Solvability and Dynamics of Superlinear Reaction Diffusion Problem with Integral Condition

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***Abstract*—In this paper, we evaluate certain type superlinear nonlocal problems that are a class of parabolic equations with the second-type integral condition. We use the Fadeo-Galarkin method to establish the existence of the weak solution and we prove the uniqueness of this solution for the problem by using an a priori estimate. In addition, we study the theoretical blow-up solution and perform several numerical simulations of finite-time blow-up of a particular example of the main problem.**

***Index Terms*—Parabolic equation, Nonlinear equations, Integral condition, Existence and uniqueness, Fadeo-Galarkin method, Blow-up solution.**

I. Introduction

THE nonlinear diffusion equations are a type of parabolic equations that originates from a diverse variety of diffusion phenomena extensively in nature [1]–[4]. The complexity of the nonlinear evolution equations, including the difficulties in analyzing them theoretically, brings the attention of many scientists and mathematicians in the field of nonlinear sciences [5]–[9]. The partial differential equations with nonlocal conditions can be used to simulate a variety of natural phenomena [10]–[12]. Many phenomena, on the other hand, are usually studied by integral conditions [13], [14]. The nonlocal and the integral formulations are used to describe the most of contemporary physics and technology problems conditions for the partial differential equations (see [15]–[22]). The first type of these formulations is given by:

$$\int_{\Omega} N(x,t)y(x,t)dx = E(t),$$

where $t \in (0,T)$, $\Omega \subset \mathbb{R}^n$ and N is a given function. The second type, where the Dirichlet or Neumann condition modeling by integral condition, is given by:

$$y(x,t)|_{\partial\Omega} = \int N(x,t)y(x,t)dx.$$

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This type can be used when it is impossible to directly measure the sought quantity on the border, where the total value or the average is known. The study of the nonlinear evolution equations with different boundary conditions types (classical and non-classical conditions) has been solved by many powerful and different methods in the nonlinear analysis, (i.e., fixed-point theorem, semigroup method, Galerkin and monotone operator method, see [23]–[27]). This, however, inspired us to study the superlinear parabolic equation with a classical Dirichlet condition coupled with an integral condition of the second type, more than any classical integral condition. In this paper, we explore the existence and the uniqueness of the weak solution for the linear problem by the Faedo-Galerkin method. Besides we apply an iterative process based on the results obtained for the linear problem to explore the existence and the uniqueness of the weak solution of the semilinear problem. Finally, we study the blow-up solution theoretically and numerically as a special case of the main problem.

II. Formulation of the problem

In this section, we consider the function $y = y(x,t)$ for $x \in \Omega$ and $t \in [0,T]$, where $\Omega = (0,l)$ is a bounded open of $\mathbb{R}$ and $Q = \Omega \times (0,T)$. In this regard, we concern with the following main problem (P_1):

$$\begin{cases} \frac{\partial y}{\partial t} - a\frac{\partial^2 y}{\partial x^2} + y^p - by = f(x,t) & \forall (x,t) \in \overline{Q} \\ y(x,0) = \varphi(x) & \forall x \in (0,l) \\ y(0,t) = 0 & \forall t \in [0,T] \\ \frac{\partial y}{\partial x}(l,t) = \int_0^l N(x,t)y(x,t)dx & \forall t \in [0,T] \end{cases}, \tag{P_1}$$

where a, b and p are positive odd integers and $p \geq 1$. In order to solve the problem (P_1), we need to introduce the following hypothesis and functional spaces that we will use later. Define the following hypothesis H as follows:

$$(H): \begin{cases} f \in L^2\left(0,T;\ L^2(\Omega)\right) & (H.1) \\ \varphi \in H^1(\Omega) \cap L^{p+1}(\Omega) & (H.2) \end{cases}$$

Define the functional space V by:

$$V = \left\{ y \in H^1(\Omega) \cap L^{p+1}(\Omega) : v(0) = 0 \right\},$$

provided with the norm:

$$\|v\|_V = \|v\|_{H^1(\Omega)} + \|v\|_{L^{p+1}(\Omega)}.$$

Definition 1: The weak solution to the problem (P_1) is a function that verifies the following properties: **(i)** $y \in L^2\left(0,T;\ H^1(\Omega)\right) \cap L^\infty\left(0,T;\ H^1(\Omega)\right)$. **(ii)** y admits a strong derivative $\frac{\partial y}{\partial t} \in L^2\left(0,T;\ H^1(\Omega)\right)$. **(iii)** $y(0) = \varphi$. **(iv)** The identity property, i.e.,

$$\begin{aligned}(y_t, v) + a(y_x, v_x) + (y^p, v) - b(y, v) \\ = (f, v) + av(l)\int_0^l N(x,t)y(x,t)dx,\end{aligned}$$

for all $v \in V$ and $t \in [0, T]$.

III. Solvability of the nonlinear problem

Herein, we aim to study the solvability of problem P_1 so that this section is divided into two parts; the existence of the solution of problem P_1 and the uniqueness of that solution.

A. The existence of the solution

In this subsection, we intend to derive a variational formulation and find the solution of problem P_1.

1) Variational formulation: In the beginning, we will derive a variational formulation of problem P_1 via multiplying the equation:

$$\frac{\partial y}{\partial t} - a\frac{\partial^2 y}{\partial x^2} + y^p - by = f(x,t), \tag{1}$$

by an element $v \in V$, and then integrating it over Ω to obtain:

$$\begin{aligned}\int_\Omega \frac{\partial y}{\partial t}\cdot v dx + a\int_\Omega \frac{\partial y}{\partial x}\cdot\frac{\partial v}{\partial x}dx + \int_\Omega y^p \cdot v dx \\ - b\int_\Omega y\cdot v dx - av(l)y_x(l,t) = \int_\Omega f\cdot v dx.\end{aligned} \tag{2}$$

By using the boundary conditions and Green's formula on (2), we get:

$$(y_t, v) + a(y_x, v_x) + (y^p, v) - b(y, v) - av(l)y(l,t) = (f, v), \tag{3}$$

for all $v \in V$, where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$.

2) Study the existence of solution of problem (P_1): In this part, we prove the existence of the solution to problem (P_1) by using the Faedo-Galerkin method that assumes the space V is separable, and so there exists a sequence $w_1, w_2, \cdots, w_m$ having the following properties:

$$\begin{cases} w_i \in V, & \forall i, \\ \forall m, w_1, w_2, ..., w_m & \text{are linearly independent,} \\ V_m = \langle\{w_1, w_2, ..., w_m\}\rangle & \text{is dense in } V. \end{cases} \tag{4}$$

In particular, we have:

$$\forall \varphi \in V, \implies \exists(\alpha_{Sm})_m \in IN^*$$

such that

$$\varphi_m = \sum_{S=1}^m \alpha_{Sm} w_S \longrightarrow \varphi, \text{as } m \longrightarrow +\infty.$$

Define the following function:

$$t \mapsto y_m(x,t) = \sum_{i=1}^m g_{im}(t) w_i(x).$$

The approximate solution for the previous function satisfies the following identities:

$$\begin{cases} ((y_m(t))_t, w_S) + a(\Delta y_m(t), w_S) + (y_m^p(t) - by_m(t), w_S) \\ \qquad = (f(t), w_S) \\ (y_m(0), w_S) = \alpha_{Sm}, \forall S = \overline{1,m}. \end{cases} \tag{P_2}$$

Note that $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$. So, we have:

$$((y_m(t))_t, w_S) = \sum_{i=1}^m (w_i, w_S)\frac{\partial g_{im}}{\partial t}(t), \tag{5}$$

and

$$\begin{aligned} a(\Delta y_m(t), w_S) = -\sum_{i=1}^m a((w_i)_x, (w_S)_x) g_{im}(t) \\ + a\sum_{i=1}^m g_{im}(t)\frac{\partial w_i}{\partial x}(l)w_S(l). \end{aligned} \tag{6}$$

Also, we have:

$$y_m(0) = \sum_{S=1}^m \alpha_{Sm} w_S(x).$$

The existence of α_{Sm} follows from $y_0 \in V \cap L^{P+1}(\Omega)$ and the fact that $\{w_S,\ S \in \mathbb{N}\}$ is the base in $V \cap L^{P+1}(\Omega)$. Thus, (P_1) is reduced to the initial value problem for a system of the first-order differential equations with respect to g_{im}, which has the form:

$$\begin{cases} \sum_{i=1}^m (w_i, w_S)\frac{\partial g_{im}}{\partial t}(t) + a\sum_{i=1}^m ((w_i)_x, (w_S)_x) g_{im}(t) \\ \quad -a\sum_{i=1}^m g_{im}(t)\frac{\partial w_i}{\partial x}(l)w_S(l) + (y_m^p - by_m, w_S) \\ \qquad = (f(t), w_S) \\ g_{Sm}(0) = \alpha_{Sm} \qquad \forall S = \overline{1,m}. \end{cases} \tag{P_3}$$

Now, consider the following vector:

$$g_m = (g_{1m}(t), \cdots, g_{mm}(t)), f_m = ((f, w_1), \cdots, (f, w_m)),$$

as well as the matrices:

$$B_m = ((w_i, w_j))_{\substack{1\le i\le m \\ 1\le j\le m}}, A_m = \left(\left(\frac{\partial w_i}{\partial x}, \frac{\partial w_j}{\partial x}\right)\right)_{\substack{1\le i\le m \\ 1\le j\le m}},$$

$$C_m = \left(\frac{\partial w_i}{\partial x}(l)\cdot w_j(l)\right)_{\substack{1\le i\le m \\ 1\le j\le m}},$$

and

$$G(g) = \left(\left(\left(\sum_{i=1}^m g_{im}(t) w_i\right)^p, w_j\right)\right)_{\substack{1\le i\le m \\ 1\le j\le m}}.$$

Define the problem (P_4) in the matrix form as follows:

$$\begin{cases} B_m\frac{\partial g_m}{\partial t}(t) + aA_m g_m - bB_m g_m + G(g) \\ \qquad = f_m + aC_m g_m, \\ g_m(0) = (\alpha_{im})_{1\le i\le m}. \end{cases} \tag{7}$$

With the help of using the Carathéodory's existence theorem for the ordinary differential equations, we can conclude that there exists t_m depends only on $|\alpha_{im}|$ in the interval $[0, t_m]$. Thus, problem (7) admits a unique local solution $g_m(t) \in$

$C\,[0,t_m]$ and $g_m^{'}\,(t)\in L^2\,[0,T]$. Now, we can study the a priory estimates for the approximate solution $y_m(x,t)$.

Theorem 1: For all $m\in\mathbb{N}^*$, and $\frac{p}{2}\geq b$. Suppose that $\varphi\in H^1\,(\Omega)\cap L^{p+1}\,(\Omega)$ and $f\in L^2\,\big(0,T,\ L^2\,(\Omega)\big)$. Then problem (P_1) has a solution y such that:

$$y\in L^2\,(0,T;\ V)\cap L^\infty\left(0,T;\ H^1\,(\Omega)\right),$$

and

$$y'\in L^2(0,T;\ L^2(\Omega)).$$

Proof: By multiplying both sides of the equations in (P_2) by $g_{im}(t)$, and then by taking the summation with respect to S, we get:

$$\sum_{S=1}^{m}\left((y_m)_t\,,w_S\right)\cdot g_{Sm}(t)-a\sum_{S=1}^{m}\left(\Delta y_m,w_S\right)\cdot g_{Sm}(t)$$
$$+\sum_{S=1}^{m}\left(y_m^p-by_m,w_S\right)\cdot g_{Sm}(t)\sum_{S=1}^{m}\left(f,w_S\right)\cdot g_{Sm}(t).$$

Thus, we find:

$$\left((y_m)_t\,,y_m\right)-a\left(\Delta y_m,y_m\right)+\left(y_m^p-by_m,y_m\right)=\left(f,y_m\right).$$

Taking the integrating from 0 to t for the above equality yields:

$$\frac{1}{2}\|y_m\|_{L^2(\Omega)}^2+a\int_0^t\left\|\frac{\partial y_m}{\partial x}\right\|_{L^2(\Omega)}^2d\tau+\int_0^t\|y_m\|_{L^{p+1}(\Omega)}^{p+1}d\tau$$
$$=\int_0^t(f,y_m)d\tau+b\int_0^t\|y_m\|_{L^2(\Omega)}\,d\tau$$
$$+a\int_0^t\frac{\partial y_m}{\partial x}(l,\tau)\cdot y_m(l,\tau)d\tau+\frac{1}{2}\|\varphi_m\|_{L^2(\Omega)}^2.$$

By using the Cauchy inequality with ε, we obtain:

$$\int_0^t\left(\frac{\partial y_m}{\partial x}\,(l,\tau)\cdot y_m\,(l,\tau)\right)d\tau$$
$$<\frac{\varepsilon}{2}\int_0^t y_m^2\,(l,\tau)\,d\tau+\frac{1}{2\varepsilon}\int_0^t y_{mx}^2\,(l,\tau)\,d\tau.$$

To evaluate the estimate, we use the following inequality:

$$y^2\,(l,t)\leqslant 2\int_x^l y_x^2dx+2y^2.$$

By using Gronwell's inequality, we get:

$$\frac{1}{2}\|y_m\|_{L^2(\Omega)}^2+(a-a\varepsilon)\int_0^t\left\|\frac{\partial y_m}{\partial x}\right\|_{L^2(\Omega)}^2$$
$$+\int_0^t\|y_m\|_{L^{p+1}(\Omega)}^{p+1}\leq\exp\left(\left(\frac{1}{2}+\frac{\varepsilon}{l}+\frac{N}{2\varepsilon}+b\right)T\right)$$
$$\times\left(\frac{1}{2}\int_0^t\|f\|^2+\frac{1}{2}\|\varphi_m\|_{L^2(\Omega)}^2\right).$$

Then by putting $\varepsilon=\frac{1}{2}$, we get:

$$C_T=\frac{1}{2}\int_0^t\|f\|^2+\frac{1}{2}\|\varphi_m\|_{L^2(\Omega)}^2. \tag{8}$$

Consequently, we have:

$$\frac{1}{2}\|y_m\,(t)\|_{L^2(\Omega)}^2+\frac{a}{2}\int_0^t\left\|\frac{\partial y_m}{\partial x}\right\|_{L^2(\Omega)}^2d\tau+\int_0^t\|y_m\|_{L^{p+1}(\Omega)}^{p+1}d\tau\leq C_T, \tag{9}$$

where C_T is a positive constant depending only on $\int_0^T\|f\,(\tau)\|_{L^2(\Omega)}^2$, $\|\varphi_m\|_{L^2(\Omega)}^2$ and T. This implies that the solution of the initial value problem for the system of the ODE given in (7) can be extended to $[0,T]$. Thus, we have the following uniform priory estimates:

$$\begin{cases} y_m \text{ uniformly bounded in } L^\infty(0,T;\ L^2\,(\Omega)\ ) \\ y_m \text{ uniformly bounded in } L^2(0,T;\ H^1\,(\Omega)) \\ y_m \text{ uniformly bounded in } L^{p+1}\left(0,T;\ L^{p+1}\,(\Omega)\right). \end{cases}$$

To get more a priory estimates, we multiply the formulation variational in (7) by $g_{Sm}^{'}(t)$, and then take bake the summation over k. This implies the following equality:

$$\left((y_m)_t\,,(y_m)_t\right)-a\left(\frac{\partial^2y_m}{\partial x^2},(y_m)_t\right)+\left(y_m^p-by_m,(y_m)_t\right)$$
$$=\left(f,(y_m)_t\right).$$

By integration on $(0,t)$, it comes:

$$\|(y_m)_t\|_{L^2(Q)}^2+\left(\frac{a}{2}-2\frac{l}{\varepsilon}\right)\|(y_m)_x\|_{L^2(\Omega)}^2$$
$$+\frac{1}{p+1}\|y_m\|_{L^{p+1}(\Omega)}^{p+1}\leq\int_0^t\left(f+by_m,(y_m)_t\right)$$
$$+a\left(\frac{1}{2}-2\frac{l}{\varepsilon}\right)\|(\varphi_m)_x\|_{L^2(\Omega)}^2+\frac{1}{p+1}\|\varphi_m\|_{L^{p+1}(\Omega)}^{p+1}$$
$$+a\left(\frac{\varepsilon l}{2}K+\frac{2}{\varepsilon}\right)\|u_m\|_{L^\infty(0,T,L^2(\Omega))}^2.$$

By putting

$$C=\frac{\max\left\{\frac{1}{2},a\left(\frac{1}{2}-l\right),a\left(lN+1+\frac{b}{2}\right),-\frac{b}{2}\right\}}{\min\left\{\frac{1}{2},\frac{a}{T}\left(\frac{1}{2}-l\right),\frac{1}{(p+1)T}\right\}},$$

we find

$$\|(y_m)_t\|_{L^2(Q)}^2+\|(y_m)_x\|_{L^2(Q)}^2+\int_0^t\|y_m\|_{L^{p+1}(\Omega)}^{p+1}\leq C\times$$
$$\left[\|f\|_{L^2(Q)}^2+\|(\varphi_m)_x\|_{L^2(\Omega)}^2+\|\varphi_m\|_{L^{p+1}(\Omega)}^{p+1}+\|\varphi_m\|_{L^2(\Omega)}^2\right]$$

Then, we get the following further priory estimates:

$$\begin{cases} y_m \text{ uniformly bounded in } L^{p+1}(0,T;\ L^{P+1}\,(\Omega)) \\ y_m \text{ uniformly bounded in } L^2(0,T;\ H^1\,(\Omega)) \\ (y_m)_t \text{ uniformly bounded in } L^2\left(0,T;\ L^2\,(\Omega)\right) \end{cases}. \tag{10}$$

There is a subsequence of y_m (see [23]), defined as follows:

$$\begin{cases} y_m\longrightarrow y \text{ weakly in } L^{p+1}(0,T;\ L^{P+1}\,(\Omega)) \\ y_m\longrightarrow y \text{ weakly in } L^2(0,T;\ H^1\,(\Omega)) \\ y_m\longrightarrow y \text{ weakly in } L^2\left(0,T;\ L^2\,(\Omega)\right) \end{cases}. \tag{11}$$

Consequently, there are subsequences denoted by $(y_{m_S})\,and\left(\frac{\partial y_{m_S}}{\partial t}\right)$ of (y_m) and $(y_m)_t$ respectively, such that:

$$y_{m_S}\rightharpoonup y\qquad\text{in } L^2(0,T;\ H^1\,(\Omega)), \tag{12}$$

$$\frac{\partial y_{m_S}}{\partial t} \rightharpoonup w \qquad \text{in } L^2\left(0,T;\ L^2(\Omega)\right). \tag{13}$$

By Relikh-Kondrachoff's theorem, the injection of $H^1(Q)$ into $L^2(Q)$ will be compact. Depends on Rellich's theorem, any weakly convergent sequence in $H^1(Q)$ has a subsequence that converges strongly in $L^2(Q)$. So, we have:

$$y_{m_S} \longrightarrow y \qquad \text{in } L^2(Q). \tag{14}$$

On the other hand, there is a subsequence of $(y_{m_S})_S$ denoted by y_{m_S} converges almost everywhere to y such that:

$$y_{m_S} \longrightarrow y \qquad \text{almost everywhere } Q. \tag{15}$$

Also, there is a subsequence of y_m, denoted by $(y_m)^p$ such that $(y_m)^p$ converges almost everywhere to y in $Q_T = \Omega \times [0,T]$. It turns out that

$$(y_m)^p \text{ almost everywhere converges to } y^p \ in\ Q_T.$$

On the other hand, (10) implies that $(y_m)^p$ is bounded in $L^{\frac{p+1}{p}}(Q_T)$. Therefore, we get:

$$y_m^p \rightharpoonup y^p \text{ is weakly in } L^{\frac{p+1}{p}}\left(0,T\ ,\ L^{\frac{p+1}{p}}(\Omega)\right).$$

Let $w = \frac{\partial y}{\partial t}$, we prove that:

$$y(t) = \varphi + \int_0^t w(\tau)d\tau. \tag{16}$$

In fact, as we have:

$$y_{m_S} \rightharpoonup y \qquad \text{in } L^2(0,T\ ;\ L^2(\Omega)).$$

Then we get:

$$y_{m_S} \rightharpoonup \varphi + \chi \qquad \text{in } L^2(0,T\ ;\ L^2(\Omega)).$$

This means:

$$\lim\,(y_{m_S} - \varphi - \chi, v)_{L^2(0,T\ ;\ L^2(\Omega))} = 0,$$

$\forall v \in L^2(0,T;\ L^2(\Omega))$. This is because of

$$\chi(t) = \int_0^t w(\tau)d\tau.$$

Now by using the equality:

$$y_{m_S} - \varphi_{m_S} = \int_0^t \frac{\partial y_{m_S}}{\partial \tau} d\tau, \forall t \in [0,T],$$

coupled with using $y_{m_S} \in L^2(0,T;V_{m_S})$ and $(y_{m_S})_t \in L^2(0,T;V_{m_S})$, we can get the following assertion:

$$\begin{aligned}&\left(y_{m_S} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\ ;\ L^2(\Omega))} \\ &= \left(\int_0^t \left(\frac{\partial y_{m_S}}{\partial \tau} - w(\tau)\right)d\tau, v\right)_{L^2(0,T\ ;\ L^2(\Omega))} \\ &\quad + (\varphi_{m_S} - \varphi, v)_{L^2(0,T\ ;\ L^2(\Omega))}, \forall t \in [0,T].\end{aligned}$$

Consequently, we get:

$$\begin{aligned}&\left(y_{m_S} - \varphi - \int_0^t w(\tau)d\tau, v\right)_{L^2(0,T\ ;\ L^2(\Omega))} \\ &= \int_0^t \left(\frac{\partial y_{m_S}}{\partial \tau} - w(\tau), v\right)_{L^2(0,T\ ;\ L^2(\Omega))} d\tau \\ &\quad + (\varphi_{m_S} - \varphi, v)_{L^2(0,T\ ;\ L^2(\Omega))}, \forall t \in [0,T].\end{aligned}$$

On the one hand, we have:

$$\lim_{S\longrightarrow\infty} \int_0^t \left(\frac{\partial y_{m_S}}{\partial \tau} - w(\tau), v\right)_{L^2(0,T\ ;\ L^2(\Omega))} d\tau = 0, \tag{17}$$

for $t \in [0,T]$. In addition, we have:

$$\lim_{S\longrightarrow\infty} (\varphi_m - \varphi, v)_{L^2(0,T\ ;\ L^2(\Omega))} = 0. \tag{18}$$

So, we can get:

$$\lim_{S\longrightarrow\infty} (y_{m_S} - \varphi - \chi, v)_{L^2(0,T\ ;\ L^2(\Omega))} = 0,$$

for all $v \in L^2\left(0,T;L^2(\Omega)\right)$. Finally, from (17) and (18), we get:

$$\lim_{k\longrightarrow\infty} (y_{m_S} - \varphi - \chi, v)_{L^2(0,T\ ;\ L^2(\Omega))} = 0,$$

for all $v \in L^2\left(0,T;L^2(\Omega)\right)$. Now, by passing to the limit in (P_2) and since each term on the left side of (P_2) is weakly convergent in $L^{\frac{p+1}{p}}(\Omega)$, we obtain that the following assertion:

$$\begin{aligned}((y_m(t))_t, w_S) + a\,(y_m(t), w_S) + (y_m^p(t) - by_m(t), w_S) \\ = (f(t), w_S),\end{aligned} \tag{19}$$

for all $S = \overline{1,m}$. In fact, the above state holds in $L^{\frac{p+1}{p}}(\Omega)$. Since $\{w_j,\ \ j \in \mathbb{N}\}$ is a base in $L^{\frac{p+1}{p}}(\Omega)$, we infer from (19) the following assertion:

$$y' - a\Delta y + y^p - by = f, \tag{20}$$

which holds in $L^{\frac{p+1}{p}}\left(0,T\ ,\ L^{\frac{p+1}{p}}(\Omega)\right)$. Since all $y', \Delta y$, and f belong to $L^2\left(0,T;L^2(\Omega)\right)$, y^p also belongs to $L^2\left(0,T;L^2(\Omega)\right)$, and (20) also holds in $L^2\left(0,T;L^2(\Omega)\right)$, then we get the desired results.

B. The uniqueness of the solution of the problem (P_1)

In this section, we aim to study the uniqueness of the solution of problem (P_1), where p is assumed here to be odd. This would be achieved by starting with the next theorem.

Theorem 2: Suppose that $\varphi \in H^1(\Omega) \cap L^{p+1}(\Omega)$ and $f \in L^2\left(0,T;L^2(\Omega)\right)$. Then problem (P_1) admits a unique solution y such that:

$$y \in L^2\left(0,T,\ H^1(\Omega)\right) \cap\ L^{p+1}(0,T;\ L^{P+1}(\Omega)),$$

and

$$y' \in L^2(0,T,\ L^2(\Omega)).$$

Proof: Suppose p is odd, and then multiply the equation of the problem (P_1) by My, we get:

$$My = y.$$

By integrating the garbed result over the domain $\Omega = (0, l)$, where y_x *and* y_t denote the partial derivative with respect to x and t respectively, we get:

$$\frac{1}{2}\frac{\partial}{\partial t}\|y\|^2_{L^2(\Omega)} + a\int\limits_{\Omega}\left(\frac{\partial y}{\partial x}\right)^2 dx + \int\limits_{\Omega} y^{p+1}dx - b\int\limits_{\Omega} y^2 dx = \int\limits_{\Omega} fydx.$$

Then, by integrating on $(0, \tau)$, where $\tau \in (0, T)$, and putting:

$$C_T = \frac{\frac{1}{2}\int\limits_0^T \|f(\tau)\|^2_{L^2(\Omega)} d\tau + \frac{1}{2}\|\varphi_m\|^2_{L^2(\Omega)} + \left(\frac{1}{2} + \frac{\varepsilon}{l} + \frac{N}{2\varepsilon} + b\right)\frac{(p+1)mes(\Omega)T}{p-1}}{\min\left(\frac{1}{2}, (a - a\varepsilon), \left(1 - \left(\frac{1}{2} + \frac{\varepsilon}{l} + \frac{N}{2\varepsilon} + b\right)\frac{2}{p+1}\right)\right)}, \tag{21}$$

we get:

$$\|y_m(t)\|^2_{L^2(\Omega)} + \int\limits_0^t \left\|\frac{\partial y_m}{\partial x}\right\|^2_{L^2(\Omega)} d\tau + \int\limits_0^t \|y_m\|^{p+1}_{L^{p+1}(\Omega)} d\tau \le C_T. \tag{22}$$

Consequently, we put:

$$\|y\|^2_{L^\infty(0,T;L^2(\Omega))} + \|y\|^2_{L^2(0,T;L^2(\Omega))} + \|y\|^{p+1}_{L^{p+1}(0,T\,;L^{p+1}(\Omega))} \equiv \|y\|_B\,.$$

Let y_1 and y_2 be two solutions to problem (P_1) such that:

$$\begin{cases} Ly_1 = \mathcal{F} \\ Ly_2 = \mathcal{F} \end{cases} \Longrightarrow Ly_1 - Ly_2 = 0,$$

where L is the differential operator of the main semilinear problem. Then, we have:

$$L(y_1 - y_2) = 0.$$

This leads to the following assertion:

$$\|y_1 - y_2\|_B \le c\,\|0\|_F = 0,$$

which gives:

$$y_1 = y_2.$$

IV. Blow-up solutions to some Fujita problems

In this section, we study the finite-time blow-up solution theoretically for a special case of the main problem:

$$\begin{cases} \frac{\partial y}{\partial t} - a\frac{\partial^2 y}{\partial x^2} - by = y^p & \forall\,(x,t) \in \overline{Q}, (P_{4.1}) \\ y(x,0) = \varphi(x) & \forall x \in (0,l)\,, (P_{4.2}) \\ y(0,t) = 0 & \forall t \in [0,T]\,, (P_{4.3}) \\ \frac{\partial y}{\partial x}(l,t) = \int_0^l N(x,t)y(x,t)dx & \forall t \in [0,T]\,, (P_{4.4}) \end{cases}. \tag{P_4}$$

A. Theoretical study of blow-up

Let we have the following Sturm Liouville problem:

$$\begin{cases} -\Delta\psi = \lambda^2\psi, \\ \psi(0) = 0, \\ \psi_x(l) = 0. \end{cases}$$

Consequently, we get:

$$\lambda = (2k+1)\frac{\pi}{2l} \text{ and } \psi(x) = B\sin\left((2k+1)\frac{\pi}{2l}x\right).$$

Herein, we use only for $k = 0$ coupled with using the following assumption:

$$\Pi(t) = \int_0^l \psi(x)y(x,t)dx.$$

Immediately, we multiply the equation (P_4) by ψ, and then integrate the result over the domain $\Omega = (0, l)$ to get:

$$\int_0^l \psi(x)\frac{\partial y}{\partial t}dx - a\int_0^l \psi(x)\frac{\partial^2 y}{\partial x^2}dx - b\int_0^l \psi(x)ydx = \int_0^l \psi(x)y^p dx.$$

Thus, we have:

$$\begin{aligned} &\Pi'(t) - b\Pi(t) - a\int_0^l y\Delta\psi dx \\ &= a\left[\int_0^l N(x,t)y(x,t)dx\right]\psi(l) + \int_0^l \psi(x)y^p dx \\ &\ge a\min_{x,y\in Q}(N(x,t))\int_0^l \psi(x)y(x,t)dx + \int_0^l \psi(x)y^p dx. \end{aligned} \tag{23}$$

Then, by applying the Jensen inequality, we obtain:

$$\frac{\pi}{2l}\int_0^l \psi(x)y^p dx \quad \ge \quad \left(\frac{\pi}{2l}\int_0^l \psi(x)ydx\right)^p.$$

So, it comes:

$$\Pi'(t) + \Big(-b + a\lambda^2 - a\min_{x,y\in Q}(N(x,t))\Big)\Pi(t) \ge \left(\frac{\pi}{2l}\right)^{p-1}(\Pi(t))^p\,.$$

Consequently, we gain the following equation:

$$\Pi'(t) + C\Pi(t) + \left(\frac{\pi}{2l}\right)^{p-1}(\Pi(t))^p = 0, \tag{24}$$

where

$$C = -b + a\lambda^2 - a\min_{x,y\in Q}(N(x,t)).$$

To solve this equation, we use the following variable $v = \Pi^{1-p}$. Finally, we get:

$$\Pi(t) = \left(\frac{1}{\left((\Pi(0))^{1-p} + \frac{1}{C}\left(\frac{\pi}{2l}\right)^{p-1}\right)e^{-(p-1)Ct} - \frac{1}{C}\left(\frac{\pi}{2l}\right)^{p-1}}\right)^{\frac{1}{p-1}}. \tag{25}$$

Like $\frac{1}{p-1} > 0$, then we have $\Pi \to \infty$, if

$$\left((\Pi(0))^{1-p} + \frac{1}{C}\left(\frac{\pi}{2l}\right)^{p-1}\right)e^{-(p-1)Ct} - \frac{1}{C}\left(\frac{\pi}{2l}\right)^{p-1} \to 0,$$

which implies:

$$T_* = \frac{-1}{(p-1)C}\ln\left(\frac{\frac{1}{C}\left(\frac{\pi}{2l}\right)^{p-1}}{(\Pi(0))^{1-p} + \frac{1}{C}\left(\frac{\pi}{2l}\right)^{p-1}}\right).$$

B. Numerical study of blow-up solution

In this section, we will study the problem at hand in numerical way. Let M be a positive integer, and divide the interval $[0, l]$ into M subintervals of equal lengths $h = 1/M$. The grids (x_i, t_n) will given by $x_i = ih$, $i = 0, 1, 2, ..., M$ and $t_0 = 0, t_{n+1} = t_n + \tau_n$, $n = 0, 1, 2, ...$, where $\tau_n > 0$ is the time steps. The reason for studying this type of time-steps as opposed to constant time-steps is to make sure that the time-step approaches zero as time is drawing close to the blow-up time. The notations u_i^n, k_i^n are used for the value of u and k at point (x_i, t_n), respectively.

1) Explicit Euler scheme: The forward difference quotient method will be used to approximate the time derivative, although the centred second-order approximation for the spatial derivative of the second order in (P_4) is used in the form:

$$\frac{y_i^{n+1}-y_i^n}{\tau_n}-a\frac{y_{i+1}^n-2y_i^n+y_{i-1}^n}{h^2}-by_i^n=(y_i^n)^p\,.$$

This scheme can be written as:

$$y_i^{n+1}=\tau_n\,(y_i^n)^p+(1-2r_h^n-b\tau_n)y_i^n+r_h^ny_{i+1}^n+r_h^ny_{i-1}^n, \tag{26}$$

for $i=1,2,...,M-1$, $n=0,1,...,N, Y_h^n=(y_0^n,y_1^n,...,y_M^n)^t$, and $r_h^n=a\frac{\tau_n}{h^2}$. The stability condition of the explicit Euler scheme given by $r_h^n\leq\frac{1}{2}$. We will take the time steps as follows to make sure we will get the best convergence:

$$\tau_n=min\left(\frac{h^2}{2},\frac{h^\alpha}{\|Y_h^n\|_\infty^p}\right),$$

where α is a fixed positive constant. From the boundary condition of problem (P_4), we get:

$$y_0^{n+1}=y(0,t_{n+1})=0. \tag{27}$$

To determinate y_M^{n+1}, we approximate the first space derivative in (P_4) by central finite difference operator of second-order and the integral formulated by the trapezoidal rule. This would obtain:

$$\begin{aligned}&\frac{\partial y}{\partial x}(1,t_{n+1})=\int_0^1 N(x,t_{n+1})y(x,t_{n+1})dx\\&\Leftrightarrow\frac{y_{M+1}^{n+1}-y_{M-1}^{n+1}}{2h}\\&=\frac{h}{2}\left(N_0^{n+1}y_0^{n+1}+N_M^{n+1}y_M^{n+1}+2\sum_{i=1}^{M-1}N_i^{n+1}y_i^{n+1}\right).\end{aligned}$$

Thus, we can obtain the following assertion:

$$\begin{aligned}&y_{M+1}^{n+1}-y_{M-1}^{n+1}\\&=h^2\left(N_0^{n+1}y_0^{n+1}+N_M^{n+1}y_M^{n+1}+2\sum_{i=1}^{M-1}N_i^{n+1}y_i^{n+1}\right).\end{aligned} \tag{28}$$

Eliminating the fictitious value y_{M+1}^{n+1} between (28) and $(26)_{i=M}$ gives:

$$\begin{aligned}y_M^{n+1}=&\frac{2r_h^ny_{M-1}^n+2\tau_n\sum_{i=1}^{M-1}k_i^{n+1}u_i^{n+1}}{(1-\tau_nk_M^{n+1})}\\&+\frac{\tau_n(y_M^n)^p)+(1-2r_h^n+b\tau_n)y_M^n}{(1-\tau_nk_M^{n+1})}.\end{aligned} \tag{29}$$

2) Linearly implicit Euler scheme: With the help of using the classical backward time-centred space finite difference scheme, we approximate the derivative in equation (P_4). In other words, we have:

$$\frac{y_i^{n+1}-y_i^n}{\tau_n}-a\frac{y_{i+1}^{n+1}-2u_i^{n+1}+y_{i-1}^{n+1}}{h^2}-by_i^{n+1}=(y_i^n)^p\,. \tag{30}$$

After some rearrangement, the equation (30) becomes:

$$-r_h^ny_{i-1}^{n+1}+(1-2r_h^n+b\tau_n)y_i^{n+1}-r_h^ny_{i+1}^{n+1}=\tau_n\,(y_i^n)^p+y_i^n, \tag{31}$$

for $i=1,2,...,M-1, n=0,1,...,N$, and $r_h^n=a\tau_n/h^2$. Depending on (31), we have $M-1$ linear equations in $M+1$ unknowns $y_0^{n+1},y_1^{n+1},...,y_M^{n+1}$. In order to solve the linear system, we need the following equations; the first equation can be obtained from the boundary condition $(P_{4.1})$:

$$y_0^{n+1}=y(0,t_{n+1})=0. \tag{32}$$

By eliminating the fictitious value y_{M+1}^{n+1} between (28) and $(31)_{i=M}$, we get:

$$\begin{aligned}&\tau_nN_0^{n+1}y_0^{n+1}+(-2r_h^n-2\tau_nk_{M-1}^{n+1})y_{M-1}^{n+1}\\&\qquad+(1-2r_h^n+b\tau_n)y_M^{n+1}-2\tau_n\sum_{i=1}^{M-2}N_i^{n+1}y_i^{n+1}\\&\qquad=\tau_n(y_M^n)^p)+y_M^n.\end{aligned} \tag{33}$$

Thus, we put:

$$\begin{aligned}a_0^{n+1}y_0^{n+1}+a_2^{n+1}y_2^{n+1}+a_3^{n+1}y_3^{n+1}+...+a_{M-1}^{n+1}y_{M-1}^{n+1}\\+a_M^{n+1}y_M^{n+1}=\tau_n(y_M^n)^p)+y_M^n,\end{aligned} \tag{34}$$

where

$$\begin{cases}a_0^{n+1}=\tau_nN_0^{n+1}\\a_{M-1}^{n+1}=-2r_h^n-2\tau_nN_{M-1}^{n+1}\\a_M^{n+1}=1-2r_h^n+b\tau_n\\a_i^{n+1}=-2\tau_nN_i^{n+1},\qquad i=1,2,...M-2.\end{cases} \tag{35}$$

Consolidating (31) and (32) with (34) yields an $M+1\times\,M+1$ linear system of equations. This system can be written in the matrix form as:

$$A^{n+1}Y^{n+1}=B^{n+1}, \tag{36}$$

where

$$A^{n+1}=$$

$$\begin{pmatrix}1&0&0&0&0&0\\-r_h^n&1-2r_h^n+b\tau_n&-r_h^n&\cdot&\cdot&0\\0&-r_h^n&1-2r_h^n+b\tau_n&0&\cdot&0\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\0&\cdot&\cdot&-r_h^n&1-2r_h^n+b\tau_n&-r_h^n\\a_0^n&a_1^n&a_2^n&a_{M-2}^n&a_{M-1}^n&a_M^n\end{pmatrix}$$

$$Y^{n+1}=\begin{pmatrix}y_0^{n+1}\\y_1^{n+1}\\y_2^{n+1}\\\cdots\\y_{M-1}^{n+1}\\y_M^{n+1}\end{pmatrix},B^{n+1}=\begin{pmatrix}0\\\tau_n\left(y_1^n\right)^p+y_1^n\\\tau_n\left(y_2^n\right)^p+y_2^n\\\cdot\cdot\\\tau_{M-1}\left(y_{M-1}^n\right)^p+y_{M-1}^n\\\tau_n(y_M^n)^p)+y_M^n\end{pmatrix}$$

where $a_0^{n+1},a_1^{n+1},a_2^{n+1},...,a_{M-1}^{n+1},a_M^{n+1}$ are the coefficients reported in (35).

C. Numerical experiments

In this section, we present some numerical approximations to the blow-up solution and the blow-up time using the two discrete finite schemes derived in section (4.2.1) and (4.2.2) (the explicit and the implicit schemes of the problem (P_4)), for $p=3,5,7$ with $\varphi(x)=100x^2$, $k(x,t)=6,a=1$ and $b=\frac{17}{8}$. For the explicit Euler scheme, the time step will be taken as follows:

$$\tau_n=min\left(\frac{h^2}{2},\frac{h^\alpha}{\|Y_h^n\|_\infty^p}\right),n\geq 0.$$

Besides, for the linear implicit Euler scheme, the time-steps will be taken as follows:

$$\tau_n=\frac{h^\alpha}{\|Y_h^n\|_\infty^p},n\geq 0.$$

Note that α is a fixed positive constant. The numerical approximation will be terminated at the first time as $\|Y_h^n\|_\infty \geq 10^6$ and the value $T_h^m = \sum_{n=0}^{m} \tau_n$ is taken as a numerical approximation to the blow-up time T_*. From Table I to III, we present the numerical results obtained for different values of the space-step using the explicit and the implicit schemes with respect to $\alpha = 1$. From Table IV to VI, we present the numerical consequences acquired for the different values of the space-step using explicit and implicit schemes with respect to $\alpha = 2$.

TABLE I: Blow-up times obtained with a explicit and implicit schemes for $p = 3$ and $\alpha = 1$.

h	Explicit scheme	Implicit scheme
$1/20$	5.06×10^{-5}	5.06×10^{-5}
$1/40$	5.055×10^{-5}	5.055×10^{-5}
$1/80$	5.0775×10^{-5}	5.0775×10^{-5}
$1/160$	5.1031×10^{-5}	5.1031×10^{-5}

TABLE II: Blow-up times obtained with explicit and implicit schemes for $p = 5$ and $\alpha = 1$.

h	Explicit scheme	Implicit scheme
$1/20$	2.53999×10^{-9}	2.53999×10^{-9}
$1/40$	2.51999×10^{-9}	2.51999×10^{-9}
$1/80$	2.51125×10^{-9}	2.51125×10^{-9}
$1/160$	2.50562×10^{-9}	2.50562×10^{-9}

TABLE III: Blow-up times obtained with a explicit and implicit schemes for $p = 7$ and $\alpha = 1$.

h	Explicit scheme	Implicit scheme
$1/20$	1.70499×10^{-13}	1.70499×10^{-13}
$1/40$	1.68749×10^{-13}	1.68749×10^{-13}
$1/80$	1.67750×10^{-13}	1.67750×10^{-13}
$1/160$	1.671875×10^{-13}	1.671875×10^{-13}

TABLE IV: Blow-up times obtained with a explicit and implicit schemes for $p = 3$ and $\alpha = 2$.

h	Explicit scheme	Implicit scheme
$1/20$	5.015749×10^{-5}	5.015749×10^{-5}
$1/40$	5.0324375×10^{-5}	5.0324375×10^{-5}
$1/80$	$5.06465625 \times 10^{-5}$	$5.06465625 \times 10^{-5}$
$1/160$	5.061453×10^{-5}	5.061453×10^{-5}

TABLE V: Blow-up times obtained with explicit and implicit schemes for $p = 5$ and $\alpha = 2$.

h	Explicit scheme	Implicit scheme
$1/20$	2.50249×10^{-9}	2.50249×10^{-9}
$1/40$	2.5006875×10^{-9}	2.5006875×10^{-9}
$1/80$	2.5001875×10^{-9}	2.5001875×10^{-9}
$1/160$	2.500132×10^{-9}	2.5001132×10^{-9}

TABLE VI: Blow-up times and obtained with an explicit and implicit scheme for $p = 7$ and $\alpha = 2$.

h	Explicit scheme	Implicit scheme
$1/20$	1.668999×10^{-13}	1.668999×10^{-13}
$1/40$	$1.6673125 \times 10^{-13}$	$1.6673125 \times 10^{-13}$
$1/80$	$1.666828125 \times 10^{-13}$	$1.666828125 \times 10^{-13}$
$1/160$	$1.66615475 \times 10^{-13}$	$1.66615475 \times 10^{-13}$

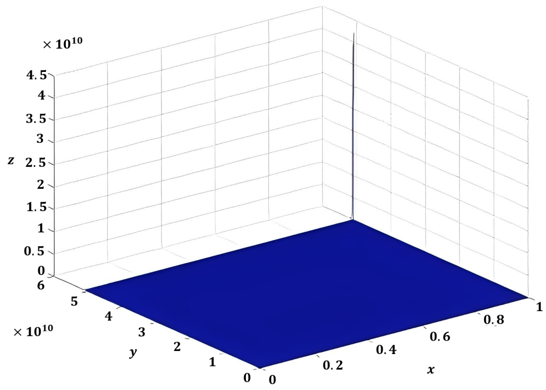

Fig. 1: Numerical solution by the explicit scheme for $p = 3$

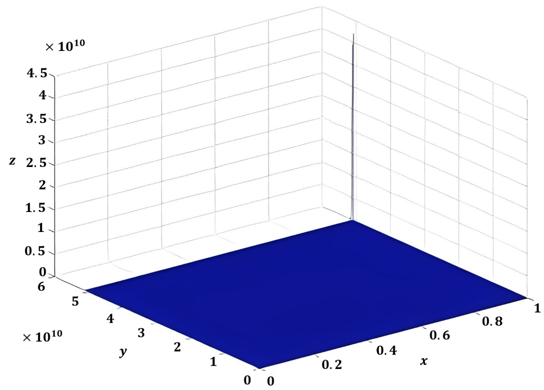

Fig. 2: Numerical solution by implicit scheme for $p = 3$

From the tables, we can see that the numerical blow-up times are decreasing as we increase the power of the nonlinear term of the equation. The numerical consequences acquired through the usage of the explicit Euler scheme are similar to that acquired through the usage of the implicit Euler scheme, however, the explicit Euler scheme given requires much less computational time than the implicit Euler scheme. Figures 1-6 present the discrete graph of the numerical solution of the problem for different values of p obtained from using the explicit and the implicit schemes, respectively.

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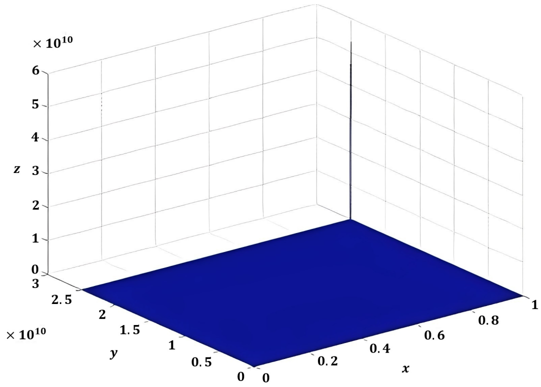

Fig. 3: Numerical solution by the explicit scheme for $p = 5$

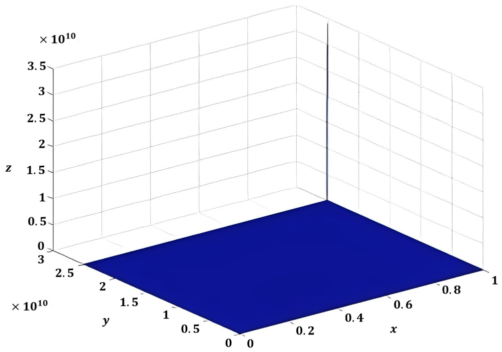

Fig. 4: Numerical solution by the implicit scheme for $p = 5$

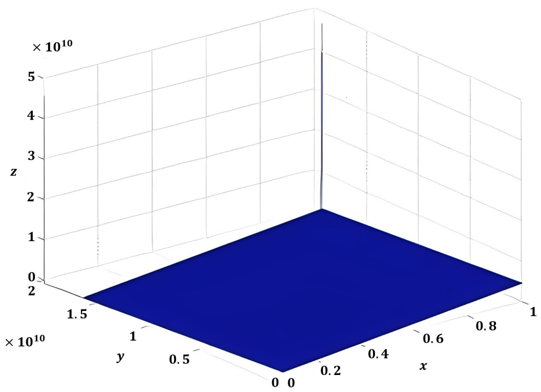

Fig. 5: Numerical solution by the explicit scheme for $p = 7$

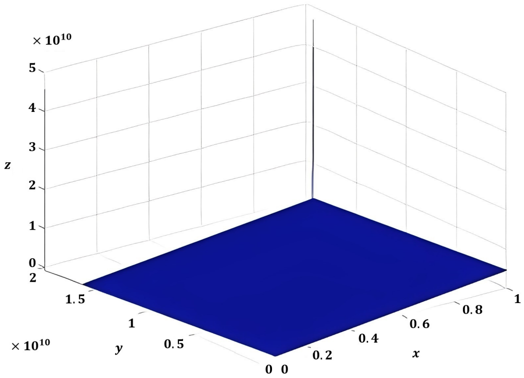

Fig. 6: Numerical solution by the implicit scheme for $p = 7$

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