Study for Local Stability Intervals in Analytic Hierarchy Process

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Abstract—**Researchers developed a method to apply the row geometric mean method to improve the consistent index of a given reciprocal comparison matrix. We provide a detailed examination of their approach to find out that sometimes their adjustments will violate the 1-9 scale rule proposed by the original restriction in Analytic Hierarchy Process. Hence, their development contained severely questionable results. We advise practitioners with caution when trying to apply their approach to adjust their comparison matrices.**

***Index Terms*—Consistent index, Analytic hierarchy process, Local stability intervals, Row geometric mean method**

I. Introduction

Many papers have discussed the consistent index and the consistent ratio problem in the analytic hierarchy process. For example, we just list a few of them, Saaty [32], Schoner and Wedley [46], Bernhard and Canada [6], Saaty [33], Apostolou and Hassell [2], Bennett and Saaty [5], Finan and Hurley [17], Saaty [36], Millet and Saaty [28], Wedley et al. [49], Chu and Liu [13], Saaty [41], Yang et al. [52], Lin et al. [24], Jung et al. [21], Ke et al. [20], and Tung et al. [47], to indicate that is a hot research topic.

In this paper, we consider the paper of Aguaron and Moreno-Jimenez [1] for the local stability intervals in the analytic hierarchy process. In the case of a single criterion, they decided the local priorities through the row geometric mean method for (a) each judgment, (b) each alternative, and (c) the matrix associated with the criterion to preserve the complete ranking and the best alternative (the $P.\gamma$ and $P.\alpha$ problems, respectively) to determine a local stability interval. We will show that their theorems contained questionable results such that their theorems cannot be applied to some comparison matrices.

Some other related papers, for example, Watson and Freeling [48], Belton and Gear [4], Azis [3], Dyer [16], Clayton and Wright [14], Saaty [34], Saaty and Kearns [35], Saaty [37], Saaty and Vargas [38], Saaty and Hu [39], Saaty and Cho [40], Blair et al. [7], Chen et al. [12], Saaty et al. [44], Saaty and Ozdemir [42, 43], Chao et al. [11], Deng et al. [15], Chang et al. [10], and Lin et al. [23, 25], which are worthy to mention, are important for the analytic hierarchy process.

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II. Review of Aguaron and Moreno-Jimenez [1]

To save precious space in this journal, we only refer to those related materials. For the detailed citation of Aguaron and Moreno-Jimenez [1], please directly refer to their paper. We recall their theorems 1, 3, and 5 as follows.

Theorem 1 of Aguaron and Moreno-Jimenez [1].
For the alternatives, the initial preference ranking is $w_1 \ge w_2 \ge \cdots \ge w_n$, for a judgment a_{rs}, $(r \ne s)$ the relative stability interval is a $P.\alpha$ problem, that is given by $\left[\underline{\alpha}_{rs}, \overline{\alpha}_{rs}\right]$ where $\underline{\alpha}_{rs} = \left(\frac{w_s}{w_1}\right)^n$, and $\overline{\alpha}_{rs} = \left(\frac{w_1}{w_r}\right)^n$ for $\forall r, s \ne 1$; $\underline{\alpha}_{1s} = \max\left\{\left(\frac{w_2}{w_1}\right)^n, \left(\frac{w_s}{w_1}\right)^{n/2}\right\}$, $\overline{\alpha}_{1s} = \infty$, $s \ne 1$; $\underline{\alpha}_{r1} = 0$, $\overline{\alpha}_{r1} = \left\{(w_1/w_2)^n, (w_1/w_r)^{n/2}\right\}$, $r \ne 1$.

Theorem 3 of Aguaron and Moreno-Jimenez [1].
The initial preference ranking for the alternatives is $w_1 \ge w_2 \ge \cdots \ge w_n$, that for the alternative A_r the relative stability interval is a $P.\alpha$ problem, $\left[\underline{\alpha}_r, \overline{\alpha}_r\right]$ and must satisfy the conditions $\underline{\alpha}_1 \ge w_2/w_1$, $\overline{\alpha}_1 = \infty$; $\underline{\alpha}_2 \ge \overline{\alpha}_2 (w_3/w_1)^n$, $\overline{\alpha}_2 \le w_1/w_2$; $\underline{\alpha}_r \ge \overline{\alpha}_r (w_2/w_1)^n$, $\overline{\alpha}_r \le w_1/w_r$ for $r \ge 3$.

Theorem 5 of Aguaron and Moreno-Jimenez [1].
For the alternatives, the initial preference ranking is $w_1 \ge w_2 \ge \cdots \ge w_n$, the relative stability interval for the matrix $A = (a_{ij})$ in a $P.\alpha$ problem is $\left[\alpha^{-1}, \alpha\right]$, where $\alpha = (w_1/w_2)^{n/(2(n-1))}$.

We recall that Aguaron and Moreno-Jimenez [1] applied the row geometric mean method so the relative weight (the local priority) for the reciprocal matrix $A = (a_{ij})$ is

$$w_j = \left(\prod_{i=1}^{n} a_{ij}\right)^{1/n}, \tag{2.1}$$

for $1 \le j \le n$.

III. Our discussion for stability interval for a judgment

We offer the following example of a reciprocal matrix $A=\left(a_{ij}\right)$ where

$$A=\begin{bmatrix} 1 & 9 & 9 \\ 1/9 & 1 & 9 \\ 1/9 & 1/9 & 1 \end{bmatrix} \tag{3.1}$$

such that $w_1 = 81^{1/3} = 4.3267$, $w_2 = 1$ and $w_3 = 81^{-1/3} = 0.2311$. Hence, the assumption in the Theorems of Aguaron and Moreno-Jimenez [1] with the requirement $w_1 \ge w_2 \ge w_3$ is satisfied.

According to Theorem 1 of Aguaron and Moreno-Jimenez [1], under a judgment a_{13} in a $P.\alpha$ problem is given by $\left[\underline{\alpha}_{13}, \overline{\alpha}_{13}\right]$, $\underline{\alpha}_{13} = \max\left\{(w_2/w_1)^3, (w_3/w_1)^{3/2}\right\} = 1/81$ and $\overline{\alpha}_{13} = \infty$. It means that Aguaron and Moreno-Jimenez [1] accepted the following revision: to change $a_{13} = 9$ to $a'_{13} = t_{13} a_{13}$ with $\underline{\alpha}_{13} \le t_{13} \le \overline{\alpha}_{13}$.

However, the entries of a comparison matrix must satisfy the rule of Saaty [32] for the entries in a comparison matrix that satisfies the 1-9 scale restriction.

Hence, the possible range for a'_{13} is denoted as $1/9 \le a'_{13} \le 9$. It indicates that the possible upper bound $\overline{\alpha}_{13} = \infty$ can not be applied to this example. Therefore, we point out our first observation.

Observation 1. The upper bound and lower bound in Theorem 1 for a judgment of Aguaron and Moreno-Jimenez [1] did not consider the restriction of Saaty [32].

To simplify the expression, we use $B=\left(b_{ij}\right)$ to denote the new matrix with $b_{13} = a'_{13}$ and $b_{ij} = a_{ij}$ otherwise.

Next, we examine the consistent ratio of B, from Saaty [36], which implies that the consistent index, say μ, satisfies

$$\mu = \frac{\lambda_{\max} - n}{n-1}, \tag{3.2}$$

where $\lambda_{\max}$ is the maximum eigenvalue for the $n \times n$ comparison matrix, B and the consistent ratio, say θ, satisfies

$$\theta = \frac{\mu}{X_{\overline{\mu}}}, \tag{3.3}$$

where $X_{\overline{\mu}}$ is the average of $n \times n$ comparison matrices with $X_{\overline{\mu}} = 0.4532$ for the 3×3 comparison matrix.

When $b_{13} = 9$, the consistent ratio $\theta = 0.62$ is greater than the acceptable consistent ratio, 0.1. It yields that the entries of B may require some modification to reduce the value of the consistent ratio.

However, if we follow Theorem 1 of Aguaron and Moreno-Jimenez [1] to revise the entry b_{13} within the permissible range of $(1/9) \le b_{13} \le 9$ then the value of the consistent ratio will become worse from $\theta = 0.62$ increasing to $\theta = 7.85$. It indicates that the discussion in Theorem 1 of Aguaron and Moreno-Jimenez [1] may contain questionable results.

From previous computation results, it reveals that all their consistent ratios are over the threshold value, 0.1, proposed by Saaty [32].

Moreover, the change of the relative weight of a judgment for the entry, say a_{13}, to its new value, $b_{13} = a'_{13} = a_{13} t_{13}$ proposed by Aguaron and Moreno-Jimenez, then the consistent index is become far away from the allowable constraint as $\theta \le 0.1$.

Based on our examination of the stability interval for judgment, the result reveals that all 17 possible modifications proposed by Theorem 1 of Aguaron and Moreno-Jimenez [1] do not satisfy the consistency test of Saaty [32]. We summarize our findings in the next observation.

Observation 2. The upper bound and lower bound in Theorem 1 for a judgment of Aguaron and Moreno-Jimenez [1] did not consider the restriction of consistent index that is proposed by Saaty [32].

IV. Our discussion of stability interval for an alternative

According to Theorem 3 of Aguaron and Moreno-Jimenez, for the alternative A_1, the relative stability interval, in a $P.\alpha$ problem, the local stability interval $\left[\underline{\alpha}_1, \overline{\alpha}_1\right]$ must satisfy the conditions $\underline{\alpha}_1 \ge \frac{w_2}{w_1}$, and $\overline{\alpha}_1 = \infty$.

It means that Aguaron and Moreno-Jimenez [1] accepted the change of $\left(a_{11}, a_{12}, a_{13}\right)$ to

$$\left(a'_{11} = 1, a'_{12}, a'_{13}\right) = \left(a_{11} = 1, a_{12} t_{12}, a_{13} t_{13}\right) \tag{4.1}$$

with $\underline{\alpha}_1 \le t_{12}, t_{13} \le \overline{\alpha}_1$. For our example, Aguaron and Moreno-Jimenez [1] claimed that

$$3^{2/3} \le a'_{12}, a'_{13} \le \infty. \tag{4.2}$$

It violates the 1-9 scale rule of Saaty [32]. Hence, we write down our next observation.

Observation 3. The upper bound and lower bound in Theorem 3 for an alternative of Aguaron and Moreno-Jimenez [1] did not consider the restriction of Saaty [32].

By the rule of the 1-9 scale of Saaty [32], it follows that

$$3^{2/3} = 2.08 < 3 \le a'_{12}, a'_{13} \le 9. \tag{4.3}$$

We use the matrix, $C=\left(c_{ij}\right)$, to denote the new matrix, with the following entries: $c_{12} = a'_{12}$, $c_{13} = a'_{13}$ and $c_{ij} = a_{ij}$ otherwise.

Next, we examine the consistent ratio of C, from $3 \le c_{12} \le 9$, and $3 \le c_{13} \le 9$ such that there are possible 49 different modifications of comparison matrix for stability interval for the first alternative, then we obtain that among them when $c_{12} = 3$ and $c_{13} = 9$, the matrix has the smallest $\lambda_{\max}$ with $\lambda_{\max} = 3.1356$ and the smallest consistent ratio $\theta = 0.15$.
It means that all 49 changes for the first row proposed by Theorem 3 of Aguaron and Moreno-Jimenez [1] will violate the consistency test proposed by Saaty [32]. We summarize our findings in the next observation.

Observation 4. The upper bound and lower bound in Theorem 3 for an alternative of Aguaron and Moreno-Jimenez [1] did not consider the restriction of consistent tests that are proposed by Saaty [32].

V. For matrices our discussion of stability interval

According to Theorem 5 of Aguaron and Moreno-Jimenez [1], for the matrix $A = (a_{ij})$ in a $P.\alpha$ problem, the relative stability interval is $[\alpha^{-1}, \alpha]$, where

$$\alpha = (w_1/w_2)^{n/(2(n-1))} = 3. \tag{5.1}$$

It means that Aguaron and Moreno-Jimenez [1] accepted the change from $A = (a_{ij})$ to $D = (a_{ij}t_{ij})$ with $1/3 \le t_{ij} \le 3$. Consequently, we still find that Aguaron and Moreno-Jimenez [1] violated the 1-9 scale rule of Saaty [32]. We write down the next observation again.

Observation 5. The upper bound and lower bound in Theorem 5 for a comparison matrix of Aguaron and Moreno-Jimenez [1] did not consider the restriction of Saaty [32].

By the rule of the 1-9 scale of Saaty [32], it shows that $1/3 \le t_{ij} \le 1$ for $j > i$ and $1 \le t_{ij} \le 3$ for $j < i$ such that $t_{ij}t_{ji} = 1$.
We use $D = (d_{ij})$ to denote the new matrix where $d_{12} = a'_{12}$, $d_{13} = a'_{13}$, $d_{23} = a'_{23}$ and $d_{ij} = a_{ij}$ otherwise.
We know that

$$3 \le d_{12}, d_{13}, d_{23} \le 9. \tag{5.2}$$

There are $7^3 = 343$ different comparison matrices that can be constructed by Theorem 5 of Aguaron and Moreno-Jimenez [1].
We examine the consistent ratio for those 343 matrices to find that 33 of them have a consistent ratio $\theta \le 0.1$ and the rest 310 matrices have a consistent ratio $\theta > 0.1$. For completeness, we will try to list those 33 comparison matrices with a consistent ratio $\theta \le 0.1$.

We will prove the following lemma, namely
(a) The maximum eigenvalue is symmetric with respect to d_{12} and d_{23},
(b) The maximum eigenvalue is an increasing function of d_{12} (d_{23}),
(c) The maximum eigenvalue is a decreasing function of d_{13}.

(Proof of Lemma)
We obtain that the characteristic polynomial of $D = (d_{ij})$, say $f(\lambda)$ satisfies

$$f(\lambda) = -\lambda^3 + 3\lambda^2 - 2 + \frac{d_{12}d_{23}}{d_{13}} + \frac{d_{13}}{d_{12}d_{23}}. \tag{5.3}$$

According to Equation (5.3), it yields that $f(\lambda)$ is symmetric with respect to d_{12} and d_{23}.
Based on

$$f'(\lambda) = -3\lambda(\lambda - 2), \tag{5.4}$$

and

$$f''(\lambda) = -6(\lambda - 1), \tag{5.5}$$

it implies that $f(\lambda)$ is decreasing and concave up from

$$\lim_{\lambda \to -\infty} f(\lambda) = \infty, \tag{5.6}$$

to

$$f(0) = \left(\sqrt{\frac{d_{12}d_{23}}{d_{13}}} - \sqrt{\frac{d_{13}}{d_{12}d_{23}}} \right)^2 \ge 0, \tag{5.7}$$

such that $f(\lambda)$ is increasing and concave up from $f(0) \ge 0$ to $f(1)$, $f(\lambda)$ is increasing and concave down from $f(1)$ to $f(2)$, and then $f(\lambda)$ is decreasing and concave down from $f(2)$ to $\lim_{\lambda \to \infty} f(\lambda) = -\infty$.
We find that

$$f(3) = \left(\sqrt{\frac{d_{12}d_{23}}{d_{13}}} - \sqrt{\frac{d_{13}}{d_{12}d_{23}}} \right)^2 \ge 0 \tag{5.8}$$

that is the same result proved by Saaty [32] as that $\lambda_{\max} \ge 3$.

We assume a new variable, say $t = \sqrt{\frac{d_{12}d_{23}}{d_{13}}}$. Form $3 \le d_{12}, d_{23}$ and $d_{13} \le 9$, it yields that $t \ge 1$. We define a new auxiliary, say

$$g(t) = \left(t - \frac{1}{t} \right)^2, \tag{5.9}$$

for $t \ge 1$.
Based on the findings of

$$g'(t) = 2\left(t - \frac{1}{t} \right)\left(1 + \frac{1}{t^2} \right) \ge 0, \tag{5.10}$$

so $g(t)$ is an increasing function of t. Consequently, when d_{12} or d_{23} is increasing then t increases so $g(t)$ also increases such that the maximum solution, say $\lambda_{\max}$, for $f(\lambda) = 0$ will increase.

Table 1. Representation for those comparison matrices satisfies the consistency test.

d_{12}	7	6	5	4	3	5	4
d_{13}	9	8	7	5	4	9	7
d_{23}	3	3	3	3	3	4	4
$\lambda_{\max}$	3.080	3.074	3.065	3.086	3.074	3.071	3.076

By a similar argument, if d_{13} is increasing then t decreases, and then $\lambda_{\max}$ will decrease.

Based on the above lemma, which shows that $\theta \le 0.1$ is equivalent to that $\lambda_{\max} \le 3.0906$. Hence, we only need to list the maximum eigenvalue of seven comparison matrices to represent those 33 matrices that satisfy the consistency test.

From $(33/343) = 9.62\%$, it implies most modifications, say 90.38%, proposed by Theorem 5 of Aguaron and Moreno-Jimenez [1] for stability interval for a matrix will fail to satisfy the consistency test of Saaty [33].

Observation 6. The upper bound and lower bound for stability interval for a matrix by Theorem 5 of Aguaron and Moreno-Jimenez [1] did not consider the restriction of consistent index that is proposed by Saaty [37].

VI. A RELATED THEORETICAL ANALYSIS

Recently, Yen [53] published a paper in Mathematical Problems in Engineering, entitled "Solving inventory models by an intuitive algebraic method" to claim that he developed a new algebraic approach to solving inventory models with linear backorder cost and completely backlogged for shortages. The purpose of this part is to point out that the argument proposed by Yen [53] contained questionable results and then we present our patch works.

First, we provide a brief review of these kinds of problems to provide motivation and explain the research gap that we try to fulfill. Since Grubbström and Erdem [18] reconsidered an EOQ model by algebraic method to help researchers not familiar with calculus to understand inventory models, there are more than one hundred papers that followed this trend to solve various inventory models by algebraic approaches without referring to an analytic method, calculus. We concentrate on the main string directly for Grubbström and Erdem [18]. Ronald et al. [31] pointed out that the algebraic method proposed by Grubbström and Erdem [18] depends on a sophisticated decomposition of one which is beyond the ability of ordinary practitioners, and then they developed a two-step algorithm to find the optimal solution. Chang et al. [9] claimed that the two-step algorithm proposed by Ronald et al. [31] is excellent work but too tedious, and then they constructed their algebraic method by traditional skill: completing the square, to locate the minimum solution. Moreover, Chang et al. [9] raised an open question for future research to solve a minimum problem containing the square root. Luo and Chou [26] provided an affirmative answer for a generalized problem of Chang et al. [9]. Yen [53] mentioned that previous papers of Luo and Chou [26], Chang et al. [9], Ronald et al. [31], and Grubbström and Erdem [18] are heavily dependent on algebraic skill without the genuine algebraic spiritual: elegance. Hence, Yen [53] created a new algebraic approach for inventory models with constant and linear backorder costs. In this part, we will show that the elegant approach proposed by Yen [53] is a mirage. There are infinite selections that can finish the algebraic approach proposed by Yen [53]. Moreover, by literature review, we point out four possible sources that implicitly influence the selection of Yen [53]. Our revisions can help researchers realize the indisputable algebraic method.

VII. NOTATION AND ASSUMPTIONS FOR INVENTORY MODELS

We follow Grubbström and Erdem [18] and Yen [53] to use the following notation. In different papers, several authors used different notations to express the same item. To be consistent within this paper, we change their expressions as mentioned in Yen [53].

$C(Q,B)$ the average cost per unit of time,
$\rho = (P-D)/P$,
Q maximum inventory level,
P production rate, under the restriction $P > D$, for the EPQ models,
K setup cost,
h holding cost per unit per unit time,
D demand rate,
c cost of production per unit,
b backorder cost per unit per unit time,
B maximum backorder level.

The research gap of Yen [53] is to solve minimum problems from a purely algebraic point of view, without referring to any analytic method, for the Economic Ordering Quantity (EOQ) model, $C(Q,B)$ of the following Equation (8.1), or the Economic Production Quantity (EPQ) model, $C(Q,B)$ of the following Equation (8.2), under the conditions $B > 0$ and $Q > 0$.

VIII. REVIEW OF YEN [53]

For an Economic Ordering Quantity (EOQ) model, Yen [53] considered that developed by Ronald et al. [31], and Grubbström and Erdem [18]:

$$C(Q,B) = \frac{D}{B+Q}\left(\frac{b}{2D}B^2 + \frac{h}{2D}Q^2 + K\right). \quad (8.1)$$

For an Economic Production Quantity (EPQ) model, Yen [53] studied that examined by Luo and Chou [26], Chang et al. [9], and Cárdenas-Barrón [8]:

$$C(Q,B)=\frac{D}{B+Q}\left(\frac{b}{2D}B^2+\frac{h}{2D}Q^2+K\rho\right)+cD \quad (8.2)$$

where $\rho=(P-D)/P$.

Yen [53] observed Equation (8.1) consider that if he added another additional restriction as

$$a_1B=a_2Q \quad (8.3)$$

and then he rewrote

$$B+Q=((a_1+a_2)/a_2)B \quad (8.4)$$

and

$$bB^2+hQ^2=((ha_1^2+ba_2^2)/a_2^2)B^2. \quad (8.5)$$

Yen [53] compared Equations (8.3) and (8.4) to find out if he assumed $a_2=h$, and $a_1=b$, then he rewrote Equations (8.3-8.5) as

$$bB=hQ, \quad (8.6)$$

$$B+Q=((b+h)/h)B, \quad (8.7)$$

and

$$bB^2+hQ^2=((h+b)b/h)B^2. \quad (8.8)$$

Therefore, he plugged Equations (8.6-8.8) into Equation (8.1) to convert $C(Q,B)$ as $C(bB/h,B)$, then

$$C(bB/h,B)=\frac{b}{2}B+\frac{hDK}{(b+h)B}. \quad (8.9)$$

According to Equation (8.9), he found

$$B^*=\sqrt{\frac{2hDK}{b(b+h)}}, \quad (8.10)$$

and then he implied

$$Q^*=\frac{b}{h}\sqrt{\frac{2hDK}{b(b+h)}}=\sqrt{\frac{2bDK}{h(b+h)}}, \quad (8.11)$$

and

$$C(Q^*,B^*)=bB^*=\sqrt{\frac{2bhDK}{b+h}}. \quad (8.12)$$

IX. Our comments for Yen [53] and our patchworks

There is an unexplained step in their derivation: why they assumed $a_1=b$ and $a_2=h$?

Without assuming $a_1=b$, and $a_2=h$, we directly substitute Equations (8.3-8.5) into Equation (8.1) to convert $C(Q,B)$ as $C(a_1B/a_2,B)$, then

$$C(a_1B/a_2,B)=\frac{(ha_1^2+ba_2^2)}{2a_2(a_1+a_2)}B+\frac{DKa_2}{(a_1+a_2)B}. \quad (9.1)$$

Based on Equation (9.1), we begin to show that the minimum solution can be derived which is suitable for any selection for a pair of a_1 and a_2.

We rewrite Equation (9.1) to derive that

$$C(a_1B/a_2,B)$$

$$=\left(\sqrt{\frac{(ha_1^2+ba_2^2)}{2a_2(a_1+a_2)}B}-\sqrt{\frac{DKa_2}{(a_1+a_2)B}}\right)^2+$$

$$\frac{\sqrt{2(ha_1^2+ba_2^2)DK}}{(a_1+a_2)}. \quad (9.2)$$

Based on our finding of Equation (9.2), we point out that in Yen [53], he assumed that $a_1=b$ and $a_2=h$, to convert his minimum problem from $C(a_1B/a_2,B,a_1,a_2)$ to $C(bB/h,B)$, and then he derived optimal solutions of Equations (8.10-8.12) which is not self-explained since we can derive the minimum solution for any selection of a pair of a_1 and a_2.

From our derivations of Equation (9.2), we still derive

$$B^*=\sqrt{\frac{2DKa_2^2}{ha_1^2+ba_2^2}}, \quad (9.3)$$

and

$$C\left(\frac{a_1B^*}{a_2},B^*\right)=\frac{\sqrt{2(ha_1^2+ba_2^2)DK}}{(a_1+a_2)}. \quad (9.4)$$

Therefore, we are facing a problem: how do we select a_1 and a_2 then minimize

$$\frac{\sqrt{(ha_1^2+ba_2^2)}}{(a_1+a_2)}. \quad (9.5)$$

Motivated by Equation (9.5), we assume an auxiliary function, say $f(x,y)$, to solve x and y, with $x>0$ and $y>0$, to minimize

$$f(x,y)=\frac{hx^2+by^2}{(x+y)^2}. \quad (9.6)$$

We recall that Ronald et al. [31] divided the minimum problem into two steps. In the first step, to minimize along each ray, $\{(x,y): y=kx\}$, we find the local minimum along the ray, denoted as (x_k,y_k), and then in the second step, we minimize $f(x_k,y_k)$, for $k>0$.

For the first step, we derive

$$f(x,kx)=\frac{hx^2+bk^2x^2}{(1+k)^2x^2}$$

$$=\frac{h+bk^2}{(1+k)^2}. \quad (9.7)$$

For the second step, we use a partial fraction to convert Equation (9.7) as

$$f(x,kx)=\frac{b+h}{(1+k)^2}-\frac{2b}{1+k}+b$$

$$=(b+h)\left(\frac{1}{1+k}-\frac{b}{b+h}\right)^2+\frac{bh}{b+h}, \quad (9.8)$$

to imply the minimum value of $f(x,y)$ as

$$f(x^*,y^*)=\frac{bh}{b+h}, \quad (9.9)$$

with

$$k^* = \frac{h}{b}. \tag{9.10}$$

We find that there are infinite pairs of a_1 and a_2 that satisfy

$$a_2 = \frac{h}{b} a_1, \tag{9.11}$$

then the minimum value will be attained. Consequently, in Yen [53], they selected $a_1 = b$ and $a_2 = h$ which is only one possible choice of infinite suitable options.

X. The Selection of A_1=B and A_2=H from the Literature Review

In this section, we provide the historical aspect of $a_1 = b$ and $a_2 = h$ from the literature review to imply $hQ = bB$. First, for the balance of the inventory holding period, T_I and backlog period, T_B, Minner [29] pointed out

$$hT_I = bT_B, \tag{10.1}$$

owing to $T_I = Q/D$ and $T_B = B/D$, and then the researcher derives that $hQ = bB$.

Second, Wee et al. [50] followed Minner [29] to develop the cost-difference comparison method (CDCM) to apply the limiting situation for a uniformly partitioned finite time horizon inventory model, its optimal solution for an infinite planning horizon inventory model to minimize its average cost for the first replenishment cycle. Wee et al. [50] derived

$$r^* = b/(b+h), \tag{10.2}$$

where r is the fill rate with $r(Q+B) = Q$.

Hence, we find $rB = (1-r)Q$ with $r^* = b/(b+h)$ to imply $hQ = bB$.

Third, we recall that Leung [22] assumed an auxiliary function for the fill rate, denoted as $\eta(r)$ with

$$\eta(r) = hr^2 + b(1-r)^2, \tag{10.3}$$

and then Leung [22] showed

$$\eta(r) = (h+b)\left(r - \frac{b}{b+h}\right)^2 - \frac{b^2}{b+h} + b, \tag{10.4}$$

to imply

$$\eta(r^*) = \frac{bh}{b+h}, \tag{10.5}$$

which is the same result as Equation (9.9).

Fourth, we recall that in Chang et al. [9], they rewrote objective function as

$$C(Q,B) = \frac{b+h}{2(Q+B)}\left(B - \frac{h(Q+B)}{b+h}\right)^2 + \frac{bh(Q+B)}{2(b+h)} + \frac{DK\rho}{Q+B} + cD, \tag{10.6}$$

to search for the minimum value. Then Chang et al. [9] found the following relation

$$B^*(Q) = \frac{h(Q+B)}{b+h}, \tag{10.7}$$

that is $bB = hQ$.

From our above discussion, we demonstrate that the optimal solution, $bB = hQ$, had appeared several times that may motivate Yen [53] to assume $a_1 = b$ and $a_2 = h$ in their derivations. However, the corrected way in Yen [53], they should assume that $a_2 = ha_1/b$ as we proposed in Equation (9.11).

We examine Yen [53] to show their selection of $a_1 = b$ and $a_2 = h$ which is questionable. We derive an infinite choice for possible selections for a_1 and a_2. Hence, Yen [53] claimed that they constructed a new algebraic method with intuitive insight which contains questionable results. Our patchworks can help the researcher understand the true meaning of algebraic methods.

At last, several related papers were worthy to mention. For example, Guo et al. [19], Luo et al. [27], Ren et al. [30], Sari et al. [45], Wijaya et al. [51], and Zhong et al. [54] are cited to help readers for the recent development of research trend.

XI. Conclusion

We study the local stability intervals for (a) an entry and (b) a comparison matrix to find out questionable results in Aguaron and Moreno-Jimenez [1]. Our examinations show that they applied the row geometric mean method to adjust entries in a given reciprocal comparison matrix which contains questionable findings. We will warn researchers to be careful to avoid unreasonable adjustments. Moreover, we point out the intuitive algebraic method proposed by Yen [53] is only one possible result among infinite possible opportunities.

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