Stochastic Differential Equation With Fuzzy Coefficients

Sumitra Panda, Jayanta K. Dash* and Golak B. Panda

Abstract—Stochastic differential equation is an efficient tool to handle the uncertain systems. But in case of hybrid uncertain systems i.e. randomness and fuzziness. Stochastic differential equation is not sufficient. To handle such situation, so we extend this tool in fuzzy environment. In this paper, different types fuzzy stochastic differential equations are discussed. Here stochastic process, Brownian motion and coefficients of such equation are taken as fuzzy random variable. The existence and uniqueness of fuzzy ssochastic differential equationt are discussed using Gronwall's inequality, Lipschitz condition linear growth condition. Moreover, linear fuzzy stochastic differential equation is derived and also provide pertinent illustrative examples of fuzzy stochastic differential equation and linear fuzzy stochastic differential equation.

Index Terms— Fuzzy Brownian motion, Lipschitz condition, Fuzzy Ito integral, Fuzzy stochastic process, Fuzzy Ito lemma, α -cut of a fuzzy number, Linear growth condition.

I. INTRODUCTION

Differential equation is a mathematical model to solve many real-world problems. In most of the cases, the coefficients of the DEs are random. In such cases, stochastic differential equation(SDE) will arise. Stochastic Calculus is one of the branches of mathematics which is a generalization of various disciplines such as Mathematical finance, Engineering, Physics, Biology, etc. In 1946 Ito [1] introduced Ito integral which plays an important role to solve SDE. Then many researchers developed different techniques to solve SDE.

Bensoussan et. al [2] discussed the approximation solution of some SDEs of parabolic type using the splitting up method.

Klebaner[3] provides a concise presentation of stochastic calculus, Finance, Engineering, and Science. Bass[4] demonostrate the weak existence and weak uniqueness of SDEs, where the path wise uniqueness and strong solution does not exist in SDEs.

Manuscript received April 19, 2022; revised October 30, 2022.

Sumitra Panda (main author) is a P.hD. scholar of Siksha O Anusandhan University, Khandagiri Square, Bhubaneswar, India-751030,
e-mail: sumitrapanda400@gmail.com.

Jayanta Kumar Dash (corresponding author) is a professor in Siksha O Anusandhan University, Khandagiri Square, Bhubaneswar, India-751030, e-mail: jkdash@gmail.com.

Golak Bihari Panda is an assistant professor in Siksha O Anusandhan University, Khandagiri Square Bhubaneswar, India-751030,
e-mail: golakmath@gmail.com.

Some analytic solutions for SDEs related to the martingale process were proposed by Farnoosh et. al [5]. Halidas and Stamatiou [6] proposed the numerical solution of nonlinear SDE with the help of semi discreet method. A scalar SDE with periodic coefficients is discussed by Boudref and Berboucha [7]. The application of SDEs in different areas were discussed in [8, 9]. Corwin and Shen [10] studied on singular stochastic partial differential equation. Qi [11] studies the existence and uniqueness of the SDEs using Ito formula and Lipschitz condition. Bibi and Merahi [27] determmine the new formula for linear stochastic differential equation(LSDE) utilizing the Adomian decomposition technique.

After the introduction of fuzzy set theory by Zadeh [12], several researchers have developed fuzzy logic in different areas of science, engineering, and management. A fuzzy random variable(FRV) was introduced by Zadeh [13]. After that Puri, Ralescu [14], and other researchers developed FRV. Puri [15] discussed the convergence theorem for fuzzy martingale. Buckley [16] developed fuzzy probability theory using fuzzy numbers as parameters in the probability density function and probability mass function. This idea is more helpful according to a computational perspectives. Panda and Dash [17] discussed the nonlinear fuzzy chance constraint programming problem(PP), where coefficient in the objective function and constraints are FRVs.

Acharya et. al [18] solved the multi objective fuzzy probabilistic PP. A new approach to fuzzy integral equations, the existence and uniqueness of solutions to the FSDEs driven by BM are discussed by Malinowski and Michta [19, 20]. The positive and negative fuzzy number is discussed by Hadi [21]. Li et. al [22] discuss fuzzy valued Gaussian processes(G.P) and martingales with continuous parameters. FII driven by an fuzzy Brownian motion(FBM) was discussed by Seya et. al [23]. Additionaly, they construct the fuzzy stochastic integral equation. Kim [24] derived the existence and uniqueness of FSDE using the Lipschitz condition. Stochastic finance problem was derived by Dash et. al [25]. Malinowski [26] discuss the solutions to the symmetric type of FSDEs using the Global Lipschitz condition. Feng [28] discussed the solution of Linear FSDE(LFSDE). Panda et. al [29] discuss the properties of fuzzy Ito integral(FII) with respect to FBM.

Here using Buckley's [16] concept of fuzzy probabilities, the existence and uniqueness of FSDEs and LFSDEs are derived. Due to the presence of randomness and fuzziness, the stochastic process(SP) ψ_t,Brownian motion(BM) $\mathcal{B}_t$, drift term $\tau(\psi_t, t)$, and diffusion term

$\upsilon(\psi_t, t)$ are taken as FRV, which are denoted as $\tilde{\psi}_t, \tilde{\mathcal{B}}(t), \tilde{\tau}(\tilde{\psi}_t, t), \tilde{\upsilon}(\tilde{\psi}_t, t)$. In LFSDEs, the coefficients are taken as FRV.

The present work is organized as follows. Section 2 contains the prerequisites of SDE and FSDE. Types of FSDE, the existence, the uniqueness of FSDE and its solution of FSDE have been discussed in section 3. Linear fuzzy stochastic differential equation(LFSDE) and example are derived in section 4. Finally conclusion and concluding remarks are given in section-5 using some references.

II. BASIC PRELIMINARIES

A. Stochastic Differential Equation(SDE) [3]

Stochastic Differential Equation(SDE) is a differential equation in which one or more term is stochastic and the resulting solution is also stochastic, this equation is called stochastic differential equation.

$$d\psi_t = \tau(\psi_t, t)dt + \upsilon(\psi_t, t)d\mathcal{B}_t \qquad (1)$$

Where the coefficient $\tau(\psi_t, t)$ and $\upsilon(\psi_t, t)$are drift and diffusion coefficients.

B. Brownian motion(BM) [3]

Brownian Motion(BM) $\mathcal{B}_t$ is a stochastic process satisfy the following property:-

(i) BM satisfies independence increment property i.e. If $t > s$, then $\mathcal{B}_t - \mathcal{B}_s$ is independent of past.
(ii) $\mathcal{B}_t$ satisfies normal increment property i.e. $\mathcal{B}_t - \mathcal{B}_s$ has a normal distribution with mean zero and variance t-s.
(iii) $\mathcal{B}_t$ is continuous function of t.

C. Fuzzy Brownian motion [23]

A Fuzzy stochastic process $\{\tilde{\mathcal{B}}_t, t \in [0, T], 0 < T < \infty\}$ is called a fuzzy Brownian motion on probability space (Ω, A, P) if and only if $\forall \alpha \in [0,1]$, the process

$$\tilde{\mathcal{B}}_t[\alpha] = [\,\tilde{\mathcal{B}}_t^L[\alpha], \tilde{\mathcal{B}}_t^U[\alpha]\,]$$

is an interval valued Brownian motion on (Ω, A, P) and

$$\tilde{\mathcal{B}}_t[\alpha] = \bigcup_{\alpha \in [0,1]} \tilde{\mathcal{B}}_t[\alpha].$$

D. White Noise [3]

White noise is defined as the informal nonexistent derivative of BM i.e. $\dot{\mathcal{B}}_t$. K. Ito used the white noise as a random noise which is independent of different times.

E. Fuzzy Ito formula [29]

Let $\tilde{\psi}_t$ be a fuzzy Ito process(FIP) satisfying the FSDE

$$d\tilde{\psi}_t = \tilde{\mu}(t, \tilde{\psi}_t)dt \oplus (\tilde{\sigma}(t, \tilde{\psi}_t) \otimes d\tilde{\mathcal{B}}_t),$$

and $\tilde{f}(\tilde{\psi}_t, t)$ be a twice- differentiable function, then $\tilde{f}(\tilde{\psi}_t, t)$ is a FPP.

Fuzzy Ito lemma is defined as

$$d(\tilde{f}(\tilde{\psi}_t, t)) = \Big(\frac{\delta \tilde{f}}{\delta t}(\tilde{\psi}_t, t)dt\Big) \oplus \Big(\frac{\delta \tilde{f}}{\delta \tilde{\psi}_t} d\tilde{\psi}_t \oplus \frac{1}{2}\frac{\delta^2 \tilde{f}}{\delta \tilde{\psi}_t^2}\tilde{\sigma}^2 \tilde{\psi}_t^2 dt\Big).$$

F. Positive fuzzy number(FN) [21]

A fuzzy number $\tilde{A}$ on $\mathbb{R}$ is called positive FN if $\tilde{A} \succ 0$ and it is the membership function $\mu_{\tilde{A}}(x)$ satisfies $\mu_{\tilde{A}}(x) = 0 \forall x \leq 0$.

G. α-cut of an FN [23]

Let $\tilde{u} \in F(\mathbb{R})$ be a fuzzy number, the α-cut of $\tilde{u}$, for every $\alpha \in [0,1]$ is the set,

$$[\tilde{u}][\alpha] = \{x \in \mathbb{R} : \tilde{u}(x) \geqslant \alpha\}$$

$$= [u^L[\alpha], u^U[\alpha]](Say)$$

$$where\, [\tilde{u}]^L[\alpha] = \inf_{x \in \mathbb{R}} \{x \in [\tilde{u}][\alpha]\}$$

$$and\, [\tilde{u}]^U[\alpha] = \sup_{x \in \mathbb{R}} \{x \in [\tilde{u}][\alpha]\},$$

the support of $\tilde{u}$ is given by

$$[\tilde{u}][0] = Support(\tilde{u}) = \{x \in \mathbb{R} : \tilde{u}(x) \succ 0\}.$$

H. Fuzzy arithmetic operations [18]

Arithmetic operation of fuzzy numbers are defined using arithmetic operations of their α-cuts. Let $\tilde{a}, \tilde{b}$ be the two fuzzy numbers whose α-cuts are

$$\tilde{a}[\alpha] = [a^L[\alpha], a^U[\alpha]], \qquad \tilde{b}[\alpha] = [b^L[\alpha], b^U[\alpha]].$$

Let $\oplus$, $\ominus$, $\odot$, and $\oslash$ denote addition, subtraction, multiplication, and division of fuzzy numbers.

- *Fuzzy addition:-*

$$(\tilde{a} \oplus \tilde{b})[\alpha] = [a^L[\alpha] + b^L[\alpha], a^U[\alpha] + b^U[\alpha]]$$

- *Fuzzy subtraction:-*

$$(\tilde{a} \ominus \tilde{b})[\alpha] = [a^L[\alpha] - b^L[\alpha], a^U[\alpha] - b^U[\alpha]].$$

- *Fuzzy multiplication:-*

$$(\tilde{a} \odot \tilde{b})[\alpha] = \tilde{a}[\alpha] \cdot \tilde{b}[\alpha]$$

$$= [\,Min\{a^L[\alpha]b^L[\alpha], a^L[\alpha]b^U[\alpha], [\alpha]b^L[\alpha], a^U[\alpha]b^U[\alpha]\},$$

$$Max\{a^L[\alpha]b^L[\alpha], a^L[\alpha]b^U[\alpha][\alpha]b^L[\alpha], a^U[\alpha]b^U[\alpha]\}\,].$$

•*Fuzzy division:-*

$(\tilde{a}/\tilde{b})[\alpha] = \tilde{a}[\alpha]/\tilde{b}[\alpha = [a^L[\alpha], a^U[\alpha]] \cdot [1/b^U[\alpha], 1/b^L[\alpha]],$

$provided\, 0 \notin \tilde{b}[\alpha].$

I. Gronwall's Inequality [3]

Let ϕ and f be nonnegative, continuous functions defined for $0 \leq t \leq T$, and let $C_0 \leq 0$ denote a constant. If

$$\phi \leq C_0 + \int_0^t f\phi ds, 0 \leq t \leq T,$$

then

$$\phi \leq C_0 e^{\int_0^t f ds}, 0 \leq t \leq T. \tag{3}$$

J. Langevin equation [3]

Langevin equation is an SDE to describe the time evaluation of a stochastic process like BM, Markov process, etc.

$$\frac{d\psi_t}{dt} = -b\psi_t + \sigma\xi.$$

$$d\psi_t = -b\psi_t dt + \sigma d\mathcal{B}_t \tag{4}$$

K . .Lemma: [28]

Let $(\tilde{\mathcal{Y}}_n)_t$ be a sequence of nonnegative fuzzy functions such that $(\tilde{\mathcal{Y}}_{\tilde{0}})_t = \tilde{\mathcal{C}}$ is fuzzy number and for +ve fuzzy number $\tilde{\mathcal{B}}$

$$(\tilde{\mathcal{Y}}_{n+1})_t \preccurlyeq \tilde{\mathcal{B}} \int_0^t (\tilde{\mathcal{Y}}_n)_s ds < \infty, \forall t \leq T$$

then

$$(\tilde{\mathcal{Y}}_n)_t \preccurlyeq \tilde{\mathcal{C}}\tilde{\mathcal{B}}^n \frac{t^n}{n!}, \forall t \leq T. \tag{5}$$

L. Lipschitz condition [3]

A stochastic process $\psi(t)$ satisfies Lipschitz condition if

$$|\psi_x - \psi_y| \leq K|x - y|.$$

M. Linear growth condition [3]

An SP ψ_t satisfy linear growth condition if $\psi_t \leq K(1 + |x|), K = constant.$

N. Linear Stochastic Differential equation [5]

The SDE can be expressed as $d\psi_t(t) = (P(t) + Q(t)\psi_t)dt + (R(t) + S(t)\psi_t)$ where P(t), Q(t), R(t) and S(t) are adapted processes and continuous function of t.

III. Fuzzy Stochastic Differential Equation(FSDE)

An equation is of the form

$$d\psi_t = \tau(\psi_t, t)dt \oplus (\upsilon(\psi_t, t) \otimes d\mathcal{B}_t) \tag{6}$$

is said to be an SDE where ψ_t is an unknown stochastic process, $\tau(\psi_t, t)$ and $\upsilon(\psi_t, t)$ are known, $\mathcal{B}_t$ is a BM process. When $\tilde{\mathcal{B}}_t$ is an FBM and $\tilde{\psi}_t$ is FSP with fuzzy drift $\tilde{\tau}(\tilde{\psi}_t, t)$ and fuzzy diffusion $\tilde{\upsilon}(\tilde{\psi}_t, t)$ is called FSDE.

For each $\alpha \in [0,1]$,

$$d\tilde{\psi}_t[\alpha] = \tilde{\tau}(\psi_t, t)[\alpha]dt + \tilde{\upsilon}(\psi_t, t)[\alpha]d\tilde{\mathcal{B}}_t[\alpha]$$

is α-cut of FSDE.

$\tilde{\psi}_t[\alpha] = \left[\tilde{\psi}_t^L[\alpha], \tilde{\psi}_t^U[\alpha]\right]$, and $\tilde{\mathcal{B}}[\alpha] = [\tilde{\mathcal{B}}_t^L[\alpha], \tilde{\mathcal{B}}_t^U[\alpha]]$

are the interval valued FSP and FBM respectively. In a particular case, let the fuzzy stochastic process $\tilde{\psi}_t$ be invested in a saving account in time t, $t \in T$, and $\tilde{R}$ be the risky free fuzzy interest rate, which is uncertain fuzzy interest rate, so $\tilde{R}$ can be perturbed by random fuzzy noise $\tilde{\xi}_t$ i.e

$$\tilde{R} = \tilde{R} \oplus (\tilde{\upsilon} \otimes \tilde{\xi}_t).$$

Then the fuzzy stochastic differential equation

$$d\tilde{\psi}_t = \tilde{R}\tilde{\psi}_t dt \oplus \tilde{\upsilon}\tilde{\psi}_t d\tilde{\mathcal{B}}_t, (\tilde{\psi}_0 = \tilde{1}) \tag{7}$$

where $\tilde{\upsilon}$ be the intensity of noise.

$Let \vartriangle \psi = \{\psi_t | \psi_t \in \tilde{\psi}_t[\alpha]\}, \vartriangle \upsilon = \{\upsilon | \upsilon \in \tilde{\upsilon}[\alpha]\},$

$\vartriangle R = \{R | R \in \tilde{R}[\alpha]\}, \vartriangle \mathcal{B} = \{\mathcal{B}_t | \mathcal{B}_t \in \tilde{\mathcal{B}}_t[\alpha]\}$

$\vartriangle \mathcal{Y} = \{\mathcal{Y}_t | \mathcal{Y}_t \in \tilde{\mathcal{Y}}_t[\alpha]\}, \quad \vartriangle \xi = \{\xi | \xi \in \tilde{\xi}[\alpha]\},$

$\vartriangle \tau = \{\tau | \tau \in \tilde{\tau}[\alpha]\}$

Then for each $\alpha \in [0,1]$,

$$\{d\psi_t | \vartriangle \psi\} = \{R\psi_t dt + \upsilon\psi_t d\mathcal{B}_t | \vartriangle \psi, \vartriangle R, \vartriangle \upsilon, \vartriangle \mathcal{B}\} \tag{8}$$

is the α-cut of equation (7) with initial condition $\tilde{\psi}(\tilde{0}) = \tilde{1}$.

Case-i $\tilde{\upsilon} = 0$, i.e. no noise. The solution of equation (8) can be obtained by separating variable method.

As $\tilde{\upsilon} = \tilde{0}$, takes the following term,

$$\{d\psi_t | \vartriangle \psi\} = \{R\psi_t dt | \vartriangle R, \vartriangle \psi\}$$

$$\Rightarrow \{\frac{d\psi_t}{\psi_t} | \vartriangle \psi\} = \{Rdt | \vartriangle R\}$$

$$\Rightarrow \{log\psi_t | \vartriangle \psi\} = \{Rt + logC | \vartriangle R, \vartriangle C\}$$

$$(integrating\ both\ sides)$$

$$\{\psi_t | \vartriangle \psi\} = \{Ce^{Rt} | \vartriangle C, \vartriangle R\} \quad (9)$$

Case-ii $\tilde{v} \neq 0$,

$Put\ f(\tilde{\psi}_t) = ln\tilde{\psi}_t$

$$f^{'}(\tilde{\psi}_t) = \frac{1}{\tilde{\psi}_t}, f^{''}(\tilde{\psi}_t) = \frac{1}{\tilde{\psi}_t^2}$$

According to fuzzy Ito formula, the α-cut of $d(ln\tilde{\psi}_t)$ becomes

$$\{d(ln\psi_t) | \vartriangle \psi\} = \{\frac{1}{\psi_t} d\psi_t + \frac{1}{2}(-\frac{1}{\psi_t^2})v^2\psi_t^2 dt$$

$$| \vartriangle \psi, \vartriangle v\}$$

$$= \{\frac{1}{\psi_t}(R\psi_t dt + v\psi_t d\mathcal{B}_t) - \frac{1}{2}v^2 dt$$

$$| \vartriangle R, \vartriangle \psi, \vartriangle \mathcal{B}, \vartriangle v\}$$

$$= \{Rdt + vd\mathcal{B}_t - \frac{1}{2}v^2 dt | \vartriangle R, \vartriangle \mathcal{B}, \vartriangle v\}$$

$$\{d(ln(\psi_t)) | \vartriangle \psi\} = \{(R - \tfrac{1}{2}v^2)dt + vd\mathcal{B}_t$$

$$| \vartriangle R, \vartriangle \mathcal{B}, \vartriangle v\}$$

Integrating two sides from 0 to t the α-cut becomes

$$\{ln\psi_t - ln\psi_0 | \vartriangle \psi\} = \{(R - \frac{1}{2}v^2)t + v\mathcal{B}_t$$
$$| \vartriangle R, \vartriangle v, \vartriangle \mathcal{B}\}$$

$$\{ln\psi_t | \vartriangle \psi\} = \{ln\psi_0 + (R - \frac{1}{2}v^2)t +$$

$$v\mathcal{B}_t | \vartriangle R, \vartriangle v, \vartriangle \mathcal{B}_1, \vartriangle \psi\}$$

$$\{\psi_t | \vartriangle \psi\} = \{\psi_0 e^{(R-\frac{1}{2}v^2)t + v\mathcal{B}_t} | \vartriangle R, \vartriangle v,$$

$$\vartriangle \mathcal{B}_1, \vartriangle \psi\}.$$

$$\{\psi_t | \vartriangle \psi\} = \{e^{(R-\frac{1}{2}v^2)t + v\mathcal{B}_t} | \vartriangle R, \vartriangle v, \vartriangle \mathcal{B}_1\} \quad (10)$$

EXAMPLE 3.1 : Find the solution of FSDE

$$d\tilde{\psi}_t = dt \oplus (\tilde{\mathcal{B}}_t \otimes d\tilde{\mathcal{B}}_t), \tilde{\psi}_0 = \tilde{1}$$

SOLUTION : Given FSDE is

$$d\tilde{\psi}_t = d\tilde{t} \oplus (\tilde{\mathcal{B}}_t \otimes d\tilde{\mathcal{B}}_t), \tilde{\psi}_0 = \tilde{1} \quad (11)$$

The α-cut of FSDE $d\tilde{\psi}_t$ is

$$d\tilde{\psi}_t[\alpha] = \{d\psi_t | \vartriangle \psi_1\}$$

$$\{d\psi_t | \vartriangle \psi_1\} = \{dt + \mathcal{B}_t d\mathcal{B}_t | \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T\}$$

Integrating two sides from 0 to t, the α-cut becomes

$$\{\int_0^t d\psi_t | \vartriangle \psi_1\} = \{\int_0^t dt + \int_0^t \mathcal{B}_t d\mathcal{B}_t | \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T\}$$

$$\{\psi_t - \psi_0 | \vartriangle \psi_1\} = \{t + \frac{\mathcal{B}_t^2}{2} - \frac{t}{2} | \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T\}$$

$$\{\psi_t - 1 | \vartriangle \psi_1, \vartriangle 1\} = \{t + \frac{\mathcal{B}_t^2}{2} - \frac{t}{2} | \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T\}$$

$$\{\psi_t | \vartriangle \psi_1\} = \{1 + t + \frac{\mathcal{B}_t^2}{2} - \frac{t}{2} | \vartriangle \mathcal{B}_1, \vartriangle \psi_1, \vartriangle T, \vartriangle 1\}$$

$$= \{1 + \frac{t}{2} + \frac{\mathcal{B}_t^2}{2} | \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T, \vartriangle 1\},$$

$$(\vartriangle 1 = \{1 | 1 \in \tilde{1}[\alpha]\})$$

$$Hence\ \tilde{\psi}_t = \tilde{1} \oplus \frac{\tilde{t}}{2} \oplus \frac{\tilde{\mathcal{B}}_t^2}{2}.$$

EXAMPLE 3.2 : Find the solution of FSDE

$$d\tilde{\psi}_t = (\tilde{\mathcal{B}}_t - 1)dt + \tilde{\mathcal{B}}_t^2 d\tilde{\mathcal{B}}_t, \tilde{\psi}_0 = \tilde{1}$$

SOLUTION : Given FSDE is

$$d\tilde{\psi}_t = (\tilde{\mathcal{B}}_t - 1)dt + \tilde{\mathcal{B}}_t^2 d\tilde{\mathcal{B}}_t, \tilde{\psi}_0 = \tilde{1}$$

The α-cut of FSDE $d\tilde{\psi}_t$ is

$$d\tilde{\psi}_t[\alpha] = \{d\psi_t | \vartriangle \psi_1\}$$

$$= \{(\mathcal{B}_t - 1)dt + \mathcal{B}_t^2 d\mathcal{B}_t | \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T\}$$

Integrating two sides from 0 to t, the α-cut becomes

$$\{\int_0^t d\psi_t | \vartriangle \psi_1\} = \{\int_0^t (\mathcal{B}_t - 1)dt + \mathcal{B}_t^2 d\mathcal{B}_t$$
$$| \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T\}$$

$$\Rightarrow \{\psi_t - \psi_0 | \vartriangle \psi_1\} = \{\textstyle\int_0^t \mathcal{B}_t dt - \int_0^t dt$$

$$+ \textstyle\int_0^t \mathcal{B}_t^2 d\mathcal{B}_t | \vartriangle \mathcal{B}, \vartriangle \psi_1\}$$

$$\Rightarrow \{\psi_t - 1 | \vartriangle \psi_1, \vartriangle 1\} = \{\textstyle\int_0^t \mathcal{B}_t dt - t + \frac{\mathcal{B}_t^3}{3}$$

$$-\frac{\mathcal{B}_t^2}{2} - \frac{t}{2} | \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T\}$$

$$\Rightarrow \{\psi_t - 1 | \vartriangle \psi_1, \vartriangle 1\} = \{\mathcal{Z}_t - t + \frac{\mathcal{B}_t^3}{3} - \frac{\mathcal{B}_t^2}{2} - \frac{t}{2}$$

$$| \vartriangle \mathcal{B}, \vartriangle \psi_1, \vartriangle T\}$$

$$(Where\ \tilde{\mathcal{Z}}_t = \textstyle\int_0^t \tilde{\mathcal{B}}_t dt)$$

$$\Rightarrow \{\psi_t \mid \triangle\,\psi_1\} = \{1 + \mathcal{Z}_t + \frac{\mathcal{B}_t^3}{3} - \frac{\mathcal{B}_t^2}{2} - \frac{3t}{2}$$

$$\mid \triangle\,\mathcal{B},\triangle\,\psi_1,\triangle\,T,\triangle\,1\}$$

$(where\ \tilde{\mathcal{Z}}_t = \int_0^t \tilde{\mathcal{B}}_t dt)$

A. Fuzzy Langevin SDE

Time evaluation of fuzzy stochastic process can be described by fuzzy Langevin SDE, which can be defined as

$$\{d\psi_t \mid \triangle\,\psi\} = \{-a\psi_t dt + v d\mathcal{B}_t$$

$$\mid \triangle\,\psi,\triangle\,v,\triangle\,\mathcal{B},\triangle\,a\} \quad (12)$$

where $\tilde{a}$ be a +ve fuzzy number and

$\triangle\,a = \{a \mid a \in \tilde{a}[\alpha]\}$

Case-i $\tilde{v} = \tilde{0}$ i.e. no noise, the α-cut of the solution (12) becomes

$\{d\psi_t \mid \triangle\,\psi\} = \{-a\psi_t \mid \triangle\,\psi\}$

$$\{\psi_t \mid \triangle\,\psi\} = \{\psi_0 e^{-at} \mid \triangle\,\psi\}$$

Case-ii $\tilde{v} \neq \tilde{0}$

$$Let\ \mathcal{Y}_t = \tilde{\psi}_t \otimes e^{at}$$

$$\{\mathcal{Y}_t \mid \triangle\,\mathcal{Y}\} = \{\psi_t e^{at} \mid \triangle\,\psi\}$$

$$\{d\mathcal{Y}_t \mid \triangle\,\mathcal{Y}_t\} = \{e^{at} d\psi_t + \psi_t e^{at} a dt \mid \triangle\,\psi\}$$

$$= \{e^{at}(-a\psi_t dt + v d\mathcal{B}_t) + \psi_t e^{at} a dt$$

$$\mid \triangle\,\psi,\triangle\,\mathcal{B},\triangle\,v\}$$

$$= \{-a e^{at}\psi_t dt + e^{at} v d\mathcal{B}_t + a\psi_t e^{at} dt$$

$$\mid \triangle\,\psi,\triangle\,\mathcal{B},\triangle\,v\}$$

$$\{d\mathcal{Y}_t \mid \triangle\,\mathcal{Y}\} = \{v e^{at} d\mathcal{B}_t \mid \triangle\,\mathcal{B},\triangle\,v\} \qquad (13)$$

Integrating both sides from 0 to 1, the α-cut of becomes

$$\{\mathcal{Y}_t \triangle\,\mathcal{Y}\} = \{\mathcal{Y}_0 + \int_0^t v e^{-at} d\mathcal{B}_s \mid \triangle\,v,\triangle\,\mathcal{B},\triangle\,v\}$$

$$So, \{\psi_t \mid \triangle\,\psi\} = \{\mathcal{Y}_t e^{-at} \mid \triangle\,\mathcal{Y}\}$$

$$\{\psi_t \mid \triangle\,\psi\} = \{e^{-at}(\mathcal{Y}_0 + \int_0^t v e^{as} d\mathcal{B}_s)$$

$$\mid \triangle\,\mathcal{B},\triangle\,v\}$$

$$\{\psi_t \mid \triangle\,\psi\} = \{e^{-at}\psi_0 + e^{-at}\int_0^t v e^{as} d\mathcal{B}_s \mid \triangle\,\psi,\triangle\,v,$$

$$= \{e^{-at}\psi_0 + e^{-at}([v e^{as}\mathcal{B}_t - \int_0^t v\mathcal{B}(s) e^{as} a ds) \mid \triangle\,v,\triangle\,\mathcal{B}\}$$

$$= \{e^{-at}\psi_0 + v e^{-at}\mathcal{B}_t e^{at} - e^{-at}\int_0^t v a e^{as}\mathcal{B}_s ds \mid \triangle\,v,\triangle\,\mathcal{B}\}$$

$$= \left\{e^{-at}\psi_0 + v e^{-at}\mathcal{B}_t e^{at} - e^{-at}\int_0^t v a e^{as}\mathcal{B}_s ds \right| \triangle\,\mathcal{B},\triangle\,v\Big\}$$

$$= \{e^{-at}\psi_0 + v\mathcal{B}_t - va\int_0^t e^{-at+vs}\mathcal{B}_s ds \mid \triangle\,\mathcal{B},\triangle\,\psi\}$$

By assuming different values of $\alpha \in [0,1]$, the solution of equation $\tilde{\psi}_t$ is

$$\{\tilde{\psi}_t \mid \triangle\,\psi\} = \{e^{-at}\psi_0 + v\mathcal{B}_t - va$$

$$\int_0^t e^{-at+vs}\mathcal{B}_s ds \mid \triangle\,\mathcal{B},\triangle\,\psi\}$$

B. Grown-wall's Inequality for fuzzy function

For the existence and uniqueness of FSDE, it is required to prove Grown-wall's inequality, in a fuzzy sense.

Let $\tilde{\mathcal{H}}(t)$ and $\tilde{\mathcal{J}}(t)$ are two positive continuous fuzzy functions that satisfy

$$\tilde{\mathcal{H}}(t) \preccurlyeq \tilde{\lambda} \oplus \int_{t_0}^t \tilde{\mathcal{J}}(s)\tilde{\mathcal{H}}(s) ds,\ t \geq t_0,$$

then there exist a fuzzy number $\tilde{\lambda} \succ \tilde{0}$ such that

$$\tilde{\mathcal{H}}(t) \preccurlyeq \tilde{\lambda} \otimes e^{\int_{t_0}^t \tilde{\mathcal{J}}(s) ds}.$$

Proof. Let $\triangle\,\mathcal{H} = \{\mathcal{H}(t) \mid \mathcal{H}(t) \in \tilde{\mathcal{H}}(t)[\alpha]\}$,

$$\triangle\,\mathcal{J} = \{\mathcal{J}(t) \mid \mathcal{J}(t) \in \tilde{\mathcal{J}}(t)[\alpha]\}.$$

For each $\alpha \in [0,1], \tilde{\mathcal{H}}(t)[\alpha]\ and \tilde{\mathcal{J}}(t)[\alpha]$ are α-cuts of fuzzy continuous functions.

$$\tilde{\mathcal{H}}(t)[\alpha] = [\tilde{\mathcal{H}}(t)^L[\alpha], \tilde{\mathcal{H}}(t)^U[\alpha]],$$

$$\tilde{\mathcal{J}}(t)[\alpha] = [\tilde{\mathcal{J}}(t)^L[\alpha], \tilde{\mathcal{J}}(t)^U[\alpha]]$$

are interval-valued continuos fuzzy function.

Given $\tilde{\mathcal{H}}(t)$ and $\tilde{\mathcal{J}}(t)$ satisfies the inequality

$$\tilde{\mathcal{H}}(t) \preccurlyeq \tilde{\lambda} \oplus \int_{t_0}^t \tilde{\mathcal{J}}(s)\tilde{\mathcal{H}}(s) ds \qquad (14)$$

For each $\alpha \in [0,1]$, the α-cuts becomes,

$$\{\mathcal{H}(t) \mid \triangle\,\mathcal{H}\} \preccurlyeq \{\lambda + \int_{t_0}^t \mathcal{J}(s)\mathcal{H}(s) ds$$

$$\mid \triangle\,\mathcal{H},\triangle\,\mathcal{J},\triangle\,\lambda\} \qquad (15)$$

$$\Rightarrow \{\frac{\mathcal{H}(t)}{\lambda + \int_{t_0}^t \mathcal{J}(s)\mathcal{H}(s) ds} \mid \triangle\,\mathcal{H},\triangle\,\mathcal{J},\triangle\,\lambda\} \preccurlyeq \{1 \mid 1 \in \tilde{1}[\alpha]\}$$

Multiplying $\tilde{\mathcal{J}}(t)[\alpha]$ both sides

$$\{\frac{\mathcal{H}(t)\mathcal{J}(t)}{\lambda+\int_{t_0}^{t}\mathcal{J}(s)\mathcal{H}(s)ds} \mid \vartriangle\mathcal{H},\vartriangle\mathcal{J},\vartriangle\lambda\} \preccurlyeq \{\mathcal{J}(t) \mid \vartriangle\mathcal{J}\} \qquad (16)$$

Integrating both sides from t_0 to t, the α-cut of equation(16) becomes

$$\{log\frac{\lambda+\int_{t_0}^{t}\mathcal{J}(s)\mathcal{H}(s)ds}{\lambda} \mid \vartriangle\mathcal{J},\vartriangle\mathcal{H},\vartriangle\lambda\}$$

$$\preccurlyeq \left\{\int_{t_0}^{t} \mathcal{J}(s)\,ds \middle| \vartriangle\mathcal{J},\vartriangle\mathcal{J}\right\}$$

$$\Rightarrow \{\frac{\lambda+\int_{t_0}^{t}\mathcal{J}(s)\mathcal{H}(s)ds}{\lambda} \mid \vartriangle\mathcal{J},\vartriangle\mathcal{H},\vartriangle\lambda\}$$

$$\preccurlyeq \{e^{\int_{t_0}^{t}\mathcal{J}(s)ds} \mid \vartriangle\mathcal{J}\}$$

$$\Rightarrow \{\lambda+\int_{t_0}^{t}\mathcal{J}(s)\mathcal{H}(s)ds \mid \vartriangle\lambda,\vartriangle\mathcal{H},\vartriangle\mathcal{J}\}$$

$$\preccurlyeq \{\lambda e^{\int_{t_0}{}^{t}\mathcal{J}(s)ds} \mid \vartriangle\mathcal{J},\vartriangle\lambda\} \qquad (17)$$

Hence $\{\mathcal{H}(t) \mid \vartriangle\mathcal{H}\} \preccurlyeq \{\lambda e^{\int_{t_0}^{t}\mathcal{J}(s)ds} \mid \vartriangle\lambda,\vartriangle\mathcal{J},\vartriangle s\}$

C. *The existence and uniqueness of FSDE*

Consider the fuzzy stochastic initial value problem

$$d\tilde{\psi}_t = \tilde{\tau}(\tilde{\psi}_t,t)dt \oplus \left(\tilde{\upsilon}(\tilde{\psi}_t,t) \otimes d\tilde{\mathcal{B}}_t\right),$$

$$\tilde{\psi}(\tilde{0}) = \tilde{\psi}_{\tilde{0}}. \qquad (18)$$

Assume that the coefficients fuzzy drift $\tilde{\tau}(\tilde{\psi}_t,t)$ and fuzzy diffusion $\tilde{\upsilon}(\tilde{\psi}_t,t)$ are uniformly continuous and satisfies the Lipschitz condition i.e.

$$\left|\tilde{\tau}(\tilde{\psi}_t,t) \ominus \tilde{\tau}(\tilde{\mathcal{Y}}_t,t)\right| \preccurlyeq \tilde{\lambda}\left|\tilde{\psi}_t \ominus \tilde{\mathcal{Y}}_t\right| \text{ and}$$

$$|\tilde{\upsilon}(\tilde{\psi}_t,t) \ominus \tilde{\upsilon}(\tilde{\lambda}(t),t)| \preccurlyeq \tilde{k}|\tilde{\psi}_t \ominus \tilde{\mathcal{Y}}_t|. \qquad (19)$$

where $\tilde{\lambda}$ and $\tilde{k}$ are fuzzy numbers. Also both coefficients $\tilde{\tau}(\tilde{\psi}_t,t)$ and $\tilde{\upsilon}(\tilde{\psi}_t,t)$ satisfy the linear growth condition.

$$|\tilde{\tau}(\tilde{\psi}_t,t)| \ominus |\tilde{\upsilon}(\tilde{\psi}_t,t)| \preccurlyeq \tilde{\lambda}(1+\tilde{\psi}_t) \qquad (20)$$

THEOREM 3.1 : Every fuzzy stochastic initial value problem of the form (18) with Lipschitz conditions (19), and linear growth condition (20) have a unique solution.

Proof. Let $\vartriangle\psi = \{\psi_t | \psi_t \in \tilde{\psi}_t[\alpha]\}$,

$\vartriangle\mathcal{Y} = \{\mathcal{Y}_t | \mathcal{Y}_t \in \tilde{\mathcal{Y}}_t[\alpha]\}$ $\vartriangle\mathcal{B} = \{\mathcal{B}_t | \mathcal{B}_t \in \tilde{\mathcal{B}}_t[\alpha]\}$,

$\vartriangle\upsilon = \{\upsilon | \upsilon \in \tilde{\upsilon}[\alpha]\}, \vartriangle\tau = \{\tau | \tau \in \tilde{\tau}[\alpha]\}$

For each $\alpha \in [0,1]$, the α-cuts of FSDE (18), Lipschitz condition (19) and linear growth condition (20) is

$$\{d\psi_t \mid \vartriangle\psi\} = \{\tau(\psi_t,t)dt + \upsilon(\psi_t,t)\,d\mathcal{B}_t \mid \vartriangle\psi,\vartriangle\mathcal{B}\},$$

$$\{|\upsilon(\psi_t,t) - \upsilon(\mathcal{Y}_t,t)|| \vartriangle\psi,\vartriangle\mathcal{Y}\} \preccurlyeq \{\lambda|\psi_t - \mathcal{Y}_t|| \vartriangle\psi,\vartriangle\mathcal{Y}\}$$

and

$$\{|\tau(\psi_t,t) + \upsilon(\psi_t,t)| \vartriangle\psi,\vartriangle\mathcal{Y}\} \preccurlyeq \{\lambda|1+\tilde{\psi}_t|| \vartriangle\psi,\vartriangle\mathcal{Y}\}$$

respectively

The existence of a fuzzy stochastic initial value problem (18) can be proved by using the iterative method.

$$d(\tilde{\psi}_{n+1})_t = \tilde{\tau}((\tilde{\psi}_n)_t,t)dt \oplus (\tilde{\upsilon}((\tilde{\psi}_n)_t,t) \otimes d\tilde{\mathcal{B}}_t)$$

integrating from 0 to t, the α-cut of the solution of (18) becomes

$$(\tilde{\psi}_{n+1})_t = \tilde{\psi}_0 \oplus \int_0^t \tilde{\tau}((\tilde{\psi}_n)_t,s)ds \oplus \int_0^t \tilde{\upsilon}((\tilde{\psi}_n)_s,s)d\tilde{\mathcal{B}}_t.$$

As $\tilde{\tau}(\tilde{\psi}_t,t)$ and $\tilde{\psi}(\tilde{\psi}_t,t)$ are uniformly continuos and satisfies Lipschitz condition, so for each $\alpha \in [0,1]$, the α-cut becomes

$\{|(\tau(\psi_n)_t) - \tau((\psi_{n-1})_t)|| \vartriangle\tau,\vartriangle\psi\}$

$$\preccurlyeq \{C_1|((\psi_n)_t) - ((\psi_{n-1})_t)|| \vartriangle\upsilon,\vartriangle\tau,\vartriangle C_1\}$$

$$\Rightarrow \{|\upsilon((\psi_n)_t) - \upsilon((\psi_{n-1})_t),|| \vartriangle\psi,\vartriangle\upsilon\}$$

$$\preccurlyeq \{C_1|(\psi_n)_t - (\psi_{n-1})_t| \vartriangle\upsilon,\vartriangle\tau,\vartriangle C_1\} \qquad (21)$$

where $\tilde{C}_1$ is a fuzzy number.

The process $\tilde{\psi}_t$ is well defined and continuos path.

$$\{E|((\psi_{n+1})_t) - ((\psi_n)_t)|^2| \vartriangle\psi\} \preccurlyeq \{C_2| \vartriangle C_2\}, \forall t \in [0,1],$$

$$(\tilde{C}_2 \textit{be a fuzzy number})$$

Now

$$\{|(\psi_{n+1})_t - (\psi_n)_t|^2| \vartriangle\psi\} =$$

$$=\{|(\int_0^t \tau(\tilde{\psi}_n)_s ds + \int_0^t \upsilon((\tilde{\psi}_n)_t)d\tilde{\mathcal{B}}_t -$$

$$(\int_0^t \tau((\tilde{\psi}_{n-1})_t ds + \int_0^t \upsilon((\tilde{\psi}_{n-1})_t)d\tilde{\mathcal{B}}_s)|^2| \vartriangle\}$$

$$(\textit{where } \vartriangle=\vartriangle\psi,\vartriangle\tau,\vartriangle\upsilon,\vartriangle\mathcal{B})$$

$$\Rightarrow \{E|(\psi_{n+1})_t - (\psi_n)_t|^2| \vartriangle\psi\}$$

$$\preccurlyeq \{E[\int_0^t (\tau((\tilde{\psi}_n)_t) - \tau((\tilde{\psi}_{n-1})_t))$$

$$ds + \int_0^t (\upsilon((\tilde{\psi}_n)_t) - \upsilon((\tilde{\psi}_{n-1})_t)d\tilde{\mathcal{B}}_t]^2| \vartriangle\}$$

$$\Rightarrow \{E|(\psi_{n+1})_t - (\psi_n)_t|^2| \vartriangle\psi\}$$

$$\preccurlyeq \{2E[\int_0^t (\tau((\tilde{\psi}_n)_t - \tau(\psi_{n-1})_s))ds]^2$$

$+2E[\int_0^t (\upsilon((\psi_n)_s) - \upsilon(\psi_{n-1})_t))d\mathcal{B}_s]^2| \vartriangle\}$
(using $(x+y)^2 \leq 2x^2 + 2y^2$)

$$\preccurlyeq \{2C_1^2 E[\int_0^t (|(\psi_n)_s - (\psi_{n-1})_s|ds)^2$$

$$+2C_1^2\int_0^t E|((\psi_n)_t) - ((\psi_{n-1})_t)|^2 ds]| \vartriangle\psi, \vartriangle C_1\},$$

(*By Cauchy schwratz inequality*)

$$\preccurlyeq \{2C_1^2 E[T\int_0^t |((\psi_n)_t) - ((\psi_{n-1})_t)|^2 ds$$

$$+2C_1^2\int_0^t E|((\psi_n)_t) - ((\psi_{n-1})_t)|^2 ds]$$

$$| \vartriangle\psi, \vartriangle C_1\}$$

$$\{E|(\psi_{n+1})_t - (\psi_n)_t|^2| \vartriangle\psi\} \leq \{2C_1^2(T+1)$$

$$\int_0^t E|((\psi_n)_t) - ((\psi_{n-1})_t)|^2 ds| \vartriangle\psi, \vartriangle C_1\} \quad (22)$$

Using (5), the α-cut becomes

$$\{E|(\psi_{n+1})_t - (\psi_n)_t|^2| \vartriangle\psi\} \preccurlyeq \{2CC_1^{2n}\frac{(T+1)^n}{n!}| \vartriangle C_1, \vartriangle C\}$$

As the fuzzy stochastic process(FSP) $\tilde{\psi}$ is uniformly continuous and satisfies Lipschitz condition.
FSP $(\tilde{\psi}_n)_t$ converges $\tilde{\psi}_t$ in L^2.

For each $\alpha \in [0,1]$, the α-cut of the limit of a fuzzy stochastic process is

$$\{\psi_t| \vartriangle\psi\} = \{L^2 - \lim_{n\to\infty}(\psi_n)_t| \vartriangle\psi\}.$$

Are bounded in L^2.
Since $(\tilde{\psi}_n)_t$ converges to $\tilde{\psi}_t$ and Lipschitz condition are bounded so
$\{\lim_{n\to\infty}[E|\tau((\psi_n)_t) - \tau(\psi_t)|^2] +$

$$[E|\upsilon((\psi_n)_t) - \upsilon(\psi_t)|^2]| \vartriangle\psi\} = \tilde{0}$$

and

$$L^2 - \{\lim_{n\to\infty}\int_0^t \upsilon((\psi_n)_s)d\mathcal{B}_s| \vartriangle\upsilon, \vartriangle\psi, \vartriangle\mathcal{B}\}$$

$$= \{\int_0^t \upsilon(\psi_s)d\mathcal{B}_s| \vartriangle\upsilon, \vartriangle\psi, \vartriangle\mathcal{B}\}$$

Similarly

$$L^2 - \{\lim_{n\to\infty}\int_0^t \tau((\psi_n)_t)ds| \vartriangle\tau, \vartriangle\psi, \vartriangle\mathcal{B}\}$$

$$= \left\{\int_0^t \tau(\psi_s)ds\right| \vartriangle\tau, \vartriangle\psi, \vartriangle\mathcal{B}\Big\}$$

Hence the solution to the fuzzy stochastic initial value problem exists.
The α-cut becomes

$$\{\psi_t| \vartriangle\psi\} = \{\psi_0 + \int_0^t \tau(\psi)_s dt$$

$$+\int_0^t \upsilon(\psi_s)d\mathcal{B}_s| \vartriangle\psi, \vartriangle\mathcal{B}, \vartriangle\upsilon, \vartriangle\tau\}$$

Next to prove Uniqueness:-

Let $\tilde{\psi}_t$ and $\tilde{\mathcal{Y}}_t$ are two solutions of (18) then $\forall t \in [0,1]$.

$$So\ \{\psi_t - \mathcal{Y}_t| \vartriangle\psi, \vartriangle\mathcal{Y}\} = \{[\int_0^t (\tau(\psi)_s - \tau(\mathcal{Y})_s)ds]$$

$$+[\int_0^t (\upsilon(\psi)_s\upsilon(\mathcal{Y})_s)d\mathcal{B}_s]| \vartriangle\psi, \vartriangle\mathcal{B}, \vartriangle\mathcal{Y}, \vartriangle\tau, \vartriangle\upsilon\}$$

$$\{E(|\psi_t - \mathcal{Y}_t|^2)| \vartriangle\psi, \vartriangle\mathcal{Y}\} \leq \{[2E|\int_0^t (\tau(\psi_s) -$$

$$\tau(\mathcal{Y}_s))ds|^2 + 2E|\int_0^t (\upsilon(\psi_s) - \upsilon(\mathcal{Y}_s))d\mathcal{B}(s)|^2]$$

$$| \vartriangle\psi, \vartriangle\tau, \vartriangle\upsilon, \vartriangle\mathcal{Y}, \vartriangle\mathcal{B}\} \quad (23)$$

$$(as(a+b)^2 \leq 2a^2 + 2b^2,$$

$$E(a+b)^2 \leq 2E(a^2) + 2E(b^2)).$$

According to Cauchy Schwartz inequality,

$$\left\{\left|\int_0^t \mathcal{H}ds|^2\right| \vartriangle\mathcal{H}\right\} \preccurlyeq \left\{T\int_0^t |\mathcal{H}|^2 ds| \vartriangle\mathcal{F}\right\}, T > 0.$$

$$So\ \{E(|\int_0^t [\tau(\psi_s) - \tau(\mathcal{Y}_s)]ds|^2)| \vartriangle\psi, \vartriangle\mathcal{Y}, \vartriangle\tau\}$$

$$\preccurlyeq \{TE(\int_0^t [\tau(\psi_s) - \tau(\mathcal{Y}_t)]^2 ds)| \vartriangle\psi, \vartriangle\mathcal{Y}, \vartriangle\tau\} \quad (24)$$

$$\preccurlyeq \{C_1^2 T\int_0^t E|\psi_t - \mathcal{Y}_t|^2 ds| \vartriangle\psi, \vartriangle\mathcal{Y}, \vartriangle C_1\}$$

and also

$$\{E(\upsilon(\psi_t) - \upsilon(\mathcal{Y}_t)d\mathcal{B}_t)^2| \vartriangle\psi, \vartriangle\mathcal{Y}\}$$

$$= \{E(\int_0^t \upsilon(\psi_t) - \upsilon(\mathcal{Y}_t))^2 ds| \vartriangle\psi, \vartriangle\mathcal{Y}\}$$

$$\preccurlyeq \{C_1^2 T\int_0^t E|\psi_t - \mathcal{Y}_t|^2 ds| \vartriangle\psi, \vartriangle\mathcal{Y}, \vartriangle C_1\}$$

So equation(19) is

$$\{E|\psi_t - \mathcal{Y}_t|^2| \vartriangle\psi, \vartriangle\mathcal{Y}\}$$

$$\preccurlyeq \{C\int_0^t E|\psi_t - \mathcal{Y}_t|^2 ds| \vartriangle\psi, \vartriangle\mathcal{Y}, \vartriangle C\}$$

$$(where\ C = (T+1)L^2 = constant)$$

Let us set

$$\tilde{\phi}(t) = E|\tilde{\psi}_t \ominus \tilde{\mathcal{Y}}_t|^2, \forall t \in [0,T]. \quad (25)$$

For each $\alpha \in [0,1]$,

$\{\phi_t | \triangle \phi\} \leq \left\{ C \int_0^t \phi_s ds \middle| \triangle C, \triangle \phi \right\}, \forall 0 \leq t \leq T.$

According to Grown-wall's lemma,

$\{\phi_t | \triangle \phi\} = \{C_0 + \int_0^t \mathcal{H} \phi ds | \triangle \mathcal{H}, \triangle C, \triangle \phi\}$

Then

$$\{\phi_t | \triangle \phi\} \leq \left\{ C_0 e^{\int_0^t \mathcal{H} ds} \middle| \triangle \mathcal{H}, \triangle C \right\}. \qquad (26)$$

Comparing (26) and (17) $\tilde{C}_0 = \tilde{0}$

S0 $\tilde{\phi} = \tilde{0}$

$$Thus\ \{E|\psi_t - \mathcal{Y}_t|^2 | \triangle \psi, \triangle \mathcal{Y}\} = \tilde{0}$$

$$\{|\psi_t - \mathcal{Y}_t|^2 | \triangle \psi, \triangle \mathcal{Y}\} = \{0 | \triangle 0\}$$

$$\{\psi_t | \triangle \psi\} = \{\mathcal{Y}_t | \triangle \mathcal{Y}\}, \forall t \in [0, T].$$

So there exists a unique solution.

IV. Linear Fuzzy Stochastic Differential Equation(LFSDE)

An equation is of the form

$$d\psi_t = (\mathcal{P}(t) + \mathcal{Q}(t)\psi_t)dt + [\mathcal{R}(t) + \mathcal{S}(t)\psi_t]d\mathcal{B}_t. \quad (27)$$

is called LSDE.

Here ψ_t is an SP that satisfy the SDEs, $\mathcal{P}(t), \mathcal{Q}(t), \mathcal{R}(t), and \mathcal{S}(t)$ are adapted processes and $\mathcal{B}_t$ is a BM, which is a continuous function of t, $t \in [0, T]$.

If the adapted process $\mathcal{P}(t), \mathcal{Q}(t), \mathcal{R}(t), and \mathcal{S}(t)$ are FSP $\tilde{\mathcal{P}}(t), \tilde{\mathcal{Q}}(t), \tilde{\mathcal{R}}(t), \tilde{\mathcal{S}}(t)$ and BM $\mathcal{B}_t$ is a FBM $\tilde{\mathcal{B}}_t$ then (28) is a LFSDE, which takes the following form

$$d\tilde{\psi}_t = \left(\tilde{\mathcal{P}}(t) \oplus \left(\tilde{\mathcal{Q}}(t) \otimes \tilde{\psi}_t\right)\right) dt$$

$$\oplus [\tilde{\mathcal{R}}(t) \oplus (\tilde{\mathcal{S}}(t) \otimes \tilde{\psi}_t)] d\tilde{\mathcal{B}}_t. \quad (28)$$

For each $\alpha \in [0,1]$

$$d\tilde{\psi}_t[\alpha] = (\tilde{\mathcal{P}}(t)[\alpha] + (\tilde{\mathcal{Q}}(t)[\alpha]\tilde{\psi}_t[\alpha]))dt$$

$$+[\tilde{\mathcal{R}}(t)[\alpha] + \left(\tilde{\mathcal{S}}(t)[\alpha]\tilde{\psi}_t[\alpha]\right)] d\tilde{\mathcal{B}}_t[\alpha]$$

is the α-cut of LFSDE where

$\tilde{\mathcal{P}}(t)[\alpha] = [\tilde{\mathcal{P}}^L(t)[\alpha], \tilde{\mathcal{P}}^U(t)[\alpha]],$

$$\tilde{\mathcal{Q}}(t)[\alpha] = [\tilde{\mathcal{Q}}^L(t)[\alpha], \tilde{\mathcal{Q}}^U(t)[\alpha]]$$

$$\tilde{\mathcal{R}}(t)[\alpha] = [\tilde{\mathcal{R}}^L(t)[\alpha], \tilde{\mathcal{R}}^U(t)[\alpha]],$$

$$\tilde{\mathcal{S}}(t)[\alpha] = [\tilde{\mathcal{S}}^L(t)[\alpha], \tilde{\mathcal{S}}^U(t)[\alpha]].$$

are the interval valued adapted process,

and $\tilde{\mathcal{B}}_t[\alpha] = [\tilde{\mathcal{B}}_t^L[\alpha], \tilde{\mathcal{B}}_t^U[\alpha]]$

is a interval valued BM.

Let $\triangle \psi = \{\psi_t | \psi_t \in \tilde{\psi}_t[\alpha]\}, \ \triangle \mathcal{P} = \{\mathcal{P} | \mathcal{P} \in \tilde{\mathcal{P}}[\alpha]\},$

$\triangle \mathcal{Q} = \{\mathcal{Q} | \mathcal{Q} \in \tilde{\mathcal{Q}}[\alpha]\}, \triangle \mathcal{R} = \{\mathcal{R} | \mathcal{R} \in \tilde{\mathcal{R}}[\alpha]\},$

$\triangle \mathcal{S} = \{\mathcal{S} | \mathcal{S} \in \tilde{\mathcal{S}}[\alpha]\}, \triangle \mathcal{Y} = \{\mathcal{Y} | \mathcal{Y} \in \tilde{\mathcal{Y}}[\alpha]\}$

$\triangle \mathcal{U} = \{\mathcal{U} | \mathcal{U} \in \tilde{\mathcal{U}}[\alpha]\}, \triangle \mathcal{V} = \{\mathcal{V} | \mathcal{V} \in \tilde{\mathcal{V}}[\alpha]\},$

$\triangle^* = \{\triangle \mathcal{P}, \triangle \mathcal{Q}, \triangle \mathcal{R}, \triangle \mathcal{S}, \triangle \mathcal{B}\},$

$\triangle^{***} = \{\triangle \mathcal{U}, \triangle \mathcal{P}, \triangle \mathcal{S}, \triangle \mathcal{V}, \triangle \mathcal{B}\}$

For each $\alpha \in [0,1]$,

$\{d\psi_t | \triangle \psi\} = \{(\mathcal{P}(t) + \mathcal{Q}(t)\psi_t)dt +$

$$(\mathcal{R}(t) + \mathcal{S}(t)\psi_t)d\mathcal{B}_t | \triangle^*\} \qquad (29)$$

To find the solution of (29), first to find the fuzzy stochastic exponential of FSDEs, with $\tilde{\mathcal{P}}(t) = \tilde{0}$ and $\tilde{\mathcal{R}}(t) = \tilde{0}$, equation (29) written as

$$\{d\psi_t | \triangle \psi\} = \{\mathcal{Q}(t)\psi_t dt + \mathcal{S}(t)\psi_t d\mathcal{B}_t | \triangle^*\} \qquad (30)$$

Let $\tilde{\mathcal{U}}(t)$ satisfy the equation.

$$\{d\mathcal{U}(t) | \triangle \mathcal{U}\} = \{\mathcal{U}(t)d\mathcal{Y}_t | \triangle \mathcal{U}, \triangle \mathcal{Y}\}, \tilde{\mathcal{U}}(\tilde{0}) = \tilde{1} \qquad (31)$$

$$So\{\mathcal{U}(t) | \triangle \mathcal{U}\} = \{\varepsilon(\mathcal{Y}_t(t)) | \triangle \mathcal{Y}\}$$

where $\tilde{\mathcal{U}}(t)$ is called the stochastic exponential of $\tilde{\mathcal{Y}}_t$.

$$\{\mathcal{U}(t) | \triangle \mathcal{U}\} = \{\mathcal{U}(0)e^{\mathcal{Y}_t - \mathcal{Y}(0) - \frac{1}{2}[\mathcal{Y}_t, \mathcal{Y}_t]} | \triangle \mathcal{Y}, \triangle \mathcal{U}\} \qquad (32)$$

$\tilde{\mathcal{Y}}_t$ is an Ito process that satisfy the FSDE.

$\{d\mathcal{Y}_t | \triangle \mathcal{Y}\} = \{\mathcal{Q}(t)dt + \mathcal{S}(t)d\mathcal{B}_t | \triangle \mathcal{Q}, \triangle \mathcal{S} \triangle \mathcal{B}\}$

$\Rightarrow \{\mathcal{Y}_t - \mathcal{Y}_0 | \triangle \mathcal{Y}\} = \{\int_0^t \mathcal{Q}(s)ds +$

$$\int_0^t \mathcal{S}(s)d\mathcal{B}_s | \triangle \mathcal{Q}, \triangle \mathcal{S}, \mathcal{B}\}$$

$$(Integrating\ both\ sides\ 0\ to\ t)$$

$\{\mathcal{Y}_t | \triangle \mathcal{Y}\} = \{\int_0^t \mathcal{Q}(s)ds + \int_0^t \mathcal{S}(s)\, d\mathcal{B}_s + \mathcal{Y}_0$

$| \triangle \mathcal{Q}, \triangle \mathcal{S}, \triangle \mathcal{B}, \triangle \mathcal{Y}\}$

So

$\{\mathcal{U}(t) | \triangle \mathcal{U}\}$

$$= \{\mathcal{U}(0)e^{\int_0^t \mathcal{Q}(s)ds+\int_0^t \mathcal{S}(s)d\mathcal{B}_s-\frac{1}{2}\int_0^t \mathcal{S}^2(s)ds} \mid \vartriangle \mathcal{Y},\vartriangle \mathcal{U}\}$$

$$\{\mathcal{U}(t)\mid \vartriangle \mathcal{U}\} = \{\mathcal{U}(0)e^{\int_0^t(\mathcal{Q}(s)-\frac{1}{2}\int_0^t \mathcal{S}^2(s))ds+\int_0^t \mathcal{S}(s)d\mathcal{B}_s}$$

$$\mid \vartriangle \mathcal{Y},\vartriangle \mathcal{U}\} \qquad (33)$$

Let the general solution of (29) be

$$\{\psi_t\mid \vartriangle \psi\} = \{\mathcal{U}(t)\mathcal{V}(t)\mid \vartriangle \mathcal{U},\vartriangle \mathcal{V}\}$$

with initial condition $\tilde{\mathcal{U}}(\tilde{0}) = \tilde{1}$

$$\tilde{\psi}(\tilde{0}) = \tilde{\mathcal{V}}(\tilde{0}).$$

where $\tilde{\mathcal{U}}(t)$ is a fuzzy stochastic exponential and $\tilde{\mathcal{V}}(t)$ satisfies FSDE,

$$\{d\mathcal{V}(t)\mid \vartriangle \mathcal{V}\} = \{a(t)dt + b(t)d\mathcal{B}_t\mid \vartriangle \mathcal{B},\vartriangle a,\vartriangle b\},$$

$$(\tilde{\mathcal{V}}(\tilde{0}) = \tilde{\psi}(\tilde{0})) \qquad (34)$$

where the coefficients a(t) and b(t) satisfy

$$\{a(t)\mathcal{U}(t)\mid \vartriangle \mathcal{U},\vartriangle a\} = \{\mathcal{P}(t) - \mathcal{S}(t)\mathcal{V}(t)\mid$$

$$\vartriangle \mathcal{U},\vartriangle \mathcal{P},\vartriangle \mathcal{S}\}$$

$$and\ \{b(t)\mathcal{U}(t)\mid \vartriangle \mathcal{U},\vartriangle b\} = \{\mathcal{R}(t)\mid \vartriangle \mathcal{R}\} \qquad (35)$$

$$\{a(t)\mid \vartriangle a\} = \{\tfrac{\mathcal{P}(t)-\mathcal{S}(t)\mathcal{V}(t)}{\mathcal{U}(t)}\mid \vartriangle \mathcal{U},\vartriangle \mathcal{P},\vartriangle \mathcal{S}\},$$

$$and\,\{b(t)\mid \vartriangle b\} = \{\tfrac{\mathcal{R}(t)}{\mathcal{U}(t)}\mid \vartriangle \mathcal{R},\vartriangle \mathcal{U}(t)\}$$

So

$$\{d\mathcal{V}(t)\mid \vartriangle \mathcal{V}\} = \{\tfrac{\mathcal{P}(t)-\mathcal{S}(t)\mathcal{V}(t)}{\mathcal{U}(t)}dt + \tfrac{\mathcal{R}(t)}{\mathcal{U}(t)}$$

$$d\mathcal{B}_t\mid \vartriangle \mathcal{U},\vartriangle \mathcal{P},\vartriangle \mathcal{S}\}$$

$$\Rightarrow\{\mathcal{V}(t)\mid \vartriangle \mathcal{V}\} = \{\mathcal{V}(0) + \int_0^t \tfrac{\mathcal{P}(s)-\mathcal{S}(s)\mathcal{V}(s)}{\mathcal{U}(s)}ds$$

$$+\int_0^t \tfrac{\mathcal{R}(s)}{\mathcal{U}(s)}d\mathcal{B}_s\mid \vartriangle^{***}\} \qquad (36)$$

$$(Integrating\ both\ sidesfrom\ 0\ to\ t)$$

The solution of LFSDE becomes

$$\{\psi_t\mid \vartriangle \psi\} = \{\ (\mathcal{U}(0)e^{\int_0^t(\mathcal{Q}(s)-\frac{1}{2}\int_0^t \mathcal{S}^2(s))ds+\int_0^t \mathcal{S}(s)d\mathcal{B}_t})$$

$$(\psi_0 + \int_0^t \frac{\mathcal{P}(s) - \mathcal{S}(s)\mathcal{V}(s)}{\mathcal{U}(s)}ds + \int_0^t \frac{\mathcal{R}(s)}{\mathcal{U}(s)}d\mathcal{B}_s)\mid \vartriangle^{***}\}$$

EXAMPLE 4.1: Find the solution of LFSDE.

$$d\tilde{\psi}_t = (2\tilde{\psi}_t \oplus \tilde{1})dt \oplus e^{2t}d\tilde{\mathcal{B}}_t$$

SOLUTION: Given LFSDE is

$$d\tilde{\psi}_t = (2\tilde{\psi}_t + 1)dt \oplus e^{2t}d\tilde{\mathcal{B}}_t \qquad (37)$$

The α-cu of $d\tilde{\psi}_t$ is

$$d\tilde{\psi}_t[\alpha] = \{d\psi_t\mid \vartriangle \psi_1\}$$

$$= \{(2\psi_t + 1)dt + e^{2t}d\mathcal{B}_t\mid \vartriangle \psi,\vartriangle \mathcal{B},\vartriangle \mathcal{T},\vartriangle 1\}$$

$$\{d\psi_t - 2\psi_t dt\mid \vartriangle \psi,\vartriangle \mathcal{B}_1\} = \{1.dt + e^{2t}d\mathcal{B}_t$$

$$\mid \vartriangle \mathcal{B},\vartriangle \mathcal{T},\vartriangle 1\}$$

$$\{e^{-2t}d\psi_t - 2e^{-2t}2\psi_t dt\mid \vartriangle \psi,\vartriangle \mathcal{B}\}$$

$$= \{e^{-2t}dt + d\mathcal{B}_t\mid \vartriangle \mathcal{B},\vartriangle \mathcal{T},\vartriangle 1\}$$

Integrating both sides from 0 to t, the α-cut becomes

$$\{\int_0^t d(e^{-2t}\psi_t)\mid \vartriangle \psi,\vartriangle \mathcal{B}\} = \{\int_0^t e^{-2t}dt + \int_0^t d\mathcal{B}_t\mid \vartriangle \mathcal{B},\vartriangle 1\}$$

$$\{\psi_t\mid \vartriangle \psi\} = \{\psi_0 e^{2t} + e^{2t}\int_0^t e^{2t}dt + e^{2t}\int_0^t d\mathcal{B}_t\mid \vartriangle \mathcal{B}\}$$

$$= \{\psi_0 e^{2t} + \tfrac{1}{2}(e^{2t} - 1) + e^{2t}\mathcal{B}_t\mid \vartriangle \mathcal{B},\vartriangle \psi\}$$

$$So\ \tilde{\psi}_t = (\tilde{\psi}_0 \otimes e^{2t}) \oplus \tfrac{1}{2}(e^{2t} - 1) \oplus (e^{2t}\tilde{\mathcal{B}}_t).$$

V. CONCLUSION

This paper mainly discussed the existence and uniqueness of FSDE using Grown wall's inequality, Lipschitz condition, linear growth condition. Moreover, the fuzzy Langevin SDE and LFSDE are discussed with suitable examples.
Here the Buckley concept of fuzzy probability.

In future, one can discuss the existence and uniqueness of FSDE that does not satisfy the Lipschitz condition, which is the future extent of the current work.

REFERENCES

[1] K. Ito, “Stochastic integral", Proceedings of the Imperial Academy, vol. 20, no. 8, 1944 , pp. 519-524.

[2] A. Bensoussan, R. Glowinski, and A. RaÅŸcanu, “Approximation of some stochastic differential equations by the splitting up method," Applied Mathematics and Optimization, vol. 25, no. 1, 1992, pp. 81-106.

[3] F. C. Klebaner, “Introduction to stochastic calculus with applications", World Scientific Publishing Company, 2012.

[4] R. Bass, “A stochastic differential equation with a sticky point Electronic Journal of Probability, vol. 19, 2014, pp. 1-22.

[5] R. Farnoosh, H. Rezazadeh, A. Sobhani and M. Behboudi, “Analytical solutions for stochastic differential equations via Martingale processes", Mathematical Sciences, vol. 9, no. 2, 2015, pp. 87-92.

[6] N. Halidias and S. I. Stamatiou, “On the numerical solution of

some non-linear stochastic differential equations using the semi-discrete method", Computational Methods in Applied Mathematics, vol. 16, no. 1, 2016, pp. 105-132.
[7] A. M. Boudref and A. Berboucha, "Existence of almost periodic solution of stochastic differential equations with periodic coefficients", Random Operators and Stochastic Equations, vol. 25, no. 1, 2017, pp.57-70.
[8] H. Ghosh, "Stochastic differential equation models and their Applications, to agriculture", An overview , 2019.
[9] T. Wang, and H. Liang, "A Stochastic Differential Equation Driven by Poisson Random Measure and Its Application in a Duopoly Market", Mathematical Problems in Engineering, 2020.
[10] I. Corwin, and H. Shen, "Some recent progress in singular stochastic Partial differential equations", Bulletin of the American Mathematical *Society*, vol. 57, no. 3, 2020, pp. 409-454.
[11] P. Qi, "Existence and Uniqueness of Solutions for Some Basic Stochastic Differential Equations", In Journal of Physics, Conference Series, , IOP Publishing.,Vol. 1802, No. 4, 2021, pp. 204-209.
[12] L. A. Zadeh, "Fuzzy sets", Inform. and control, vol. 8, 1965, pp. 338-353.
[13] L. A. Zadeh, "Probability measures of fuzzy events", Journal of Mathematical analysis and applications, vol. 23, no.2, 1968, pp. 421-427.
[14] M. L. Puri, D. A. Ralescu and L. A. Zadeh, "Fuzzy random variables", in Readingsin fuzzy sets for intelligent systems, 1999,pp. 265-271.
[15] M .L. Puri and D. A. Ralescu, "Convergence theorem for fuzzy martingales", Journal of Mathematical Analysis and Applications, vol. 160, no. 1, 1991, pp. 107-122.
[16] J. J. Buckley, "Fuzzy probabilities new approach and applications" Springer Science Business Media , vol.115, 2005.
[17] G. Panda and J. K. Dash, "Nonlinear fuzzy chance constrained programming problem", Opsearch, vol. 51, no. 2,2014, pp. 270-279
[18] S Acharya, N. Ranarahu, J. K. Dash and Acharya, M. M., "Solving multi-objective fuzzy probabilistic programming problem", Journal of Intelligent Fuzzy Systems, vol. 26, no. 2, 2014 , pp. 935-948.
[19] M. T. Malinowski and M. Michta, "Fuzzy stochastic integral equations", Dynamic Systems and Applications, vol. 19, no. 3, 2010, pp-473.
[20] M. T. Malinowski and M. Michta, "Stochastic fuzzy differential equations with an application", *Kybernetika*, vol. 47, no. 1, 2011, pp. 123-143.
[21] H. Nasseri, "Fuzzy numbers: Positive and non negative", In International Mathematical Forum , vol. 36, no. 3, 2008, pp-1777-1780.
[22] S. Li, Y. Ogura and H. T. Nguyen, "Gaussian processes and martingales for fuzzy valued random variables with continuous parameter", Information Sciences, vol. 133, no. 2, 2001, pp. 7-21.
[23]] D. K. Seya, R. M. Makengo, M. RÃ©mon and W. O. Rebecca, "Fuzzy Ito integral driven by a fuzzy Brownian motion", Journal of Fuzzy Set Valued Analysis, vol. 3,2015, pp. 232-244.
[24] J. K. Kim, "On fuzzy stochastic differential equations", Journal of the Korean Mathematical Society, vol. 42, no. 1, 2005, pp. 153-169.
[25] J. K. Dash, S. Panda and G. B. Panda, "A new method to solve fuzzy stochastic finance problem", Journal of Economic Studies, vol. 49, no. 2, 2021, pp. 243-258.
[26] M. T. Malinowski, "Symmetric fuzzy stochastic differential equations with generalized global Lipchitz condition", Symmetry, vo. 12, no.5, 2020, pp. 819.
[27] A. Bibi and F. Merahi, "Adomian decomposition method applied to linear stochastic differential equations", International Journal of Pure and Applied Mathematics ,vol. 118, no. 3, 2018, pp. 501-510.
[28] Y. Feng, "The solutions of linear fuzzy stochastic differential systems", Fuzzy Sets and Systems, vol. 140, no. 3, 2003 , pp. 541-554.
[29] S. Panda, J. K. Dash and G. B. Panda, "Properties of fuzzy Ito integrals With respect to fuzzy Brownian motion", Journal of Intelligent and Fuzzy Systems, 2022, pp. 1-12.