)^k\left(\sqrt[3]{a-\sqrt{b}}\right)^{3-k},$$

$$=a+\sqrt{b}+3\left(\sqrt[3]{a+\sqrt{b}}\right)^2\left(\sqrt[3]{a-\sqrt{b}}\right)$$

$$+3\left(\sqrt[3]{a+\sqrt{b}}\right)\left(\sqrt[3]{a-\sqrt{b}}\right)^2+a-\sqrt{b}$$

$$=2a+3c\left(\sqrt[3]{a+\sqrt{b}}+\sqrt[3]{a-\sqrt{b}}\right)$$

$$=2a+3cx. \tag{6.5}$$

Based on Equation (6.5), we know the Cardan polynomial of degree 3 as

$$C_3(c,x)=x^3-3cx=2a. \tag{6.6}$$

When $n=3$, $a=2$, and $b=5$, then $c=-1$ such that the above derivations of Equations (6.3-6.6) show that

$$x=\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}, \tag{6.7}$$

satisfies the Cardan polynomial of degree 3,

$$x^3+3x=4. \tag{6.8}$$

Osler [8] claimed that $x^3+3x=4$ has only one real root, $x=1$ to imply that

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=x=1. \tag{6.9}$$

Referring to the Cardan polynomial of degree 3, with $a=2$, and $b=5$, Osler [8] proved Equation (6.9) is valid which is the goal of this paper as Equation (5.1).

Osler [8] proposed an open question: To verify Equation (6.9) without referring to Cardan polynomial of degree 3, with $a=2$, and $b=5$, that is Equation (6.8).

The purpose of the second part of this paper is to solve the open question proposed by Osler [8].

Next, we examine how did Osler [8] try to answer the open question that was proposed by himself.

Osler[8] claimed that he would derive that

$$\sqrt[n]{a\pm\sqrt{b}}=\frac{x\pm\sqrt{x^2-4c}}{2}, \tag{6.10}$$

where

$$x=\sqrt[n]{a+\sqrt{b}}+\sqrt[n]{a-\sqrt{b}}, \tag{6.11}$$

and

$$c=\sqrt[n]{a^2-b}, \tag{6.12}$$

We recall the derivations proposed by Osler [8] for Equation (6.10). Osler [8] assumed that

$$y=\sqrt[n]{a+\sqrt{b}}, \tag{6.13}$$

and then multiply both sides of Equation (6.13) by $\sqrt[n]{a-\sqrt{b}}$ to obtain

$$y\sqrt[n]{a-\sqrt{b}}=c, \tag{6.14}$$

that is

$$\sqrt[n]{a-\sqrt{b}}=\frac{c}{y}. \tag{6.15}$$

Osler [8] applied Equations (6.13) and (6.15) to Equation (6.11) to derive that

$$x=y+\frac{c}{y}, \tag{6.16}$$

that is

$$y^2-xy+c=0, \tag{6.17}$$

as a quadratic equation in variable *y*, such that

$$y=\frac{x\pm\sqrt{x^2-4c}}{2}. \tag{6.18}$$

Osler [8] claimed that based on Equations (6.13) and (6.18), then Equation (6.10) is verified.

However, we must point out that there are two possible solutions for y in Equation (6.18) such that

$$y=\frac{x+\sqrt{x^2-4c}}{2}, \quad (6.19)$$

or

$$y=\frac{x-\sqrt{x^2-4c}}{2}. \quad (6.20)$$

Hence, Osler [8] only found that

$$\sqrt[n]{a+\sqrt{b}}=\frac{x+\sqrt{x^2-4c}}{2}, \quad (6.21)$$

or

$$\sqrt[n]{a+\sqrt{b}}=\frac{x-\sqrt{x^2-4c}}{2}. \quad (6.22)$$

Based on our findings of Equations (6.21) and (6.22), we show that the proof of Equation (6.10) is not finished.

On the other hand, Osler [8] did not provide any discussion for the proof of

$$\sqrt[n]{a\text{-}\sqrt{b}}=\frac{x-\sqrt{x^2-4c}}{2} \quad (6.23)$$

such that the proof of Equation (6.10) is far from completion by Osler [8].

Last, but not least, even though we accept that Equation (6.10) is valid for the moment when applying Equation (6.10), researchers only find that

$$\sqrt[n]{a+\sqrt{b}}+\sqrt[n]{a-\sqrt{b}}=x. \quad (6.24)$$

which is identical to Equation (6.11) so that the result of Equation (6.24) is true. However, the results of Equations (6.24) or (6.11) are useless to verify Equation (5.1).

In Example 1 of Osler [8], he mentioned that using Equations (6.10) and (6.11) with $n = 3$, $a = 2$, and $b = 5$, and then $c = -1$ to claim that

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-5}=x, \quad (6.25)$$

and

$$\sqrt[3]{2\pm\sqrt{5}}=\frac{x\pm\sqrt{x^2+4}}{2}. \quad (6.26)$$

Osler [8] asserted that the Cardan polynomial of degree 3 as $x^3+3x=4$ has a unique real root of x =1, and then he found that

$$\sqrt[3]{2\pm\sqrt{5}}=\frac{1\pm\sqrt{5}}{2}. \quad (6.27)$$

Applying the result of Equation (5.27), Osler [8] claimed that he derived that

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=1. \quad (6.28)$$

We must point out that the computation in Example 1 of Osler [8] contained severe questionable results because he referred to Equation (5.2).

On the other hand, if Osler [8] already accepted that x =1, then Osler [8] recalled Equation (6.25), then he directly verified Equation (6.28), without referring to Equation (6.27). Hence, we can claim that the assertion of Osler [8] to obtain Equation (6.10) is not valid.

Based on the above discussion, we can say that Osler [8] cannot prove Equation (6.28) without referring to Equation (5.2).

Next, we will present a review of Osler [13].

Osler [13] studied a generalized version of the formulated solution for

$$x^3-3cx-2a=0, \quad (6.29)$$

under the condition, $b=a^2-c^3\geq 0$, then a real root is denoted as

$$x=\sqrt[3]{a+\sqrt{b}}+\sqrt[3]{a-\sqrt{b}}. \quad (6.30)$$

When Osler [13] applied Equations (6.29) and (6.30) to solve Equation (5.2), with $a=2$, $b=5$, and $c=-1$, Osler [13] obtained the solution $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$ and then he raised an open question:

How do we use the algebraic method directly to show that

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=1. \quad (6.31)$$

Osler [13] mentioned that "The reader might try to simplify this difference of two cube roots into the number 1. But all attempts to do this simply lead back to the original cubic $x^3+3x-4=0$."

The discussions of Sofo [12], Osler [8], and Osler [13] related to examining the identity of Equation (5.1) under the restriction, without referring to Equation (5.2) are the original open question for the second part of our article.

VII. Motivation for our approach

In the following, we will provide a genuine algebraic approach without referring to $x^3+3x-4=0$, which is Equation (5.2), to show that Equation (5.1) is valid. First, we will present a reasonable motivation for our approach.

In the next section, we will assume that

$$\sqrt[3]{2+\sqrt{5}}=a+b\sqrt{5}, \quad (7.1)$$

and

$$\sqrt[3]{2-\sqrt{5}}=a-b\sqrt{5}. \quad (7.2)$$

In this section, we begin to present a reasonable motivation for why we can assume Equations (7.1) and (7.2).

We still assume that

$$x=\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}, \quad (7.3)$$

with two auxiliary expressions, we assume that

$$y=\sqrt[3]{2+\sqrt{5}}, \quad (7.4)$$

and

$$z=\sqrt[3]{2-\sqrt{5}}, \quad (7.5)$$

to yield that

$$x=y+z. \quad (7.6)$$

Here, we must point out that

$$y>0, \quad (7.7)$$

and

$$z<0. \quad (7.8)$$

We compute that

$$yz=\sqrt[3]{2+\sqrt{5}}\sqrt[3]{2-\sqrt{5}}=\sqrt[3]{-1}=-1, \quad (7.9)$$

that is consistent with Equations (7.7) and (7.8).

Based on Equation (7.9), we derive that

$$z=-1/y, \quad (7.10)$$

and then we plug our findings of Equation (7.10) into Equation (7.6) to yield that

$$x=y+\frac{-1}{y}. \quad (7.11)$$

We rewrite Equation (7.11) in the quadratic form in the variable of y to imply that

$$y^2 - xy - 1 = 0, \tag{7.12}$$

and then it follows that

$$y = \frac{x \pm \sqrt{x^2+4}}{2}. \tag{7.13}$$

Owing to Equation (7.7), we know that $y > 0$ and $\left(x - \sqrt{x^2+4}\right)/2 < 0$, such that we obtain a positive solution as

$$y = \frac{x + \sqrt{x^2+4}}{2}. \tag{7.14}$$

Similarly, we use Equation (7.9) again to derive that

$$y = -1/z, \tag{7.15}$$

and then we plug our result of Equation (7.15) into Equation (7.6) to derive that

$$x = z + \frac{-1}{z}. \tag{7.16}$$

We rewrite Equation (7.16) in the quadratic form in the variable of z to imply that

$$z^2 - xz - 1 = 0, \tag{7.17}$$

and then it follows that

$$z = \frac{x \pm \sqrt{x^2+4}}{2}. \tag{7.18}$$

Owing to Equation (7.8), we know that $z < 0$, and on the other hand, it is trivial $\left(x + \sqrt{x^2+4}\right)/2 > 0$, so we obtain a negative value of z as

$$z = \frac{x - \sqrt{x^2+4}}{2}. \tag{7.19}$$

From our derivations of Equations (7.14) and (7.19), we know that if we assume that $a = \frac{x}{2}$ and $b = \frac{1}{2}\left(\sqrt{\frac{x^2+4}{5}}\right)$, then use Equations (7.4) and (7.14), to obtain that

$$\begin{aligned}\sqrt[3]{2+\sqrt{5}} &= y \\ &= \frac{x + \sqrt{x^2+4}}{2} \\ &= \frac{x}{2} + \frac{1}{2}\left(\sqrt{\frac{x^2+4}{5}}\right)\sqrt{5} \\ &= a + b\sqrt{5},\end{aligned} \tag{7.20}$$

which is Equation (7.1). Moreover, using Equations (7.5) and (7.19), we find that

$$\begin{aligned}\sqrt[3]{2-\sqrt{5}} &= z \\ &= \frac{x - \sqrt{x^2+4}}{2} \\ &= \frac{x}{2} - \frac{1}{2}\left(\sqrt{\frac{x^2+4}{5}}\right)\sqrt{5} \\ &= a - b\sqrt{5},\end{aligned} \tag{7.21}$$

which is Equation (7.2).

Therefore, we provide a reasonable motivation for our assumptions of Equations (7.1) and (7.2).

VIII. Our solution approach

We will assume the following two relationships of (8.1) and (8.2) to derive the desired goal of Equation (5.1).

$$\sqrt[3]{2+\sqrt{5}} = a + b\sqrt{5}, \tag{8.1}$$

and

$$\sqrt[3]{2-\sqrt{5}} = a - b\sqrt{5}. \tag{8.2}$$

We cube both sides of Equations (8.1) and (8.2) to derive that

$$2+\sqrt{5} = a^3 + 3\sqrt{5}a^2b + 15ab^2 + 5\sqrt{5}b^3, \tag{8.3}$$

and

$$2-\sqrt{5} = a^3 - 3\sqrt{5}a^2b + 15ab^2 - 5\sqrt{5}b^3. \tag{8.4}$$

We take the sum and difference of Equations (8.3) and (8.4) to imply that

$$2 = a^3 + 15ab^2, \tag{8.5}$$

and

$$1 = 3a^2b + 5b^3. \tag{8.6}$$

From Equations (8.5) and (8.6), we find that

$$a^3 + 15ab^2 - 2\left(3a^2b + 5b^3\right) = 0. \tag{8.7}$$

We can factor the left-hand side of Equation (8.7) as

$$\begin{aligned}&a^3 - 6a^2b + 15ab^2 - 10b^3 = \\ &(a-b)\left(\left(a - (5b/2)\right)^2 + \left(15b^2/4\right)\right).\end{aligned} \tag{8.8}$$

We observe the right-hand side of Equation (8.8) to know that there is a unique real coefficient relationship between a and b as

$$a = b. \tag{8.9}$$

We plug Equation (8.9) into Equation (8.5) to obtain that

$$a^3 - \frac{1}{8} = \left(a - \frac{1}{2}\right)\left(\left(a + \frac{1}{4}\right)^2 + \frac{3}{16}\right), \tag{8.10}$$

to yield a unique real solution as

$$a = 1/2, \tag{8.11}$$

and then recall Equation (8.9),

$$b = 1/2. \tag{8.12}$$

Consequently, we know that

$$\sqrt[3]{2+\sqrt{5}} = 0.5 + 0.5\sqrt{5}, \tag{8.13}$$

and

$$\sqrt[3]{2-\sqrt{5}} = 0.5 - 0.5\sqrt{5}. \tag{8.14}$$

After we take the sum of Equations (8.13) and (8.14) we show that Equation (5.1) is valid, without referring to Equation (5.2).

IX. Application to solve questions in Çalışkan [9]

In this section, we will study a related problem to present a comprehensive study for Çalışkan [9] to examine approximated inventory models, and then we will show that comparing minimum values for those approximated models is invalid to choose the best-approximated inventory model. We need the following extra notation to discuss Çalışkan [9].

c: is the unit cost per item.
D: is the demand per unit of time.
H: is the holding cost per item per unit of time, with $H = ic$.
N: is the number of intervals (cycles) per unit time with $N=1/T$.

Q: is the ordering quantity.
S: is the ordering cost per order.
T: is the time between consecutive orders.
T_1: is the time when the inventory level drops to zero. During $[T_1, T]$, the shortages occur.
δ : is the rate of deterioration per unit of the item per unit time.

For an inventory system with setup cost, purchasing cast, and holding cost, as mentioned in Ghare and Schrader [14], the average cost is derived as follows,

$$AC_1(T)=\frac{s}{T}+\frac{cD}{\delta T}(e^{\delta T}-1)+\frac{DH}{\delta^2 T}(e^{\delta T}-1-\delta T) \quad (9.1)$$

and then Ghare and Schrader [14] constructed a simplified inventory model to locate an optimal solution in a closed-form expression.

Ghare and Schrader [14] used the first three terms of Taylor's series expansion for the exponential function to simplify the ordering quantity as

$$Q=\frac{D}{\delta}(e^{\delta T}-1)\approx D(T+\frac{\delta T^2}{2}) \quad (9.2)$$

and then Ghare and Schrader [14] computed the holding cost of $Q=DT(1+(\delta T/2))$ for the entire replenishment cycle $[0, T]$ to find an over-estimated holding cost as

$$HD(T+\frac{\delta T^2}{2})T\,, \quad (9.3)$$

and then the average holding cost is derived as

$$HD(T+\frac{\delta T^2}{2})\,. \quad (9.4)$$

We know that the exact average purchasing cost is expressed as

$$c\frac{I(0)}{T}=\frac{cD}{\delta T}(e^{\delta T}-1)\,. \quad (9.5)$$

Applying the estimation of Equation (9.2), Ghare and Schrader [14] obtained an estimated average purchasing cost as

$$cD(1+\frac{\delta T}{2})\,. \quad (9.6)$$

Based on our above discussion, Ghare and Schrader [14] obtained their approximated inventory system,

$$AC_2(T)=cD+\frac{s}{T}+\frac{c\delta D}{2}T+DHT+\frac{\delta DH}{2}T^2\,, \quad (9.7)$$

that had appeared in Çalışkan [9].

The closed-form minimum solution of $AC_2(T)$ is the roots of a cubic polynomial and then Çalışkan [9] showed an implicit expression,

$$T=\sqrt{\frac{s}{\frac{(c\delta+2H)D}{2}+H\delta DT}}\,. \quad (9.8)$$

Çalışkan [9] constructed the next approximated inventory system,

$$AC_3(T)=\frac{s}{T}+\frac{[c(r+\delta)2H]DT}{2}+\frac{H\delta DT^2}{2}+cD\,. \quad (9.9)$$

In Çalışkan [9], he did not inform readers how did he locate the minimum solution for his approximated inventory system $AC_2(T)$.

Moreover, Çalışkan [9] claimed that the exact total cost function for the deteriorating items EOQ problem can be derived as

$$AC_4(T)=TC(Q,T)=\frac{Cq+S}{T}+HQ+\frac{rc^2}{\delta T}(Q-DT)\,, \quad (9.10)$$

where $Q=\frac{D}{\delta}(e^{\delta T}-1)$.

Similarly, concerning to AC_4 (T), Çalışkan [9] did not tell researchers how to derive the minimum solution.

At last, motivated by Widyadana et al. [15], Çalışkan [9] developed another approximated inventory system,

$$AC_5(Q)=\frac{S(D+\delta)}{Q}+(2H+rc)\frac{Q}{2}\,. \quad (9.11)$$

The optimal solution of AC_5 (Q) is obtained as

$$Q^*=\sqrt{\frac{2S(D+\delta)}{2H+rc}}\,. \quad (9.12)$$

In Tables 1 and 2 of Çalışkan [9], he showed the minimum points of $AC_2(T)$, $AC_3(T)$, $AC_4(T)$, and $AC_5(Q)$ as T_2^*, T_3^*, T_4^*, and Q_5^*, respectively, and then he computed $AC_2(T_2^*)$, $AC_3(T_3^*)$, $AC_4(T_4^*)$, and $AC_5(Q_5^*)$ to list them in Tables 1 and 2 of Çalışkan [9].

We will show that those comparisons are useless.

Consequently, we will not cite Tables 1 and 2 of Çalışkan [9] in our paper to avoid wasting the precious space of this journal.

Our goal is to point out that after finding T_2^*, T_3^*, T_4^*, and Q_5^*, researchers should plug into a genuine inventory system, for example, $AC_1(T)$, and then compare values of AC_1 (T_2^*), $AC_1(T_3^*)$, $AC_1(T_4^*)$, and $AC_1(Q_5^*)$ to decide among four approximated inventory systems, $AC_2(T)$, $AC_3(T)$, $AC_4(T)$, and $AC_5(Q)$, which one can provide the best estimated minimum value.

If we assume that the genuine inventory system has the following expression,

$$AC_6(T)=\frac{s}{T}+\frac{cD}{\delta T}(e^{\delta T}-1)+\frac{D(H+rc)}{\delta^2 T}(e^{\delta T}-1-\delta T)\,, \quad (9.13)$$

We construct four approximated inventory systems as follows:

$$AC_7(T)=\frac{s}{T}+(0.9)(e^{\delta T}-1)+\frac{D(H+rc)}{\delta^2 T}(e^{\delta T}-1-\delta T)\,, \quad (9.14)$$

$$AC_8(T)=\frac{s}{T}+(1.1)(e^{\delta T}-1)+\frac{D(H+rc)}{\delta^2 T}(e^{\delta T}-1-\delta T)\,, \quad (9.15)$$

$$AC_9(T)=\frac{s}{T}+\frac{cD}{\delta T}(e^{\delta T}-1)+(0.9)\frac{D(H+rc)}{\delta^2 T}(e^{\delta T}-1-\delta T)\,, \quad (9.16)$$

and

$$AC_{10}(T)=\frac{s}{T}+\frac{cD}{\delta T}(e^{\delta T}-1)+(1.1)\frac{D(H+rc)}{\delta^2 T}(e^{\delta T}-1-\delta T)\,. \quad (9.17)$$

After we develop AC_j (T), for j=7,8,9, and 10, we give a family of parameters, S=100, δ=0.05, r=0.01, D=1000, c=10, i=0.2, and H=ic, and then we begin to find the minimum solution for AC_j (T), denoted as T_j^*, for j=7,8,9, and 10 to list them in the next table 1.

Based on Table 1, we claim that

$$TC_7(T_7^*)< TC_8(T_8^*)< TC_9(T_9^*)< TC_{10}(T_{10}^*). \quad (9.18)$$

From the construction of Equations (9.14-9.17), we immediately know that

$$TC_7(T_7^*)< TC_8(T_8^*), \quad (9.19)$$

and

$$TC_9(T_9^*)< TC_{10}(T_{10}^*). \quad (9.20)$$

We put the results of Equations (9.18-9.20) together to reveal that $cD(e^{\delta T}-1)/\delta T$ is the dominated term, and $D(H+rs)(e^{\delta T}-1-\delta T)/\delta^2 T$ is the liberated term, that is,

Table 1. Comparison among T_j^* and AC_H (T_j^*), for j=7,8,9, and 10.

T_7^*	T_8^*	T_9^*	T_{10}^*	$AC_7(T_7^*)$	$AC_8(T_8^*)$	$AC_9(T_9^*)$	$AC_{10}(T_{10}^*)$
0.2788	0.2735	0.2877	0.2656	9715.81	11729.68	10693.04	10751.33

$$\frac{D(H+rc)}{\delta^2 T}(e^{\delta T}-1-\delta T)<\frac{cD}{\delta T}(e^{\delta T}-1), \quad (9.21)$$

because AC_7 reduces the dominated term, and AC_8 increases the dominated term.

However, in the following, we will point out that the comparisons of Equation (9.18) did not provide an answer to the most important question: which approximated system can generate the best-approximated minimum solution?

On the other hand, we can estimate the second term of AC_6 as follows,

$$\frac{cD}{\delta T}(e^{\delta T}-1)\approx\frac{cD}{\delta T}(\delta T)\approx CD=1000, \quad (9.22)$$

and estimate the third term of AC_6 as follows,

$$\frac{D(H+rc)}{\delta^2 T}(e^{\delta T}-1-\delta T)\approx\frac{D(H+rc)}{\delta^2 T}(\frac{\delta^2 T^2}{2})\approx\frac{T}{2}D(H+rc)\approx 283.5 \quad (9.23)$$

in Equation (9.22), where we estimate $T\approx 0.27$.

We can conclude the results of Equation (9.21) directly by observing our estimations of Equations (9.22) and (9.23) without computing AC_j (T_j^*) for j=7,8,9, and 10.

Hence, we can say that numerical comparisons among approximated models are a little useful because Table 1 only tells us some information that can be obtained by other easy methods, but those comparisons did not tell us among T_j^* for j=7,8,9, and 10 which one provides the best approximation.

Researchers should compute AC_6 (T_j^*), for j=7,8,9, and 10 to compare them to decide which approximated system can generate an accurate estimation for the original inventory model.
Hence, we improve Table 1 to the desired comparisons as to the Table 2.

Based on Table 2, we observe that

$$AC_6(T_8^*)< AC_6(T_7^*)< AC_6(T_{10}^*)< AC_6(T_9^*), \quad (9.24)$$

to indicate that based on the approximated system $AC_8(T)$ can generate the best-approximated estimation, T_8^*.

We recall that $AC_8(T_8^*)$ is the maximum among $\left\{AC_j(T_j^*):=7,8,9 \text{ and } 10\right\}$. Hence, Based on Table 1, we cannot expect that researchers dare to predict that TC_8 (T) will produce the best-approximated solution, T_8^*.

Moreover, we can say that under the setting of parameters with S=100, δ=0.05, r=0.01, D=1000, c=10, i=0.2, and H=ic, the numerical comparisons in Table 2 can also imply that

$$\frac{D(H+rc)}{\delta^2 T}(e^{\delta T}-1-\delta T)<\frac{cD}{\delta T}(e^{\delta T}-1), \quad (9.25)$$

without referring to numerical comparisons in Equations (9.22) and (9.23), because if we derive an optimal solution for the approximated system with the dominated term then the resulting cost will be smaller than those approximated systems with the liberated term.
From the above discussions, we demonstrate the usefulness to compute T_j^* and then compare $AC_6(T_j^*)$, for j=7,8,9, and 10.

X. Further study for Çalışkan [10]

A recently published paper, Çalışkan [10] discussed inventory systems that had been studied by Cárdenas-Barrón [3], Chang et al. [5], and Luo and Chou [6]. Except for the notation used in Section IX, we need other expressions in the following,

b: is the cost of backorder per unit per unit time.
B: is the number of units to backorder in each production cycle.
h: is the cost of inventory holding per unit per unit time.
K: is the cost of setup per production batch.
Q: is the economic production quantity.
I_{max}: is the maximum inventory level.

The objective function examined by Cárdenas-Barrón [3], Chang et al. [5], Luo and Chou [6], and Çalışkan [10] is denoted as follows,

$$C(B,Q)=cD+\frac{KD}{Q}+\frac{h\rho Q}{2}-hB+\frac{h+b}{2\rho Q}B^2, \quad (10.1)$$

with an abbreviation,

$$\rho=(P-D)/P. \quad (10.2)$$

Cárdenas-Barrón [3] rewrote Equation (10.1) as

$$C(B,Q)=cD+\frac{D}{\rho Q}\left[\frac{b}{2D}B^2+\frac{h}{2D}(\rho Q-B)^2+\rho K\right]. \quad (10.3)$$

Based on

$$1=\frac{bh}{b(b+h)}+\frac{bh}{h(b+h)}, \quad (10.4)$$

Cárdenas-Barrón [3] obtained that

$$\rho K=\frac{2bh\rho DK}{2h(b+h)D}+\frac{2bh\rho DK}{2b(b+h)D}. \quad (10.5)$$

Cárdenas-Barrón [3] plugged Equation (10.5) into Equation (10.3), and completed the square for the following two items,

$$\frac{b}{2D}B^2+\frac{2bh\rho DK}{2b(b+h)D}=\frac{b}{2D}\left(B-\sqrt{\frac{2\rho hDK}{b(b+h)}}\right)^2+\frac{b}{D}B\sqrt{\frac{2\rho hDK}{b(b+h)}}, \quad (10.6)$$

and

$$\frac{h}{2D}(\rho Q-B)^2+\frac{2bh\rho DK}{2h(b+h)D},$$
$$=\frac{h}{2D}\left(\rho Q-B-\sqrt{\frac{2\rho bDK}{h(b+h)}}\right)^2+\frac{h}{D}(\rho Q-B)\sqrt{\frac{2\rho bDK}{h(b+h)}}. \quad (10.7)$$

Owing to

$$\frac{b}{D}B\sqrt{\frac{2\rho hDK}{b(b+h)}}+\frac{h}{D}(\rho Q-B)\sqrt{\frac{2\rho bDK}{h(b+h)}},$$
$$=\sqrt{\frac{2\rho bhDK}{(b+h)}}\frac{\rho Q}{D}. \quad (10.8)$$

Cárdenas-Barrón [3] derived

$$C(B,Q)=cD+\frac{b}{2D}\left(B-\sqrt{\frac{2\rho hDK}{b(b+h)}}\right)^2,$$
$$+\frac{h}{2D}\left(\rho Q-B-\sqrt{\frac{2\rho bDK}{h(b+h)}}\right)^2+\sqrt{\frac{2\rho bhDK}{(b+h)}}. \quad (10.9)$$

Table 2. Comparison among T_j^* and AC_6 (T_j^*), for j=7,8,9, and 10.

T_7^*	T_8^*	T_9^*	T_{10}^*	AC_6 (T_7^*)	AC_6 (T_8^*)	AC_6 (T_9^*)	AC_6 (T_{10}^*)
0.2788	0.2735	0.2877	0.2656	10722.8090	10722.8077	10723.3939	10723.3189

Therefore, Cárdenas-Barrón [3] claimed that

$$B^* = \sqrt{\frac{2\rho hDK}{b(b+h)}}, \quad (10.10)$$

$$Q^* = \frac{1}{\rho}\left(B^* + \sqrt{\frac{2\rho bDK}{h(b+h)}}\right), \quad (10.11)$$

and

$$C(B^*, Q^*) = cD + \sqrt{\frac{2\rho bhDK}{(b+h)}}. \quad (10.12)$$

The above algebraic approach is too sophisticated that is too difficult for common practitioners to image Equations (10.4) and (10.5). Hence, Ronald et al. [4] tried to provide a two-stage solution procedure. However, Chang et al. [5] pointed out that the two-stage solution procedure proposed by Ronald et al. [4] is still too complicated for ordinary readers, and then Chang et al. [5] developed the following solution method.

Chang et al. [5] rewrote Equation (10.1) as

$$C(B, Q) = cD + \frac{KD}{Q} + \frac{h\rho Q}{2},$$
$$+ \frac{h+b}{2\rho Q}\left(B^2 - \frac{2\rho Q}{h+b}hB\right). \quad (10.13)$$

Chang et al. [5] completed the square for the variable, B, then

$$C(B, Q) = cD + \frac{KD}{Q} + \frac{h\rho Q}{2},$$
$$+ \frac{h+b}{2\rho Q}\left(B - \frac{\rho Q}{h+b}h\right)^2 - \frac{\rho h^2 Q}{2(h+b)}, \quad (10.14)$$

and then they simplify the expression of Equation (10.14) to obtain that

$$C(B, Q) = cD + \frac{KD}{Q} + \frac{hb\rho Q}{2(h+b)} + \frac{h+b}{2\rho Q}\left(B - \frac{\rho Q}{h+b}h\right)^2. \quad (10.15)$$

Chang et al. [5] completed the square for the variable, Q, then

$$C(B, Q) = cD + \left(\sqrt{\frac{KD}{Q}} - \sqrt{\frac{hb\rho Q}{2(h+b)}}\right)^2,$$
$$+ \frac{h+b}{2\rho Q}\left(B - \frac{\rho Q}{h+b}h\right)^2 + 2\sqrt{\frac{KD}{Q}}\sqrt{\frac{hb\rho Q}{2(h+b)}}. \quad (10.16)$$

Based on Equation (10.16), Chang et al. [5] derived the same optimal solutions for the economic production quantity, the backorder quantity, and the minimum cost.

Chang et al. [5] raised an open question as follows.

If they first completed the square for the variable, Q, then

$$C(B, Q) = cD + \frac{1}{Q}\left(KD + \frac{h+b}{2\rho}B^2\right) + \frac{h\rho Q}{2} - hB,$$
$$= cD + 2\sqrt{\left(KD + \frac{h+b}{2\rho}B^2\right)}\sqrt{\frac{h\rho}{2}} + -hB,$$
$$+ \left(\sqrt{\frac{1}{Q}\left(KD + \frac{h+b}{2\rho}B^2\right)} - \sqrt{\frac{h\rho Q}{2}}\right)^2. \quad (10.17)$$

Based on Equation (10.17), Chang et al. [5] claim that

$$Q^*(B) = \sqrt{\left(KD + \frac{h+b}{2\rho}B^2\right)\frac{2}{h\rho}}, \quad (10.18)$$

and the remaining minimum problem,

$$C(B, Q^*(B)) = cD + \sqrt{2h\rho KD + h(h+b)B^2} - hB. \quad (10.19)$$

Referring to Equation (10.19), Chang et al. [5] raise the following open question:

How to solve the minimum problem of $f(B)$ with

$$f(B) = \sqrt{(1+\alpha)B^2 + \beta} - B, \quad (10.20)$$

with $\alpha > 0$, $\beta > 0$, $\alpha = b/h$, and $\beta = 2\rho KD/h$.

Later, Lau et al. [11] generated the minimum problem of Equation (10.20) as follows,

$$f(x) = \sqrt{ax^2 + bx + c} - x, \quad (10.21)$$

for $x > 0$, to find the criterion among parameters a, b and c.

Chiu et al. [16] pointed out that the generalized minimum problem proposed by Lau et al. [11] did not answer by themselves.

Luo and Chou [6] first showed that the solution procedure of Chiu et al. [16] is still questionable, and then presented their revisions.

Recently, Çalışkan [10] mentioned that he developed a new approach to solving the open question proposed by Lau et al. [11].

We recall his lemma 1 that $f(x)$ has a unique interior minimum solution only if $ax^2 + bx + c$ has at most one real root.

In his proof, we cite that "To be convex".

Çalışkan [10] added another restriction for the minimum problem proposed by Lau et al. [11] which is $f(x)$ being a convex function.

We will construct an example to illustrate that "To be convex" is not a good research direction.

We assume that $a > 1$, $b = 0$, and $c = 0$, then

$$f(x) = (\sqrt{a} - 1)x, \quad (10.22)$$

for $x > 0$.

$f(x)$ is a convex function, but the minimum problem of $f(x)$ does not have a solution, because when $x > 0$, then

$$f(x) > f\left(\frac{x}{2}\right) > 0 = \lim_{x\to 0} f(x), \quad (10.23)$$

such that if x^* is the minimum point, then

$$f(x^*) > f\left(\frac{x^*}{2}\right) > 0, \quad (10.24)$$

to imply that $f(x^*)$ is not the minimum value.

Based on our example, we show that discussing the convex property is useless to answer the open question proposed by Lau et al. [11].

Moreover, Çalışkan [10] considered the following inequality:

$$2ax_1x_2 + b(x_1+x_2) + 2c,$$
$$\leq 2\sqrt{(ax_1^2 + bx_1 + c)(ax_2^2 + bx_2 + c)}. \quad (10.25)$$

and an auxiliary function, $h(x_1, x_2)$, with

$$h(x_1, x_2) = 2ax_1x_2 + b(x_1+x_2) + 2c. \quad (10.26)$$

Çalışkan [10] assumed that $h(x_1, x_2) \geq 0$, and then he squared both sides of Equation (10.25) to verify hie lemma 1.

We must claim that adding another condition by Çalışkan [10] is against the purpose of Lau et al. [11].

On the other hand, Lau et al. [11] tried to find conditions for parameters a, b, and c, and then the minimum problem of $f(x)$ can be solved.

Çalışkan [10] did not discuss the conditions for parameters a, b and c. On the other hand, he directly assumed $h(x_1, x_2) \geq 0$ without any supported explanation which is violated academic principles.

The proper solution procedure is to divide into two cases:

Case (a): For those (x_1, x_2), satisfying $h(x_1, x_2) \geq 0$;

Case (b): For those (x_1, x_2), satisfying $h(x_1, x_2) < 0$.

Under Case (a), researchers can apply the square operation on both sides of Equation (10.25).

Otherwise, for Case (b), a new solution procedure must be developed by Çalışkan [10].

We check the convex property of $f(x)$ if and only if $\frac{d^2f}{dx^2} \geq 0$.

We recall $f(x)$ of Equaiton (10.21), then

$$\frac{d^2f}{dx^2} = \frac{4ac - b^2}{4(ax^2+bx+c)^{3/2}}. \quad (10.27)$$

We derive that $f(x)$ is a convex function if and only if

$$4ac - b^2 \geq 0. \quad (10.28)$$

On the other hand, we recall the findings of Luo and Chou [6] to know the following theorems for parameters:

$a > 1$, and $c > 0$,

If $b \geq 0$, then

$$4c > b^2; \quad (10.29)$$

If $b < 0$, then

$$4(a-1)c > b^2. \quad (10.30)$$

When $b \geq 0$, owing to $a > 1$ and Equation (10.29), we find that

$$4ac > 4c > b^2. \quad (10.31)$$

When $b < 0$, using Equation (10.30), we obtain that

$$4ac > 4(a-1)c > b^2. \quad (10.32)$$

We combine the results of Equations (10.31) and (10.32), and then we derive that

$$4ac > b^2. \quad (10.33)$$

We obtain a result that is stronger than "convex" such that "strictly convex" is needed. Therefore, in Çalışkan [10], he only considered "convex" which is not enough.

XI. Real-life implication and managerial insights

Studying inventory systems by algebraic methods can help those researchers who are not familiar with the expertise of calculus and differential equations. Some researchers using calculus treated the real world as a continuous environment and then they can take derivatives and integrations to solve the modularized system that was a reflection of the real-world instance. However, during the modularized process, many complicated issues must be simplified or overlooked to achieve a tractable system. Consequently, many beautiful relations and lengthy derivations did not have practical applications. On the contrary, many researchers tried to build complex models to include many variables and many conditions to approximate the genuine world as much as possible. Consequently, seldom researchers can solve their chaos and mythical system, and then they applied numerical approaches to derive a result that declared their findings are the best result for the time being. However, those complicated models still cannot have practical applications owing to their parameters are (a) constant, (b) vector values, (c) stochastic data, and (d) fuzzy numbers that cannot describe the real-world instance and only stand for a glance of the genuine world.

We recall that Cárdenas-Barrón [3] followed Grubbström and Erdem [2] to apply algebraic methods to solve the optimal solution for inventory models. Ronald et al. [4] pointed out that Cárdenas-Barrón [3] already knew the final result in advance, and then developed his algebraic approach which is elegant but beyond the imagination of ordinary researchers. Chang et al. [5] claimed that the solution procedure of Ronald et al. [4] was too complicated and then Chang et al. [5] provided a revised method. At the end of their paper, Chang et al. [5] proposed an open question for a minimum problem related to a square root of a quadratic polynomial by algebraic solutions. Lau et al. [11] tried to solve the open question raised by Chang et al. [5]. Chiu et al. [16] mentioned that the solution approach proposed by Lau et al. [11] contained severe doubtful derivations and then presented improvements. Luo and Chou [6] claimed that there are still suspicious findings in Chiu et al. [16] and then Luo and Chou [6] offered an algebraic method to solve the open question. Yen [1] observed the list of papers of Cárdenas-Barrón [3], Ronald et al. [4], Chang et al. [5], Lau et al. [11], Chiu et al. [16], and Luo and Chou [6] to assert that those previously published algebraic methods are too sophisticated to reflect the true meaning of the intuitive spirit of an algebraic method such that Yen [1] constructed a new procedure to answer the open question proposed by Chang et al. [5]. Moreover, at the end of Yen [1], he also raised another open question for future researchers. Our derivation shows the true spirit of the algebras to approximate the genuine world as much as possible. Based on the above discussion, we can say that algebraic approaches look simple but still contain many mathematical skills that are deserved to be investigated by future practitioners.

Up to now, verifying Equation (4.1) without referring to Equation (4.2) does not have an apparent real-world application. However, many scientific advances did not have immediate usefulness in their pioneering period. After many improvements by several generations, some theoretical results have their real-world application. We still believe that our results will be blooms and outgrowth in the future. Six related papers are worthy to mention: Othata, and Pochai [17], Ramirez-Juidias et al. [18], Meng et al. [19], Challita and Abdo [20], Zhao et al. [21], and Alaoui and Ettaouil [22] to help researchers realize the current research trend.

XII. Conclusion

We provide a positive answer to the open question proposed by Yen [1]. Our algebraic method will help researchers to realize the material discussed in Yen [1] concerning algebraic approaches. Moreover, we study another open question proposed by Osler [8] to obtain a pure algebraic method to derive the desired identity that satisfies the restriction proposed by Osler [8, 13]. We also show that comparisons among minimum values of several approximated inventory models in Çalışkan [9] do not provide meaningful results. The revised method should be plugged into the original model to compare those values to decide the best-approximated inventory model. Last, but not least, we point out questionable findings in Çalışkan [10] to show that "convex property" is not enough for the open question proposed by Lau et al. [11]. Our solution approach will help researchers in the future to solve operational research problems with algebraic skills.

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