Exact Average Run Length Evaluation for an ARMAX(p,q,r) Process Running on a Modified EWMA Control Chart

Korakoch Silpakob, Yupaporn Areepong, Saowanit Sukparungsee and Rapin Sunthornwat

***Abstract*—In this study, we apply the Fredholm-type integral equation method to derive the explicit formulas of the average run length (ARL) for an autoregressive moving average process with explanatory variables (ARMAX(p,q,r)) with exponential white noise running on a modified exponentially weighted moving average (EWMA) control chart. As a performance measure, we compared the computational times of calculating the ARL based on explicit formulas and the classical numerical integral equation (NIE) method. We found that although the ARLs using both methods were very close with an absolute percentage difference of less than 0.00001%, their calculational times were less than 0.01 and 10 seconds, respectively. Furthermore, the comparison of the performances of the ARL methods for ARMAX(p,q,r) processes with exponential white noise by practical application for time series data comprising exchange rates and the price of energy running on modified and standard EWMA and cumulative sum (CUSUM) control charts using the relative mean index (RMI) criteria. The results show that the explicit formulas method for the ARL of the process on the modified EWMA control chart is more powerful than the CUSUM and standard EWMA control charts.**

***Index Terms*— Autoregressive process, moving average process, explanatory variable, explicit formulas**

I. Introduction

STATISTICAL process control (SPC) is a powerful set of tools that are used to inspect, control, and improve the quality of products or services that plays an essential role in business and manufacturing sectors. Control charts are one of the key tools in SPC widely used in various fields, such as health [1], medicine [2], and finance [3]. Shewhart [4] presented the first control chart that is still widely used for monitoring and detecting large shifts in a process mean. Later, the cumulative sum (CUSUM) control chart [5] and the standard exponentially weighted moving average (EWMA) control chart [6] were found to be more suitable for detecting small shifts in a process mean. Moreover, Khan et al. [7] modified the EWMA control chart by adding an extra constant (k) in the last term of the modified EWMA statistic, which was further modified by [8]. The authors compared its performance with the originally modified and standard EWMA control charts in terms of the average run length (ARL) and found that it was able to detect shifts in a process mean more quickly.

The ARL is a popular measure for comparing control chart performance. It is the average number of observations until the first observation is detected outside the control limits. There are two components: ARL_0 is called an in-control ARL and ARL_1 is called an out-of-control ARL. ARL_0 is the average number of observations before an out-of-control observation is detected when the process is in-control and should be large while ARL_1 is the average number of observations before an out-of-control signal is received when the process has shifted to the out-of-control state and should be small [9]. Several methods to calculate the ARL for many control charts, such as explicit formulas, Monte Carlo simulation, Markov chain, Martingale, and numerical integration equations (NIEs) methods [10]. Crowder [11] used a Fredholm integral equation to develop an approximation for the ARL of a Gaussian process on an EWMA control chart. Champ and Rigdon [12] employed the NIE and Markov chain approaches for the ARL of processes on CUSUM and EWMA control charts. Various researchers have aimed at approximating the ARL to measure the performance of control charts by using different methods. Robert [6] introduced the standard EWMA control chart using Monte Carlo simulation to evaluate the ARL. Areepong and Novikov [13] presented an explicit formula for the ARL and the average delay for a process running on an EWMA control chart while assuming that the observations follow an exponential distribution by using the Martingale approach. Phanyaem et al. [14] used a Fredholm integral equation technique to derive an exact expression of the ARL for the first-order autoregressive moving average (ARMA(1,1)) process running on the CUSUM control chart and compare the performance of control charts with the exact expression for the EWMA control chart. Sukparungsee and Areepong [15] derived explicit formulas for the ARL on an EWMA control chart for an autoregressive of order p (AR(p)) process. Sunthornwat et

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al. [16] solved explicit formulas and optimal parameters for evaluating the ARL on an EWMA control chart for a long-memory AR fractionally integrated moving average (MA) (ARFIMA) process. Peerajit and Areepong [17] derived an exact solution for the ARL for an ARMA process with exogenous variables (ARMAX(p.q.r)) with exponential white noise running on a CUSUM control chart. Supharakonsakun et al. [18] suggested explicit formulas for the ARL of an MA(1) process running on a modified EWMA control chart. Phanthuna et al. [19] studied the run length distribution for the ARL of a stationary AR(p) process running on a modified EWMA control chart. After that, Silpakob et al. [20] derived an exact solution for the ARL of AR with explanatory variables (ARX(p,r)) processes running on a modified EWMA control chart. Most recently, Phanthuna and Areepong [21] studied the detection sensitivity of a modified EWMA control chart with a time series model for integrated MA (IMA) and fractional integrated MA (FIMA) models.

The main purpose of the present study is to derive explicit formulas for the ARL of an ARMA process with explanatory variables (ARMAX(p,q,r)) with exponential white noise running on a modified EWMA control chart based on Khan et al.'s [7] derivation. We apply Fredholm-type integral equations to derive an exact equation for two components of the ARL. This paper is organized as follows. An introduction to the control charts is provided in Section II. The explicit formulas and the NIE for the ARL of the process on the modified EWMA control chart are shown in Sections III and IV. Next, numerical results for comparing the performances of the ARLs derived by using integral equations and the NIE method are offered in Sections V and VI, respectively. The practical application of the presented explicit formulas with real data is reported in Section VII. Finally, conclusions are given in Section VIII.

II. Properties of the Control Charts Used in the Study

A. The CUSUM Control Chart

This has been widely used to detect small shifts in process means in the same way as the EWMA control chart [5]. The CUSUM control chart can be defined as

$$C_t = \max\left\{0, C_{t-1} + Y_t - a\right\} \quad ; t = 1,2,3,\ldots, \tag{1}$$

where C_t is the CUSUM statistic, Y_t is the sequence of the ARMAX(p,q,r) process with exponential white noise, a is a constant. $C_0 = u$ is the initial value when $u \in [0,b]$, where 0 is the lower control limit (LCL) and b is the upper control limit (UCL).

B. The Standard and Modified EWMA Control Charts

The modified EWMA control chart by defined as [7]

$$M_t = (1-\lambda)M_{t-1} + \lambda Y_t + k\left(Y_t - Y_{t-1}\right) \quad ; t = 1,2,3,\ldots, \tag{2}$$

where M_t is the modified EWMA statistic, Y_t is the sequence of the ARMAX(p,q,r) process with exponential white noise, λ is an exponential smoothing parameter $(0 < \lambda \le 1)$, and k is a constant $(k > 0)$. Meanwhile, mean $E(M_t) = \mu_0$ and variance $Var(M_t) = \dfrac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}\sigma^2$.

From (2), the modified EWMA statistic is reduced to the standard EWMA statistic in [6] when $k = 0$ (i.e., $M_t = (1-\lambda)M_{t-1} + \lambda Y_t$) and reduced to the primary modified EWMA statistic in [8] when $k = 1$ (i.e., $M_t = (1-\lambda)M_{t-1} + \lambda Y_t + (Y_t - Y_{t-1})$). Thus, we can derive the LCL and UCL of the two EWMA control charts as follows.

The respective LCL and UCL of the standard EWMA control chart with a control width limit L_S are

$$\text{LCL} = \mu_0 - L_S\sigma\sqrt{\frac{\lambda}{2-\lambda}} \tag{3a}$$

$$\text{and UCL} = \mu_0 + L_S\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \tag{3b}$$

while the respective LCL and UCL of the modified EWMA control chart with a control width limit L_M are

$$\text{LCL} = \mu_0 - L_M\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}} \tag{4a}$$

$$\text{and UCL} = \mu_0 + L_M\sigma\sqrt{\frac{(\lambda + 2\lambda k + 2k^2)}{(2-\lambda)}}, \tag{4b}$$

where μ_0 is the target mean, σ is the standard deviation of process, and $L_S, L_M > 0$.

III. Explicit Formulas for the ARL of the Process

A. The ARMAX(p,q,r) Process

This is defined as

$$\begin{aligned} Y_t &= \omega + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t - \theta_1\varepsilon_{t-1} \\ &\quad - \theta_2\varepsilon_{t-2} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl} \quad ; t = 1,2,3,\ldots, \end{aligned} \tag{5}$$

where ω is a constant $(\omega \ge 0)$, ϕ_i is an AR coefficient for $i = 1,2,\ldots,p$ $(|\phi_i| < 1)$, θ_j is a MA coefficient for $j = 1,2,\ldots,q$ $(|\theta_j| < 1)$, ε_t are independent and identically distributed (iid) observations in an exponential distribution ($\varepsilon_t \sim Exp(\alpha)$), X_{tl} are explanatory variables of Y_t, and B_l is a coefficient for $l = 1,2,\ldots,r$. The initial value for the ARMAX(p,q,r) process is 1.

B. Explicit Formulas

Explicit formulas for the ARL of an ARMAX(p,q,r) process running on the CUSUM control chart are shown in [17]. Explicit formulas for the ARL of an ARMAX(p,q,r) process on the modified EWMA control chart are derived as follows:

$$M_t = (1-\lambda)M_{t-1} + (\lambda+k)\varepsilon_t - kY_{t-1} + (\lambda+k)\begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\end{pmatrix}$$

If Y_1 signals the out-of-control state for M_1 when $M_0 = u$, then

$$M_1 = (1-\lambda)u + (\lambda+k)\varepsilon_1 - kY_0 + (\lambda+k)\begin{pmatrix}\omega + \phi_1 Y_0 + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_0 - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\end{pmatrix}$$

If ε_1 is the in-control state for M_1, then $0 \le M_1 \le b$; 0 is LCL and b is UCL. Consider the Fredholm integral equation of the second kind [22] following

$$H(u) = 1 + \int H(M_1) f(\varepsilon_1) d(\varepsilon_1), \quad (6)$$

Moreover, $H(u)$ can be written as

$$H(u) = 1 + \int_0^b L\left\{\begin{matrix}(1-\lambda)u - kY_{t-1} + (\lambda+k)y \\ +(\lambda+k)\begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots \\ -\theta_1\varepsilon_{t-1} - \ldots + \sum_{l=1}^{r}\beta_l X_{tl}\end{pmatrix}\end{matrix}\right\} f(y)dy .$$

Let

$$w = (1-\lambda)u - kY_{t-1} + (\lambda+k)y + (\lambda+k)\begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\end{pmatrix}.$$

By changing the variable, we can obtain the integral equation as follows:

$$H(u) = 1 + \frac{1}{\lambda+k} \bullet \int_0^b H(w) f\left\{\begin{matrix}\frac{w-(1-\lambda)u}{\lambda+k} + \frac{kY_{t-1}}{\lambda+k} \\ -\begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\end{pmatrix}\end{matrix}\right\} dw . \quad (7)$$

If $Y_t \sim Exp(\alpha)$ and $f(y) = \frac{1}{\alpha}e^{\frac{-y}{\alpha}}$; $y \ge 0$, then

$$H(u) = 1 + \frac{1}{\lambda+k} \bullet \int_0^b H(w)\frac{1}{\alpha} e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u}{\lambda+k} + \frac{kY_{t-1}}{\lambda+k} - \begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l x_{tl}\end{pmatrix}\right\}} dw . \quad (8)$$

Let function $F(u) = e^{\frac{(1-\lambda)u - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\end{pmatrix}}$, then we obtain

$$H(u) = 1 + \frac{F(u)}{\alpha(\lambda+k)}\int_0^b H(w) e^{\frac{-w}{\alpha(\lambda+k)}} dw \qquad ; 0 \le u \le b .$$

Let $B = \int_0^b H(w) e^{\frac{-w}{\alpha(\lambda+k)}} dw$, then $H(u) = 1 + \frac{F(u)}{\alpha(\lambda+k)} \cdot B$. Consequently, we obtain

$$H(u) = 1 + \frac{1}{\alpha(\lambda+k)} e^{\frac{(1-\lambda)u - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\end{pmatrix}} \cdot B . \quad (9)$$

By solving for constant B, we obtain

$$\begin{aligned}B &= \int_0^b H(w) e^{\frac{-w}{\alpha(\lambda+k)}} dw \\ &= \int_0^b \left[1 + \frac{B}{\alpha(\lambda+k)} F(w)\right] e^{\frac{-w}{\alpha(\lambda+k)}} dw \\ &= \int_0^b e^{\frac{-w}{\alpha(\lambda+k)}} dw + \int_0^b \frac{Be^{\frac{(1-\lambda)w - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l x_{tl}\end{pmatrix}}}{\alpha(\lambda+k)} \cdot e^{\frac{-w}{\alpha(\lambda+k)}} dw \\ &= \frac{-\alpha(\lambda+k)\left(e^{\frac{-b}{\alpha(\lambda+k)}} - 1\right)}{1 + \frac{e^{\frac{-kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\left(\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l x_{tl}\right)}}{\lambda}\left(e^{\frac{-\lambda b}{\alpha(\lambda+k)}} - 1\right)} .\end{aligned}$$

By substituting constant B into Eq. (23), we arrive at

$$H(u) = 1 + \frac{e^{\frac{(1-\lambda)u - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\left(\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\right)}}{\alpha(\lambda+k)} \bullet \left(\frac{-\lambda\alpha(\lambda+k)\left[e^{\frac{-b}{\alpha(\lambda+k)}} - 1\right]}{\lambda + e^{\frac{-kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\begin{pmatrix}\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} \\ -\theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\end{pmatrix}}\left[e^{\frac{-\lambda b}{\alpha(\lambda+k)}} - 1\right]}\right) . \quad (10)$$

Hence, the one-sided explicit formulas for the ARL on a modified EWMA control chart for an ARMAX(p,q,r) process can be derived by using the Fredholm integral equation of the second kind. Let $\alpha = \alpha_0$ for the process is in the in-control state, and $\alpha = \alpha_1$ for the process is in the out-of-control state, the one-sided explicit formulas for ARL_0 and ARL_1 can be written as follows:

$$ARL_0 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_0(\lambda+k)}}\left[e^{\frac{-b}{\alpha_0(\lambda+k)}} - 1\right]}{\lambda e^{\frac{kY_{t-1}}{\alpha_0(\lambda+k)} + \frac{1}{\alpha_0}\left(\begin{array}{l}-\omega-\phi_1 Y_{t-1}-\ldots-\phi_p Y_{t-p}\\ +\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{l=1}^{r}\beta_l X_{tl}\end{array}\right)} + \left[e^{\frac{-\lambda b}{\alpha_0(\lambda+k)}} - 1\right]}, \quad (11)$$

and

$$ARL_1 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha_1(\lambda+k)}}\left[e^{\frac{-b}{\alpha_1(\lambda+k)}} - 1\right]}{\lambda e^{\frac{kY_{t-1}}{\alpha_1(\lambda+k)} + \frac{1}{\alpha_1}\left(\begin{array}{l}-\omega-\phi_1 Y_{t-1}-\ldots-\phi_p Y_{t-p}\\ +\theta_1\varepsilon_{t-1}+\ldots+\theta_q\varepsilon_{t-q}-\sum_{l=1}^{r}\beta_l X_{tl}\end{array}\right)} + \left[e^{\frac{-\lambda b}{\alpha_1(\lambda+k)}} - 1\right]}. \quad (12)$$

C. The Existence and Uniqueness of the Explicit Formulas

Here, we show the existence and uniqueness of the solution in (8). First, we define

$$T(H(u)) = 1 + \frac{1}{\lambda+k}\int_0^b H(w)\frac{1}{\alpha} e^{-\frac{1}{\alpha}\left\{\begin{array}{l}\frac{w-(1-\lambda)u}{(\lambda+k)} + \frac{kY_{t-1}}{(\lambda+k)} - \omega - \phi_1 Y_{t-1} - \ldots\\ -\phi_p Y_{t-p} + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{l=1}^{r}\beta_l X_{tl}\end{array}\right\}} dw \quad (13)$$

Theorem 1. Banach's fixed-point theorem [23].

Let $C[0,b]$ be a set of all of the continuous functions on complete metric (X,d), and assume that $T: X \to X$ is a contraction mapping with contraction constant $0 \le s < 1$; i.e., $\|T(H_1) - T(H_2)\| \le s\|H_1 - H_2\| \ \forall H_1, H_2 \in X$. Subsequently, $H(.) \in X$ is unique at $T(H(u)) = H(u)$; i.e., it has a unique fixed point in X.

Proof: To show that T defined in (13) is a contraction mapping for $H_1, H_2 \in C[0,b]$, we use the inequality $\|T(H_1) - T(H_2)\| \le s\|H_1 - H_2\| \ \forall H_1, H_2 \in C(0,b)$ with $0 \le s < 1$. Consider (8) and (13), then

$$\|T(H_1) - T(H_2)\|_\infty = \sup_{u\in[0,b]}\left|\frac{F(u)}{\alpha(\lambda+k)}\int_0^b (H_1(w) - H_2(w)) e^{\frac{-w}{\alpha(\lambda+k)}} dw\right|$$

$$\le \sup_{u\in[0,b]}\left|\|H_1 - H_2\|_\infty F(u)\left(1 - e^{\frac{-b}{\alpha(\lambda+k)}}\right)\right|$$

$$= \|L_1 - L_2\|_\infty \left|1 - e^{\frac{-b}{\alpha(\lambda+k)}}\right| \sup_{u\in[0,b]}|F(u)|$$

$$\le s\|L_1 - L_2\|_\infty,$$

where $s = \left|1 - e^{\frac{-b}{\alpha(\lambda+k)}}\right| \sup_{u\in[0,b]}|F(u)|$ and

$$F(u) = e^{\frac{(1-\lambda)u - kY_{t-1}}{\alpha(\lambda+k)} + \frac{1}{\alpha}\left(\omega + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} - \theta_1\varepsilon_{t-1} - \ldots - \theta_q\varepsilon_{t-q} + \sum_{l=1}^{r}\beta_l X_{tl}\right)}; \ 0 \le s < 1.$$

Therefore, as confirmed by applying Banach's fixed-point theorem, the solution exists and is unique.

IV. THE NIE FOR THE ARL OF THE PROCESS

The NIE approach is widely used for evaluating the ARL. It can be based on one of various quadrature rules (midpoint, trapezoidal, Simpson's, and Gauss-Legendre), all of which give ARLs that are very close to each other [24]. In the present study, we use the Gauss-Legendre rule to evaluate the ARL. The Fredholm integral equation of the second kind for the ARL for the ARMAX(p,q,r) process running on the modified EWMA control chart in (10) can be evaluated using the quadrature formula. We apply the Gauss-Legendre rule as follows:

Given that

$$f(a_j) = f\left\{\begin{array}{l}\frac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \frac{kY_{t-1}}{(\lambda+k)} - \omega - \phi_1 Y_{t-1} - \ldots\\ -\phi_p Y_{t-p} + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{l=1}^{r}\beta_l X_{tl}\end{array}\right\}, \quad (14)$$

the estimation for the integral equation by using Gauss-Legendre rule is in the form

$$\int_0^b H(w) f(w) dw \approx \sum_{j=1}^{m} w_j f(a_j), \quad (15)$$

where $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right)$ and $w_j = \frac{b}{m}; j = 1, 2, \ldots, m$.

The numerical approximation $\tilde{H}(u)$ for the integral equations can be found as the solution to the following equations:

$$\tilde{H}(a_i) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{H}(a_j)\bullet f\left\{\begin{array}{l}\frac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \frac{kY_{t-1}}{(\lambda+k)} - \omega - \phi_1 Y_{t-1} - \ldots\\ -\phi_p Y_{t-p} + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{l=1}^{r}\beta_l X_{tl}\end{array}\right\}$$

$$\tilde{H}(a_m) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{H}(a_j)\bullet f\left\{\begin{array}{l}\frac{a_j - (1-\lambda)a_m}{(\lambda+k)} + \frac{kY_{t-1}}{(\lambda+k)} - \omega - \phi_1 Y_{t-1} - \ldots\\ -\phi_p Y_{t-p} + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{l=1}^{r}\beta_l X_{tl}\end{array}\right\}.$$

This set of m equations with m unknowns can be rewritten in matrix form. The column vector of $\tilde{H}(a_i)$ is $\mathbf{L}_{m\times 1} = \left(\tilde{H}(a_1), \tilde{H}(a_2), \ldots, \tilde{H}(a_m)\right)'$. Since $\mathbf{1}_{m\times 1} = (1, 1, \ldots, 1)'$ is a column vector of ones and $\mathbf{R}_{m\times m}$ is a matrix, we can define m to m^{th} composition of matrix $\mathbf{R}$ as follows:

$$\left[R_{ij}\right] \approx \frac{1}{\lambda+k} w_j f\left\{\begin{array}{l}\frac{a_j - (1-\lambda)a_i}{(\lambda+k)} + \frac{kY_{t-1}}{(\lambda+k)} - \omega - \phi_1 Y_{t-1} - \ldots\\ -\phi_p Y_{t-p} + \theta_1\varepsilon_{t-1} + \ldots + \theta_q\varepsilon_{t-q} - \sum_{l=1}^{r}\beta_l X_{tl}\end{array}\right\},$$

and $\mathbf{I}_m = diag(1,1,...,1)$ as a unit matrix order m. If $(\mathbf{I}-\mathbf{R})^{-1}$ exists, the numerical approximation for the integral equation in matrix terms can be written as $\mathbf{G}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}$.

Finally, by substituting a_i with u in $\tilde{H}(a_i)$, the numerical integration equation for function $\tilde{H}(u)$ can be obtained as

$$\tilde{H}(u) = 1 + \frac{1}{\lambda+k}\sum_{j=1}^{m} w_j \tilde{H}(a_j)\bullet f\left\{\begin{array}{l}\dfrac{a_j-(1-\lambda)u}{(\lambda+k)} + \dfrac{kY_{t-1}}{(\lambda+k)} - \omega - \phi_1 Y_{t-1} - ... \\ -\phi_p Y_{t-p} + \theta_1\varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q} - \sum_{l=1}^{r}\beta_l X_{tl}\end{array}\right\}. \tag{16}$$

V. Comparison of the Efficacies of the NIE and Explicit Formulas Methods

Here, the details of the simulation study to compare the efficacies for the ARL on the modified EWMA control chart of an ARMAX(p,q,r) process for the explicit formulas $(H(u))$ and the NIE method $(\tilde{H}(u))$ are provided. The parameter of the modified EWMA control chart ($\lambda = 0.05$, 0.1 and $k=1$) and ARMAX(p,q,r) process with the in-control process $\alpha_0 = 1$; where the shift size (δ) varied as 0.001, 0.003, 0.005, 0.007, 0.01, 0.03, 0.05, 0.07, 0.1, 0.3, or 0.5. given $ARL_0 = 370$. The absolute percentage difference between the ARL methods is defined as

$$Diff(\%) = \frac{\left|H(u) - \tilde{H}(u)\right|}{H(u)} \times 100. \tag{17}$$

Equations (10) and (16) were used to evaluate the ARL on the modified EWMA control chart for the

TABLE I

Comparison of the ARL for an ARMAX(1,1,1) Process on the Modified EWMA Control Chart by Using Explicit Formulas and the NIE Method with $\hat{\omega} = 2$.

λ	δ	$\hat{\phi}=0.1,\ \hat{\theta}=-0.1,\ \hat{\beta}=0.1$ and $b=0.546791$				$\hat{\phi}=0.2,\ \hat{\theta}=-0.2,\ \hat{\beta}=0.2$ and $b=0.404322$			
		Explicit	NIE	Time[a]	Diff%	Explicit	NIE	Time	Diff%
0.05	0.00	370	370	10.812	0.00000000	370	370	10.531	0.00000000
	0.001	229.904260	229.904259	11.297	0.00000041	221.949849	221.949848	10.531	0.00000022
	0.003	130.988167	130.988167	12.750	0.00000036	123.419607	123.419607	10.751	0.00000019
	0.005	91.687880	91.687879	11.484	0.00000034	85.574008	85.574008	11.531	0.00000018
	0.007	70.589029	70.589028	11.625	0.00000032	65.549877	65.549876	11.468	0.00000017
	0.01	52.539577	52.539577	10.979	0.00000031	48.579748	48.579748	11.063	0.00000017
	0.03	19.696613	19.696612	10.750	0.00000027	18.073970	18.073970	10.687	0.00000015
	0.05	12.302525	12.302525	11.328	0.00000025	11.273029	11.273029	10.609	0.00000014
	0.07	9.042091	9.042091	10.593	0.00000023	8.282850	8.282850	10.469	0.00000013
	0.10	6.563506	6.563506	12.156	0.00000021	6.014209	6.014209	10.467	0.00000011
	0.30	2.688419	2.688419	11.328	0.00000011	2.483831	2.483831	11.687	0.00000006
	0.50	1.937998	1.937998	10.656	0.00000007	1.808533	1.808533	11.656	0.00000003
		$b=0.550849$				$b=0.4066401$			
0.1	0.00	370	370	10.438	0.00000000	370	370	11.188	0.00000000
	0.001	227.026409	227.026407	11.344	0.00000076	218.788511	218.788510	11.078	0.00000040
	0.003	128.207062	128.207061	11.531	0.00000055	120.550648	120.550647	11.031	0.00000028
	0.005	89.435741	89.435741	10.437	0.00000046	83.301325	83.301325	10.828	0.00000024
	0.007	68.734285	68.734285	10.407	0.00000042	63.699735	63.699735	10.687	0.00000022
	0.01	51.087218	51.087218	10.703	0.00000038	47.145277	47.145277	11.156	0.00000020
	0.03	19.122925	19.122925	11.188	0.00000029	17.518459	17.518459	10.969	0.00000016
	0.05	11.952362	11.952362	11.281	0.00000026	10.936191	10.936191	11.531	0.00000014
	0.07	8.793327	8.793327	11.156	0.00000024	8.044608	8.044608	11.313	0.00000013
	0.10	6.392839	6.392839	10.860	0.00000021	5.851618	5.851618	11.484	0.00000011
	0.30	2.640331	2.640331	11.562	0.00000011	2.439131	2.439131	11.078	0.00000005
	0.50	1.912903	1.912903	10.906	0.00000006	1.785612	1.785612	11.047	0.00000003

[a]The calculations for the NIE method are based on Windows 10 Professional with an Intel Core i5 CPU with number of nodes 1000 iterations

TABLE II

Comparison of the ARL for an ARMAX(1,2,2) Process on the Modified EWMA Control Chart by Using Explicit Formulas and the NIE Method with $\hat{\omega} = 2$, $\hat{\phi} = 0.1$.

λ	δ	$\hat{\theta}_1=-0.1,\ \hat{\theta}_2=0.2,\ \hat{\beta}_1=0.1,\ \hat{\beta}_2=-0.1$ and $b=0.739943.$				$\hat{\theta}_1=-0.3,\ \hat{\theta}_2=0.2,\ \hat{\beta}_1=0.1,\ \hat{\beta}_2=-0.2$ and $b=0.6689122$			
		Explicit	NIE	Time	Diff%	Explicit	NIE	Time	Diff%
0.05	0.00	370	370	10.937	0.00000000	370.000000	370.000000	11.515	0.00000000
	0.001	238.302931	238.302929	11.672	0.00000078	235.456965	235.456964	12.063	0.00000063
	0.003	139.365880	139.365879	11.703	0.00000067	136.474732	136.474731	11.828	0.00000055
	0.005	98.588803	98.588802	12.156	0.00000063	96.190221	96.190221	11.875	0.00000051
	0.007	76.337605	76.337605	11.390	0.00000060	74.331849	74.331849	12.36	0.00000049
	0.01	57.098633	57.098632	11.281	0.00000058	55.502639	55.502639	12.078	0.00000047
	0.03	21.598196	21.598196	11.625	0.00000051	20.928321	20.928321	11.406	0.00000041
	0.05	13.515006	13.515005	11.110	0.00000047	13.087170	13.087170	11.344	0.00000038
	0.07	9.938871	9.938871	11.235	0.00000043	9.622141	9.622141	12.016	0.00000035
	0.10	7.214336	7.214336	11.678	0.00000039	6.984252	6.984252	11.532	0.00000032
	0.30	2.934234	2.934234	11.969	0.00000021	2.846976	2.846976	12.344	0.00000017
	0.50	2.095283	2.095283	11.781	0.00000012	2.039270	2.039270	11.813	0.00000010
		$b=0.747141$				$b=0.6748496$			
0.1	0.00	370	370	11.765	0.00000000	370.000000	370.000000	12.031	0.00000000
	0.001	235.720048	235.720045	11.594	0.00000146	232.736927	232.736925	11.875	0.00000117
	0.003	136.733055	136.733053	11.453	0.00000106	133.772024	133.772023	11.328	0.00000085
	0.005	96.410183	96.410182	11.891	0.00000089	93.973650	93.973649	10.906	0.00000072
	0.007	74.522127	74.522127	11.906	0.00000080	72.493444	72.493443	11.297	0.00000064
	0.01	55.662295	55.662295	11.891	0.00000072	54.054049	54.054049	11.703	0.00000058
	0.03	21.018871	21.018871	11.469	0.00000055	20.348695	20.348695	11.563	0.00000044
	0.05	13.158986	13.158986	11.328	0.00000048	12.731843	12.731843	12.218	0.00000039
	0.07	9.684822	9.684822	11.593	0.00000044	9.368979	9.368979	11.235	0.00000036
	0.10	7.039143	7.039143	11.344	0.00000039	6.809976	6.809976	11.547	0.00000032
	0.30	2.883711	2.883711	11.594	0.00000021	2.797098	2.797098	11.938	0.00000017
	0.50	2.068491	2.068491	11.406	0.00000012	2.012954	2.012954	11.781	0.00000010

ARMAX(p,q,r) process with exponential white noise. The results are shown in Tables I and II.

The results in Tables I and II shows that the ARLs derived by the explicit formulas are close to the NIE method ($Diff(\%)$ was less than 0.00001%). However, the computational time for the NIE method was more than 10 seconds while that of the explicit formulas was less than 1 second.

VI. The ARL Results Using Various Control Charts

The performances of the ARL derived using explicit formulas for an ARMAX(p,q,r) process with exponential white noise running on the standard and modified EWMA and CUSUM control charts were compared by using the relative mean index (RMI). The ARL with the lowest value indicates the best performance. The RMI is defined as

$$RMI(r) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{ARL_i(r) - Min\left[ARL_i(s)\right]}{Min\left[ARL_i(s)\right]}\right), \qquad (18)$$

where $ARL_i(r)$ is the ARL of the control chart for the shift size in row i and $Min\left[ARL_i(s)\right]$ denotes the smallest ARL of the three control charts in comparison to the shift size in row i, for $i = 1, 2, ..., n$. The control chart with the smallest RMI is the best for a particular set of criteria.

For the simulation study with ARMAX(1,1,1) and ARMAX(2,2,2) processes, the parameter of the modified EWMA control chart ($\lambda =$ 0.05, 0.1 and $k = 1, 2, 3$) and the in-control process $\alpha_0 = 1$; where $\delta =$ 0.001, 0.003, 0.005, 0.007, 0.01, 0.03, 0.05, 0.07, 0.1, 0.3, or 0.5 and $ARL_0 = 370$. The results for the ARMAX(1,1,1) processes are reported in Table III and plot in Fig. 1 while those for the ARMAX(2,2,2) process are reported in Table IV and plot in Fig. 2.

TABLE III

Comparison of the ARL for the ARMAX(1,1,1) Process on CUSUM, Standard, and Modified EWMA Control Charts with $\hat{\omega} = 2$, $\hat{\phi} = 0.2$, and $\hat{\beta} = 0.1$.

λ	$\hat{\theta}$	δ	CUSUM	EWMA	Modified EWMA		
					k=1	*k=2*	*k=3*
			a=5, *b*=3.1466	*b*=0.00000001266	*b*=0.3339873	*b*=0.6689124	*b*=1.003798
		0.00	370	370	370	370	370
		0.001	367.719	361.985	269.720	235.469	222.534
		0.003	363.149	346.437	174.876	136.479	124.045
		0.005	358.654	331.615	129.375	96.192	86.130
		0.007	354.231	317.482	102.661	74.333	66.050
	0.2	0.01	347.731	297.502	78.381	55.503	49.021
		0.03	308.174	194.794	30.426	20.928	18.381
		0.05	274.357	129.684	18.898	13.087	11.542
		0.07	245.299	87.737	13.728	9.622	8.531
		0.10	208.949	50.275	9.766	6.984	6.242
		0.30	86.574	3.180	3.554	2.847	2.651
		0.50	45.429	1.217	2.373	2.039	1.944
0.05	**RMI**		**15.246**	**4.210**	**0.503**	**0.150**	**0.050**
			a=5, *b*=3.6681	*b*=0.000000008486	*b*=0.2232272	*b*=0.447094	*b*=0.6709292
		0.00	370	370	370	370	370
		0.001	367.627	361.830	260.325	224.551	211.263
		0.003	362.880	346.014	163.397	125.860	113.915
		0.005	358.213	330.947	119.051	87.534	78.117
		0.007	353.624	316.594	93.629	67.160	59.516
	-0.2	0.01	346.883	296.321	70.907	49.841	43.932
		0.03	305.978	192.543	27.069	18.588	16.324
		0.05	271.182	127.253	16.735	11.599	10.237
		0.07	241.426	85.494	12.128	8.523	7.567
		0.10	204.414	48.514	8.611	6.188	5.542
		0.30	82.204	2.988	3.143	2.548	2.384
		0.50	42.452	1.190	2.123	1.849	1.771
	RMI		**16.378**	**4.715**	**0.494**	**0.141**	**0.041**
			a=5, *b*=3.1466	*b*=0.00053475	*b*=0.3376842	*b*=0.6748497	*b*=1.012368
		0.00	370	370	370	370	370
		0.001	367.719	365.532	263.023	232.743	221.178
		0.003	363.149	356.751	166.655	133.774	122.783
		0.005	358.654	348.215	121.983	93.975	85.123
		0.007	354.231	339.915	96.207	72.494	65.227
	0.2	0.01	347.731	327.893	73.063	54.054	48.381
		0.03	308.174	259.296	28.126	20.349	18.130
		0.05	274.357	206.884	17.473	12.732	11.388
		0.07	245.299	166.465	12.713	9.369	8.421
		0.10	208.949	121.951	9.071	6.810	6.167
		0.30	86.574	22.409	3.367	2.797	2.629
		0.50	45.429	6.907	2.280	2.013	1.932
0.1	**RMI**		**14.249**	**7.812**	**0.374**	**0.086**	**0.000**
			a=5, *b*=3.6681	*b*=0.0003582	*b*=0.2251526	*b*=0.4498835	*b*=0.674842
		0.00	370	370	370	370	370
		0.001	367.627	365.432	253.108	221.486	209.568
		0.003	362.880	356.357	155.096	123.016	112.452
		0.005	358.213	347.543	111.810	85.264	76.981
		0.007	353.624	338.980	87.418	65.305	58.601
	-0.2	0.01	346.883	326.589	65.872	48.398	43.229
		0.03	305.978	256.213	24.965	18.026	16.055
		0.05	271.182	202.873	15.446	11.257	10.073
		0.07	241.426	162.053	11.216	8.281	7.451
		0.10	204.414	117.499	7.992	6.022	5.463
		0.30	82.204	20.484	2.982	2.502	2.361
		0.50	42.452	6.160	2.044	1.825	1.759
	RMI		**15.620**	**8.608**	**0.378**	**0.087**	**0.000**

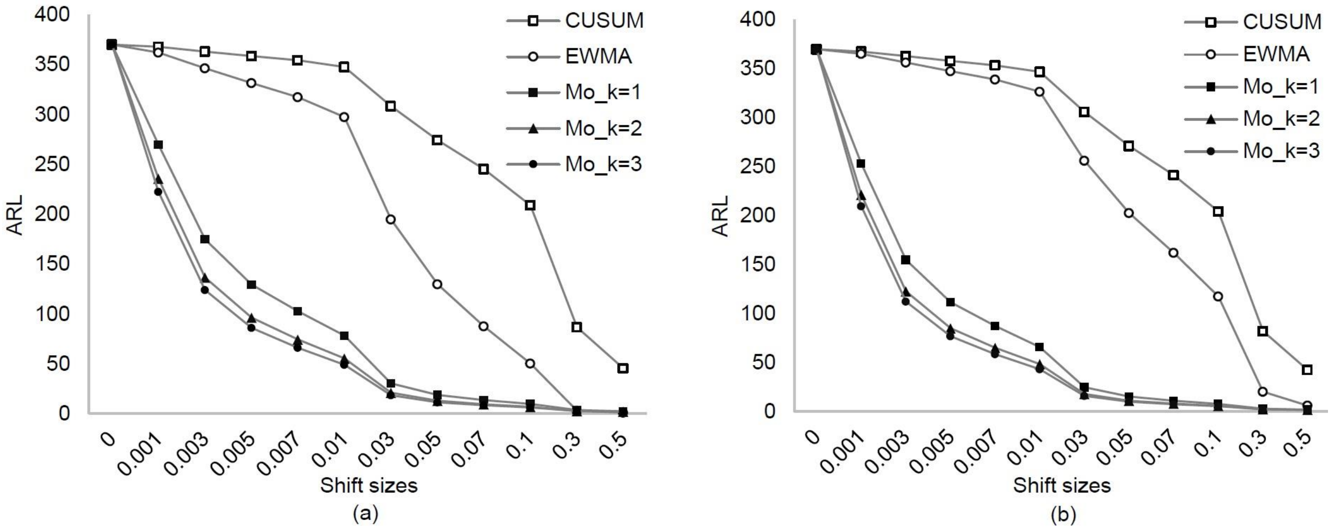

Fig. 1. The ARL for an ARMAX(1,1,1) process running on CUSUM, standard, and modified EWMA control charts. (a) The ARL for $\lambda = 0.05$ and $\hat{\theta} = 0.2$ and (b) the ARL for $\lambda = 0.1$ and $\hat{\theta} = -0.2$.

TABLE IV

COMPARISON OF THE ARL FOR THE ARMAX(2,2,2) PROCESS ON CUSUM, STANDARD, AND MODIFIED EWMA CONTROL CHARTS WITH $\hat{\omega} = 2$, $\hat{\phi}_1 = 0.1$, $\hat{\phi}_2 = 0.2$, $\hat{\theta}_2 = 0.2$, $\hat{\beta}_1 = 0.1$ AND $\hat{\beta}_2 = 0.1$.

λ	$\hat{\theta}_1$	δ	CUSUM	EWMA	Modified EWMA		
					k=1	*k=2*	*k=3*
			a=5, *b*=3.0274	*b*=0.000000013991	*b*=0.369456	*b*=0.739944	*b*=1.11039
		0.00	370	370	370	370	370
		0.001	367.767	362.009	272.145	238.359	225.506
		0.003	363.227	346.529	177.954	139.385	126.825
		0.005	358.761	331.768	132.196	98.598	88.363
		0.007	354.367	317.692	105.156	76.343	67.885
	0.3	0.01	347.907	297.786	80.466	57.102	50.461
		0.03	308.582	195.352	31.382	21.599	18.971
		0.05	274.934	130.294	19.517	13.515	11.917
		0.07	245.998	88.304	14.187	9.939	8.809
		0.10	209.765	50.724	10.098	7.214	6.445
		0.30	87.380	3.231	3.674	2.934	2.729
		0.50	45.994	1.224	2.447	2.095	1.995
0.05	**RMI**		**14.922**	**4.082**	**0.505**	**0.153**	**0.052**
			a=5, *b*=3.8159	*b*=0.000000007679	*b*=0.2018705	*b*=0.4043218	*b*=0.6067437
		0.00	370	370	370	370	370
		0.001	367.599	361.820	258.046	221.933	208.579
		0.003	362.787	345.935	160.712	123.414	111.594
		0.005	358.056	330.807	116.679	85.572	76.309
		0.007	353.405	316.397	91.576	65.548	58.054
	-0.3	0.01	346.575	296.050	69.225	48.579	42.801
		0.03	305.176	192.000	26.329	18.074	15.872
		0.05	270.024	126.662	16.261	11.273	9.951
		0.07	240.019	84.949	11.778	8.283	7.356
		0.10	202.776	48.088	8.360	6.014	5.390
		0.30	80.684	2.943	3.054	2.484	2.326
		0.50	41.449	1.184	2.069	1.809	1.734
	RMI		**16.600**	**4.841**	**0.492**	**0.139**	**0.039**
			a=5, *b*=3.0274	*b*=0.0005912	*b*=0.373837	*b*=0.747141	*b*=1.120839
		0.00	370	370	370	370	370
		0.001	367.767	365.602	265.607	235.720	224.244
		0.003	363.227	356.894	169.783	136.733	125.634
		0.005	358.761	348.426	124.787	96.410	87.407
		0.007	354.367	340.192	98.655	74.522	67.102
	0.3	0.01	347.907	328.261	75.086	55.662	49.849
		0.03	308.582	260.106	29.033	21.019	18.730
		0.05	274.934	207.926	18.057	13.159	11.769
		0.07	245.998	167.609	13.145	9.685	8.703
		0.10	209.765	123.107	9.384	7.039	6.372
		0.30	87.380	22.923	3.480	2.884	2.708
		0.50	45.994	7.112	2.350	2.068	1.983
0.1	**RMI**		**13.864**	**7.608**	**0.373**	**0.086**	**0.000**
			a=5, *b*=3.8159	*b*=0.00032407	*b*=0.2035172	*b*=0.4066402	*b*=0.609968
		0.00	370	370	370	370	370
		0.001	367.599	365.406	250.711	218.797	206.830
		0.003	362.787	356.258	152.411	120.553	110.105
		0.005	358.056	347.374	109.489	83.303	75.158
		0.007	353.405	338.745	85.435	63.700	57.129
	-0.3	0.01	346.575	326.263	64.264	47.146	42.092
		0.03	305.176	255.448	24.272	17.519	15.602
		0.05	270.024	201.883	15.003	10.936	9.788
		0.07	240.019	160.969	10.890	8.045	7.241
		0.10	202.776	116.413	7.758	5.852	5.311
		0.30	80.684	20.031	2.900	2.439	2.304
		0.50	41.449	5.988	1.994	1.786	1.722
	RMI		**15.902**	**8.802**	**0.379**	**0.087**	**0.000**

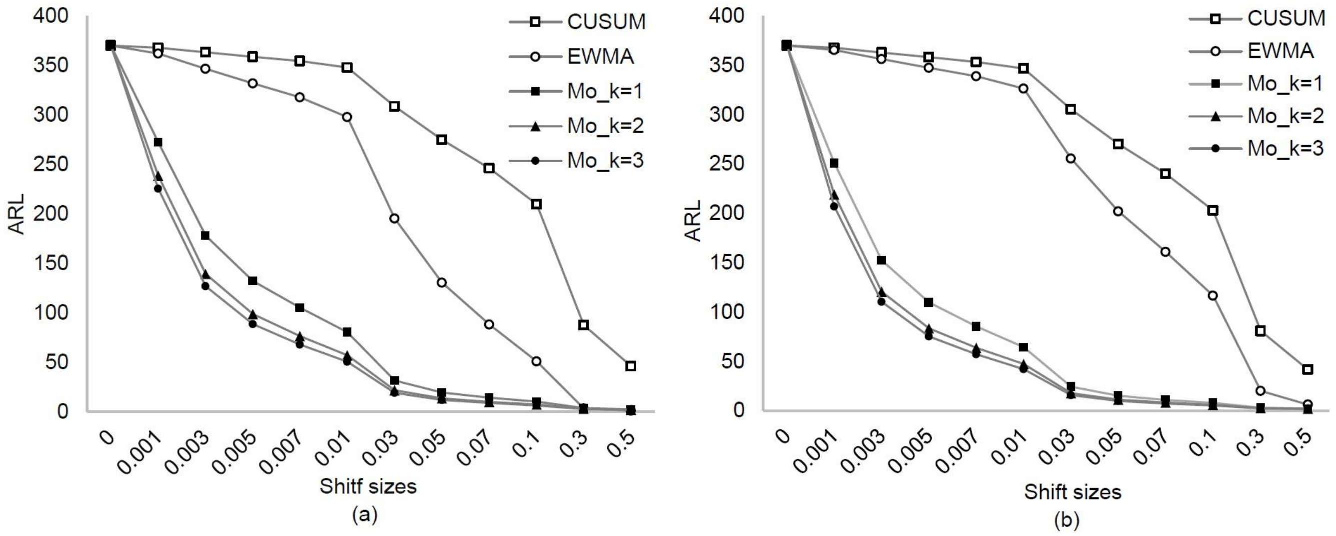

Fig. 2. The ARL for an ARMAX(2,2,2) process running on CUSUM, standard, and modified EWMA control charts. (a) The ARL for $\lambda = 0.05$ where $\hat{\theta}_1 = 0.3$ and (b) the ARL for $\lambda = 0.1$ where $\hat{\theta}_1 = -0.3$.

According to Tables III and IV and Figs. 1 and 2, it is evident from the results that the ARL values for the explicit formulas method on the modified EWMA control chart were lower than those for the standard EWMA and CUSUM control charts for all λ , shift sizes and values of constants k, and thus its RMI values were lower.

VII. Practical Applications with Real Data

We applied the explicit formulas for the ARL of an ARMAX(1,1,1) process using 72 real data observations of the price of gasoline (Unit: USD per barrel [25]) and crude oil (Unit: USD per gallon [26]) from January 2015 to December 2020, with the latter being the explanatory variable, on CUSUM, and standard and modified EWMA control charts. The parameters were set as $\lambda = 0.05$; the various parameter values listed in Tables V and VI; and a shift size of 0.001, 0.003, 0.005, 0.007, 0.01, 0.03, 0.05, 0.07, 0.1, 0.3, or 0.5. The results are summarized in Table VII.

We also performed another comparison for the ARL of an ARMAX(1,2,1) process involving 72 real-world data observations of the exchange rate of 100 JPY to THB from January 2015 to December 2020, and with the USD to THB exchange rate over the same time period as the explanatory variable [27] on CUSUM, and standard and modified EWMA control charts. The parameters were set as $\lambda = 0.05$; the various parameter values listed in Tables V and VI; and the same shift size as for the ARMAX(1,1,1) process. The results are summarized in Table VIII.

From Tables VII and VIII, it can be seen that the ARL values obtained from the explicit formulas running on the modified EWMA control chart were less than those for the CUSUM and standard EWMA control charts for all shift sizes and all values of *k*. Furthermore, as k increased, ARL_1 and the RMI decreased. Because the ARL of the CUSUM control chart is very different from EWMA and modified EWMA control chart, we compared the detection of shifts in the process means for the ARMAX(1,1,1) and ARMAX(1,2,1) processes with real data on the two types of EWMA control charts only, the results for which are displayed in Figs. 3 and 4, respectively.

TABLE V
Fitting Statistics for the Real-World Datasets to ARMAX(1,1,1) and ARMAX(1,2,1) models.

Data	Variable	Coefficient	Std. Error	t	Sig.
Gasoline (ARMAX(1,1,1))	Constant ($\hat{\omega}$)	0.2345	0.0845	2.7740	0.0072
	AR(1) ($\hat{\phi}$)	0.5347	0.1367	3.9107	0.0002
	MA(1) ($\hat{\theta}$)	0.3361	0.1535	2.1904	0.0320
	Crude Oil ($\hat{\beta}$)	0.0267	0.0016	16.7076	0.0000
JPY (ARMAX(1,2,1))	Constant ($\hat{\omega}$)	15.4721	4.4510	3.4761	0.0009
	AR(1) ($\hat{\phi}$)	0.8267	0.0783	10.5550	0.0000
	MA(1) ($\hat{\theta}_1$)	0.3982	0.1323	3.0107	0.0037
	MA(2) ($\hat{\theta}_2$)	0.2744	0.1306	2.1009	0.0395
	USD ($\hat{\beta}$)	0.4365	0.1348	3.2378	0.0019

TABLE VI
Checking that Exponential Distributions Fit the White Noise of the Real-World Datasets

Data	Mean (α_0)	Kolmogorov-Smirnov Z	Sig.
Gasoline (ARMAX(1,1,1))	0.0567	0.8942	0.4008
JPY (ARMAX(1,2,1))	0.3331	0.4560	0.9854

TABLE VII
COMPARISON OF THE ARL FOR THE ARMAX(1,1,1) PROCESS FOR REAL DATA RUNNING ON CUSUM, STANDARD, AND MODIFIED EWMA CONTROL CHARTS.

λ	δ	CUSUM	EWMA	Modified EWMA		
				k=1	*k=2*	*k=3*
		a=60, b=45.458	b=1.989×10^{-16}	b=0.021604	b=0.044232	b=0.0668341
	0.00	370	370	370	370	370
	0.001	367.811	357.104	313.520	267.902	247.382
	0.003	363.379	331.803	239.977	172.499	148.884
	0.005	359.017	308.387	194.209	127.224	106.576
	0.007	354.725	286.710	162.982	100.788	83.047
0.05	0.01	348.413	256.956	131.168	76.855	62.445
	0.03	309.928	126.171	56.204	29.841	23.787
	0.05	276.914	63.833	35.256	18.587	14.867
	0.07	248.452	33.382	25.446	13.543	10.907
	0.10	212.704	13.500	17.761	9.676	7.885
	0.30	90.643	1.067	5.565	3.595	3.132
	0.50	48.538	1.001	3.314	2.423	2.202
RMI		**18.415**	**1.625**	**1.301**	**0.470**	**0.261**

TABLE VIII
COMPARISON OF THE ARL FOR THE ARMAX(1,2,1) PROCESS FOR REAL DATA RUNNING ON CUSUM, STANDARD, AND MODIFIED EWMA CONTROL CHARTS.

λ	δ	CUSUM	EWMA	Modified EWMA		
				k=1	*k=2*	*k=3*
		a=10, b=8.6369	b= 3.975×10^{-13}	b=0.231849	b=0.470822	b=0.709694
	0.00	370	370	370	370	370
	0.001	367.750	358.642	313.296	275.493	259.122
	0.003	363.261	336.976	239.605	182.407	162.178
	0.005	358.844	316.701	193.887	136.406	118.143
	0.007	354.498	297.720	162.756	108.976	92.989
0.05	0.01	348.109	271.490	131.089	83.770	70.560
	0.03	309.185	148.934	56.649	33.183	27.415
	0.05	275.844	83.758	35.868	20.849	17.252
	0.07	247.141	48.296	26.123	15.288	12.712
	0.10	211.152	22.215	18.468	11.006	9.232
	0.30	89.012	1.257	6.149	4.191	3.703
	0.50	47.332	1.010	3.771	2.836	2.591
RMI		**15.977**	**1.724**	**1.147**	**0.472**	**0.293**

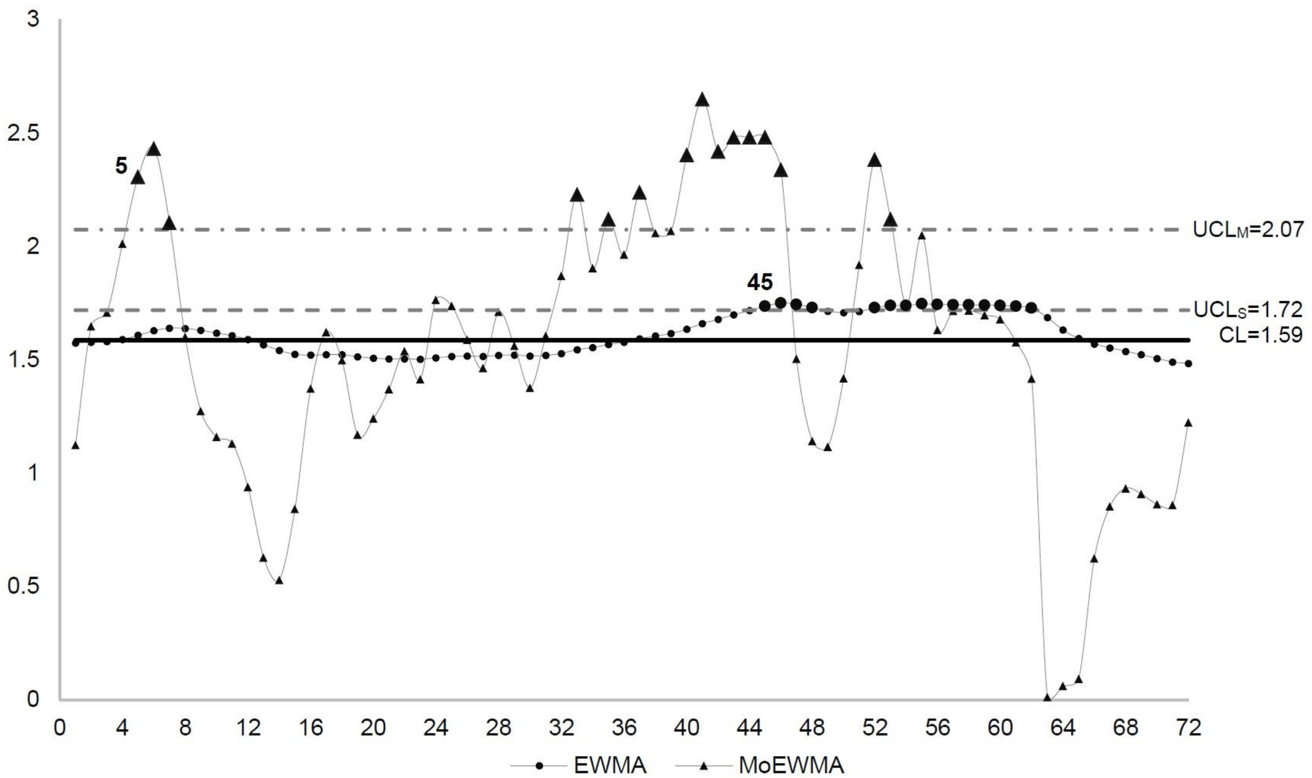

Fig. 3. Mean shift detection for the ARMAX(1,1,1) process for the price of gasoline and crude oil is explanatory variable.

The results in Fig. 3 display that the modified EWMA control chart was able to detect a change in the price of gasoline for the first time at the 5th observation while the standard EWMA control chart achieved this at the 45th observation.

The results in Fig. 4 indicate that the modified EWMA control chart can be detect a change in the exchange rate of

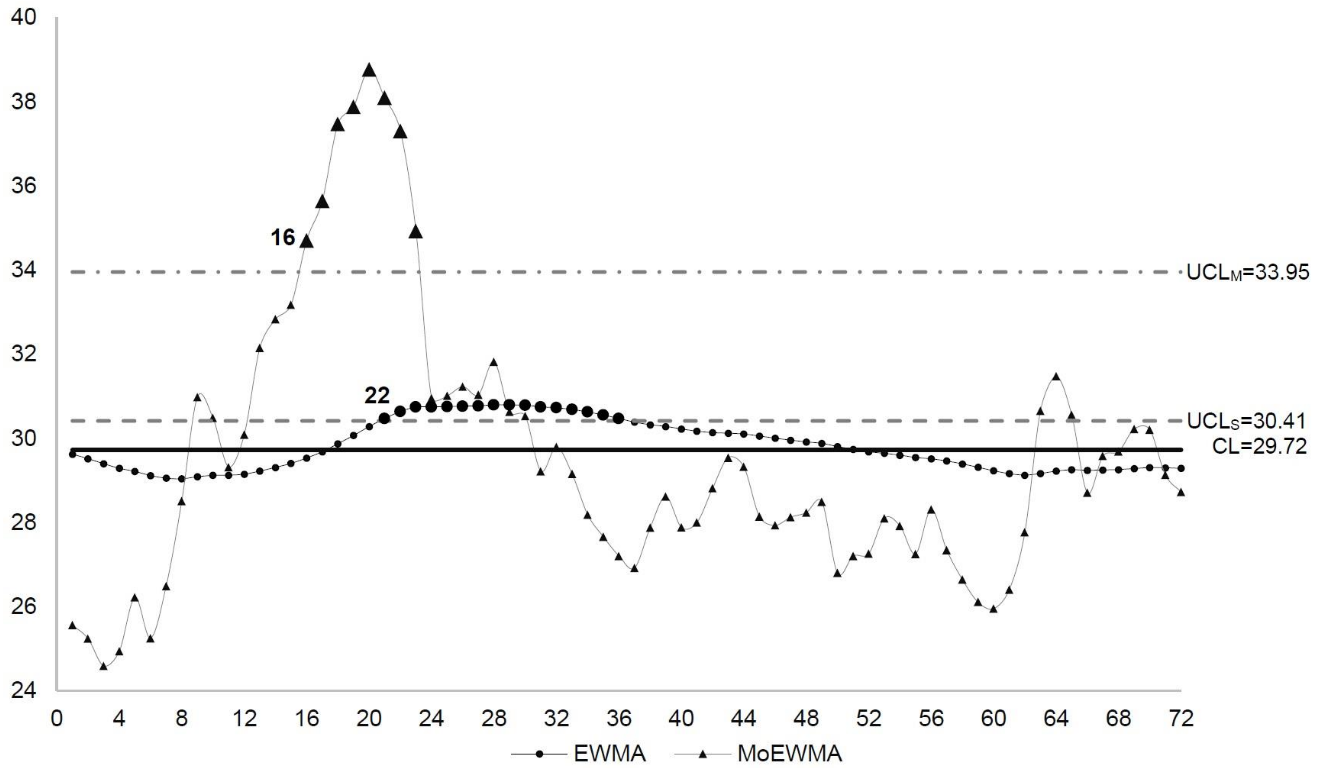

Fig. 4. Mean shift detection for the ARMAX(1,2,1) process for the exchange rate of Japanese Yen and US Dollars is explanatory variable.

JPY for the first time at the 16th observation while the standard EWMA control chart achieved this at the 22nd observation. Hence, in both cases, the performance of the modified EWMA control chart is better than of the standard EWMA control chart for detecting shifts in the process mean, thus the former is more efficient than the latter.

VIII. Conclusions

We derived explicit formulas for the ARL of the modified EWMA control chart for an ARMAX(p,q,r) process. and used simulated data to check its accuracy by comparing it with the ARL derived from the NIE method by using an absolute percentage difference . The results indicate that although both methods yielded very close ARL values with an absolute percentage difference of less than 0.00001%, the explicit formula method took much less time to calculate them. A comparison of the ARL derived by using explicit formulas for the ARL of an ARMAX(p,q,r) process with exponential white noise running on CUSUM, and standard and modified EWMA control charts, indicate that the proposed explicit formulas was more effective than on the CUSUM and standard EWMA control charts in terms of RMI. Practical application with real data for ARMAX(p,q,r) processes with exponential white noise running on the three control charts indicate that the method on the modified EWMA control chart performed much better than on the other two for a one-sided shift. In addition, as k increased, its ARL_1 and the RMI decreased. Based on the findings, the ARL derived by using explicit formulas of an ARMAX(p,q,r) process with exponential white noise running on the modified EWMA control chart were the most efficient.

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