Dynamic Behaviors of a Discrete Commensal Symbiosis Model with Holling Type Functional Response

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***Abstract*—In this paper, we investigate the dynamics of a discrete commensal symbiosis model with Holling type functional response and density dependent birth rate. Sufficient conditions are obtained for the permanence, partial extinction, and global attractivity of the system. Our analysis presents that the density dependent birth rate plays a crucial role on the dynamic behaviors of the system. Finally, numerical simulations and graphical illustrations are given to indicate the analytical results.**

***Index Terms*—Discrete, Commensal Symbiosis; Density Dependent Birth Rate, Permanence, Extinction, Global attractivity.**

I. Introduction

THE mutualism model and commensalism model have been extensively investigated by many scholars in recent years, see [1-30]. Some important conclusions have been obtained, for instance the persistent, the global attractivity, the existence of the positive periodic solutions and the stability of the positive equilibrium and so on.

It is well known that mathematical ecological model with some kind of functional response is more suitable. Wu et al. [8] incorporated the following model of commensal symbiosis with functional response of Holling type:

$$\begin{aligned} x'(t) &= x\big(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\big), \\ y'(t) &= y\big(a_2 - b_2 y\big), \end{aligned} \tag{1.1}$$

where the constants $a_i, b_i, i=1,2, p$ and c_1 are all positive, and $p \geq 1$. They obtained sufficient conditions which ensure the unique globally stable positive of equilibrium system (1.1).

Further, Wu and Lin [9] considered the commensalism system which incorporate ratio-dependent functional response:

$$\begin{aligned} x'(t) &= x\big(-a_1 - b_1 x + \frac{c_1 y}{x+y}\big), \\ y'(t) &= y\big(a_2 - b_2 y\big). \end{aligned} \tag{1.2}$$

For the autonomous case and non-autonomous case, they got the results of partial extinction and other dynamic behaviors.

Brauer and Castillo-Chavez[27], Tang and Chen[28], Berezansky, Braverman and Idels[29] had presented that the species is more appropriate which incorporate density dependent birth rate in some cases. Recently, Chen [10] proposed the functional response of Holling type to the two species autonomous commensal symbiosis model with density dependent birth rate as follows:

$$\begin{aligned} \frac{dx}{dt} &= x\big(\frac{a_{11}}{a_{12}+a_{13}x} - a_{14} - b_1 x + \frac{c_1 y^p}{1+y^p}\big), \\ \frac{dy}{dt} &= y\big(a_2 - b_2 y\big), \end{aligned} \tag{1.3}$$

where $a_{1j}, b_i, i=1,2, j=1,2,3,4$ and a_2, c_1 are continuous functions whose lower and upper bounds are positive, and the constant $p \geq 1$. For the autonomous case, his study focused on the global and local stability of the positive equilibrium. And he showed that system (1.3) admits the persistent property and extinction, owns positive periodic solution, for non-autonomous case.

Stimulated by the works of [8], [10], and since the environment is vary with seasonal, we establish the following discrete time version of system (1.3) :

$$\begin{aligned} x_1(n+1) = x_1(n)\exp\{&\frac{a_{11}(n)}{a_{12}(n)+a_{13}(n)x_1(n)} \\ &-a_{14}(n) - b_1(n)x_1(n) \\ &+\frac{c_1(n)x_2^p(n)}{1+x_2^p(n)}\} \\ x_2(n+1) = x_2(n)\exp\{&a_2(n) - b_2(n)x_2(n)\} \end{aligned} \tag{1.4}$$

where $x_1(n)$, $x_2(n)$ are separately the population density of species x_1 and x_2 at the $n-$th generation, $\frac{a_{11}(n)}{a_{12}(n)+a_{13}(n)x_1(n)}$ and $a_{14}(n)$ are the birth rate and death rate of the species x_1 respectively. The coefficients $a_{ij}(n), b_i(n), i=1,2, j=1,2,3,4$ and $a_2(n), c_1(n)$ are all bounded nonnegative sequences, $p \geq 1$ is a positive constant.

Our main objective is to study the discrete model of commensal symbiosis. The paper is organized as follows. In Section 2, we discuss the persistent property and extinction, and the global attractivity is investigated in Section 3. Then, we give two examples with computer simulations and a brief conclusion.

For biological reasons, we base on the following initial conditions $x_1(0) > 0$, $x_2(0) > 0$ and we can immediately see that the solutions of system (1.4) are positive.

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II. Extinction and Permanence

We denote

$$h^l = \inf_{n\in N}\{h(n)\}, \quad h^u = \sup_{n\in N}\{h(n)\},$$

for any $\{h(n)\}$ which is bounded sequence defined on Z.

Lemma 2.1.([24])

(1) *Assume that sequences $a(n)$ and $b(n)$ are nonnegative and the upper and lower bounds are positive constants, if $\{x(n)\}$ satisfies*

$$x(n+1) \leq x(n)\exp\{a(n) - b(n)x(n)\}, \quad n \in N,$$

where $x(n) > 0$. Then, we have

$$\limsup_{n\to+\infty} x(n) \leq \frac{1}{b^l}\exp(a^u - 1).$$

(2) *Suppose that nonnegative sequences $a(n)$ and $b(n)$ bounded above and below by positive constants, and $\{x(n)\}$ satisfies $\limsup\limits_{n\to+\infty} x(n) \leq x^*$, $x(N_0) > 0$, and*

$$x(n+1) \geq x(n)\exp\{a(n) - b(n)x(n)\}, \quad n \geq N_0,$$

where $N_0 \in N$. That, we have

$$\liminf_{n\to+\infty} x(n) \geq \min\{\frac{a^l}{b^u}\exp(a^l - b^u x^*), \frac{a^l}{b^u}\}.$$

Theorem 2.2. *If (H_1)*

$$\frac{a_{11}^u}{a_{12}^l} - a_{14}^l + \frac{c_1^u M_2^p}{1+M_2^p} < 0$$

holds, where $M_2 = \dfrac{1}{b_2^l}\exp(a_2^u - 1)$, then for each positive solution $(x_1(n), x_2(n))^T$ of system (1.4) satisfies

$$\lim_{n\to+\infty} x_1(n) = 0,$$

$$m_2 \leq \liminf_{n\to+\infty} x_2(n) \leq \limsup_{n\to+\infty} x_2(n) \leq M_2,$$

where $m_2 = \dfrac{a_2^l}{b_2^u}\exp(a_2^l - b_2^u M_2)$. That is, the species x_1 is extinct, and the second species x_2 is permanent.

Proof. From (H_1), for any constant $\varepsilon > 0$ small enough, we have

$$\frac{a_{11}^u}{a_{12}^l} - a_{14}^l + \frac{c_1^u(M_2+\varepsilon)^p}{1+(M_2+\varepsilon)^p} \stackrel{\text{def}}{=} -\delta < 0. \tag{2.1}$$

From the second equation of (1.4), it implies that

$$x_2(n+1) \leq x_2(n)\exp\{a_2^u - b_2^l x_2(n)\}. \tag{2.2}$$

We get from Lemma 2.1 that

$$\limsup_{n\to+\infty} x_2(n) \leq \frac{1}{b_2^l}\exp(a_2^u - 1) \stackrel{\text{def}}{=} M_2. \tag{2.3}$$

For the above ε, from (2.3), there is a positive constant N_1 such that

$$x_2(n) < M_2 + \varepsilon, \quad n \geq N_1. \tag{2.4}$$

By (2.4) and the first equation of (1.4), it immediately follows that, for any $k > N_1$,

$$\begin{aligned}\ln\frac{x_1(k+1)}{x_1(k)} &= \frac{a_{11}(k)}{a_{12}(k)+a_{13}(k)x_1(k)} - a_{14}(k) \\ &\quad -b_1(k)x_1(k) + \frac{c_1(k)x_2^p(k)}{1+x_2^p(k)} \\ &\leq \frac{a_{11}^u}{a_{12}^l} - a_{14}^l + \frac{c_1^u(M_2+\varepsilon)^p}{1+(M_2+\varepsilon)^p} \\ &= -\delta < 0\end{aligned} \tag{2.5}$$

We sum both sides of the above inequalities from $N+1$ to $n-1$, then we can get

$$\ln\frac{x_1(n)}{x_1(N+1)} < -\delta(n-N-1),$$

hence

$$x_1(n) < x_1(N+1)\exp\{-\delta(n-N-1)\} \to 0, \quad (n \to +\infty). \tag{2.6}$$

By the second equation of (1.4), we obtain

$$x_2(n+1) \geq x_2(n)\exp\{a_2^l - b_2^u x_2(n)\}. \tag{2.7}$$

According to Lemma 2.1 and (2.7), we have

$$\liminf_{n\to+\infty} x_2(n) \geq \min\{\frac{a_2^l}{b_2^u}\exp(a_2^l - b_2^u M_2), \frac{a_2^l}{b_2^u}\}. \tag{2.8}$$

By calculation, one can easily get

$$\begin{aligned}a_2^l - b_2^u M_2 &= a_2^l - b_2^u\frac{\exp(a_2^u - 1)}{b_2^l} \\ &\leq a_2^l - \exp(a_2^u - 1) \\ &\leq a_2^l - a_2^u \leq 0.\end{aligned} \tag{2.9}$$

Inequality (2.9) together with (2.8) lead to

$$\liminf_{n\to+\infty} x_2(n) \geq \frac{a_2^l}{b_2^u}\exp(a_2^l - b_2^u M_2) \stackrel{\text{def}}{=} m_2. \tag{2.10}$$

Thus from (2.3), (2.6) and (2.10), the conclusions hold. That is the proof ends.

Theorem 2.3. *Assume that (H_2)*

$$\frac{a_{11}^l}{a_{12}^u + a_{13}^u M_1} - a_{14}^u + \frac{c_1^l m_2^p}{1+m_2^p} > 0 \tag{2.11}$$

holds, then, all the positive solutions $(x_1(n), x_2(n))^T$ of (1.4) satisfy

$$m_1 \leq \liminf_{n\to+\infty} x_1(n) \leq \limsup_{n\to+\infty} x_1(n) \leq M_1,$$
$$m_2 \leq \liminf_{n\to+\infty} x_2(n) \leq \limsup_{n\to+\infty} x_2(n) \leq M_2,$$

i.e., system (1.4) is permanent, where m_1, M_1 will be defined in (2.14) and (2.22) respectively, and m_2, M_2 are defined in Theorem 2.2.

Proof. For any constant $\varepsilon_1 > 0$ small enough, it follows from (2.3) that

$$x_2(n) < M_2 + \varepsilon_1, \quad n \geq N_2 \tag{2.12}$$

where N_2 is a positive constant. For $n > N_2$, the first equation of (1.4) together with (2.12) lead to

$$\begin{aligned}x_1(n+1) \leq x_1(n)\exp\{&\frac{a_{11}^u}{a_{12}^l} - a_{14}^l - b_1^l x_1(n) \\ &+\frac{c_1^u(M_2+\varepsilon_1)^p}{1+(M_2+\varepsilon_1)^p}\}.\end{aligned} \tag{2.13}$$

By using Lemma 2.1, it yields

$$\limsup_{n\to+\infty} x_1(n) \leq \frac{1}{b_1^l}\exp\{\frac{a_{11}^u}{a_{12}^l} - a_{14}^l + \frac{c_1^u(M_2+\varepsilon_1)^p}{1+(M_2+\varepsilon_1)^p} - 1\}.$$

Setting $\varepsilon_1 \to 0$ in the above inequality, one can obtain

$$\limsup_{n\to+\infty} x_1(n) \leq \frac{1}{b_1^l}\exp\{\frac{a_{11}^u}{a_{12}^l} - a_{14}^l + \frac{c_1^u M_2^p}{1+M_2^p} - 1\} \stackrel{\text{def}}{=} M_1. \tag{2.14}$$

By condition (2.11), there exists $\varepsilon_2 > 0$ satisfying

$$\frac{a_{11}^l}{a_{12}^u + a_{13}^u(M_1+\varepsilon_2)} - a_{14}^u + \frac{c_1^l(m_2-\varepsilon_2)^p}{1+(m_2-\varepsilon_2)^p} > 0 \quad (2.15)$$

For the above ε_2, from (2.10) and (2.14), one can choose $N_3 > 0$ such that

$$x_2(n) > m_2 - \varepsilon_2, \quad x_1(n) < M_1 + \varepsilon_2, \quad n \geq N_3. \quad (2.16)$$

For $n > N_3$, the first equation of (1.4) and (2.16) yield that

$$x_1(n+1) \geq x_1(n)\exp\{A_{\varepsilon_2} - b_1^u x_1(n)\}, \quad (2.17)$$

where

$$A_{\varepsilon_2} \overset{\text{def}}{=} \frac{a_{11}^l}{a_{12}^u + a_{13}^u(M_1+\varepsilon_2)} - a_{14}^u + \frac{c_1^l(m_2-\varepsilon_2)^p}{1+(m_2-\varepsilon_2)^p}. \quad (2.18)$$

By applying Lemma 2.1 to (2.18), we derive

$$\liminf_{n\to+\infty} x_1(n) \geq \min\{\frac{A_{\varepsilon_2}}{b_1^u}\exp(A_{\varepsilon_2} - b_1^u M_1), \frac{A_{\varepsilon_2}}{b_1^u}\}. \quad (2.19)$$

Setting $\varepsilon_2 \to 0$ in (2.19), it leads to

$$\liminf_{n\to+\infty} x_1(n) \geq \min\{\frac{A}{b_1^u}\exp(A - b_1^u M_1), \frac{A}{b_1^u}\}, \quad (2.20)$$

where

$$A \overset{\text{def}}{=} \frac{a_{11}^l}{a_{12}^u + a_{13}^u M_1} - a_{14}^u + \frac{c_1^l m_2^p}{1+m_2^p}.$$

By calculation, we have

$$\begin{aligned} A - b_1^u M_1 &= \frac{a_{11}^l}{a_{12}^u + a_{13}^u M_1} - a_{14}^u + \frac{c_1^l m_2^p}{1+m_2^p} \\ &\quad -\frac{b_1^u}{b_1^l}\exp\{\frac{a_{11}^u}{a_{12}^l} - a_{14}^l + \frac{c_1^u M_2^p}{1+M_2^p} - 1\} \\ &\leq \frac{a_{11}^l}{a_{12}^u + a_{13}^u M_1} - a_{14}^u + \frac{c_1^2 m_2^p}{1+m_2^p} \\ &\quad -\{\frac{a_{11}^u}{a_{12}^l} - a_{14}^l + \frac{c_1^u M_2^p}{1+M_2^p}\} \\ &\leq 0. \end{aligned} \quad (2.21)$$

Then, (2.20) and (2.21) show that

$$\liminf_{n\to+\infty} x_1(n) \geq \frac{A}{b_1^u}\exp(A - b_1^u M_1) \overset{\text{def}}{=} m_1. \quad (2.22)$$

The proof is end.

III. Global Attractivity

By constructing a nonnegative discrete Lyapunov function, we will derive the global attractivity of the system in this part.

Theorem 3.1. *In addition to (H_2), further suppose that there exist constants $\lambda_1 > 0$ and $\lambda_2 > 0$, and the following statements hold*

$$\begin{aligned} &\lambda_1\mu_1 - \lambda_1\frac{a_{11}^u a_{13}^u}{(a_{12}^l + a_{13}^l m_1)^2} > 0, \\ &\lambda_2\mu_2 - \lambda_1\frac{c_1^u p M_2^{p-1}}{(1+m_2^p)^2} > 0, \end{aligned} \quad (H_3)$$

where $\mu_i = \min\{b_i^l, \frac{2}{M_i} - b_i^u\}$, $i = 1, 2$. Then, for any two positive solutions $(x_1(n), x_2(n))^T$ and $(x_1^*(n), x_2^*(n))^T$ of (1.4), one has

$$\begin{aligned} &\lim_{n\to+\infty}|x_1(n) - x_1^*(n)| = 0, \\ &\lim_{n\to+\infty}|x_2(n) - x_2^*(n)| = 0, \end{aligned} \quad (3.1)$$

that is, system (1.4) has global attraction.

Proof. From (H_3), there exist $\varepsilon > 0$ and $\delta > 0$ such that

$$\begin{aligned} &\lambda_1\mu_1^\varepsilon - \lambda_1\frac{a_{11}^u a_{13}^u}{(a_{12}^l + a_{13}^l(m_1-\varepsilon))^2} > \delta, \\ &\lambda_2\mu_2^\varepsilon - \lambda_1\frac{c_1^u p(M_2+\varepsilon)^{p-1}}{(1+(m_2-\varepsilon))^2} > \delta, \end{aligned} \quad (3.2)$$

where $\mu_i^\varepsilon = \min\{b_i^l, \frac{2}{M_i+\varepsilon} - b_i^u\}$, $i = 1, 2$.

For the above ε, by (H_2) and Theorem 2.3, there exists a constant $N_1 > 0$, and for all $n \geq N_1$, it has

$$\begin{aligned} &m_i - \varepsilon \leq x_i(n) \leq M_i + \varepsilon, \\ &m_i - \varepsilon \leq x_i^*(n) \leq M_i + \varepsilon, \quad i = 1, 2, \end{aligned} \quad (3.3)$$

where $(x_1(n), x_2(n))^T$ and $(x_1^*(n), x_2^*(n))^T$ are any positive solutions of system (1.4).

Let us define

$$V_1(n) = |\ln x_1(n) - \ln x_1^*(n)|.$$

From the first equation of (1.4), we can get

$$\begin{aligned} V_1(n+1) &= |\ln x_1(n+1) - \ln x_1^*(n+1)| \\ &\leq |\ln x_1(n) - \ln x_1^*(n) - b_1(n)(x_1(n) - x_1^*(n))| \\ &+ \frac{a_{11}(n)a_{13}(n)}{(a_{12}(n) + a_{13}(n)x_1(n))(a_{12}(n) + a_{13}(n)x_1^*(n))}|x_1(n) - x_1^*(n)| \\ &+ c_1(n)|\frac{|x_2^p(n) - x_2^{*p}(n)|}{(1+x_2^p(n))(1+x_2^{*p}(n))}. \end{aligned} \quad (3.4)$$

Applying the Mean Value Theorem, one has

$$\begin{aligned} &\ln x_1(n) - \ln x_1^*(n) = \tfrac{1}{\theta_1(n)}(x_1(n) - x_1^*(n)), \\ &x_2^p(n) - x_2^{*p}(n) = p\xi^{(p-1)}(n)(x_2(n) - x_2^*(n)), \end{aligned} \quad (3.5)$$

where $\theta_1(n)$ is among $x_1(n)$ and $x_1^*(n)$, $\xi(n)$ is between $x_2(n)$ and $x_2^*(n)$. (3.4) and (3.5) lead to

$$\begin{aligned} \Delta V_1(n) &\leq -(\frac{1}{\theta_1(n)} - |\frac{1}{\theta_1(n)} - b_1(n)|)|x_1(n) - x_1^*(n)| \\ &+ \frac{a_{11}(n)a_{13}(n)}{(a_{12}(n) + a_{13}(n)x_1(n))(a_{12}(n) + a_{13}(n)x_1^*(n))}|x_1(n) - x_1^*(n)| \\ &+ |\frac{c_1(n)p\xi^{(p-1)}(n)}{(1+x_2^p(n))(1+x_2^{*p}(n))}|x_2(n) - x_2^*(n)| \\ &\leq -\min\{b_1^l, \frac{2}{M_1+\varepsilon} - b_1^u\}|x_1(n) - x_1^*(n)| \\ &+ \frac{a_{11}^u a_{13}^u}{(a_{12}^l + a_{13}^l(m_1-\varepsilon))^2}|x_1(n) - x_1^*(n)| \\ &+ |\frac{c_1^u p(M_2+\varepsilon)^{p-1}}{(1+(m_2-\varepsilon))^2}|x_2(n) - x_2^*(n)|. \end{aligned} \quad (3.6)$$

Define

$$V_2(n) = |\ln x_2(n) - \ln x_2^*(n)|.$$

Be similar to the discussion of (3.4)-(3.6), the second equation of (1.4) implies that

$$\begin{aligned}\Delta V_2(n) &\leq -\Big(\frac{1}{\theta_2(n)} - \Big|\frac{1}{\theta_2(n)} - b_2(n)\Big|\Big)|x_2(n) - x_2^*(n)| \\ &\leq -\min\{b_2^l, \ \frac{2}{M_2+\varepsilon} - b_2^u\}|x_2(n) - x_2^*(n)|\end{aligned} \tag{3.7}$$

where $\theta_2(n)$ lies between $x_2(n)$ and $x_2^*(n)$.

In the following, we denote the Lyapunov function:

$$V(n) = \lambda_1 V_1(n) + \lambda_2 V_2(n).$$

From (3.6) and (3.7), for any $n \geq N_1$, we obtain

$$\begin{aligned}\Delta V(n) \leq& -\Big(\lambda_1\mu_1^\varepsilon - \lambda_1\frac{a_{11}^u a_{13}^u}{(a_{12}^l + a_{13}^l(m_1-\varepsilon))^2}\Big)|x_1(n) - x_1^*(n)| \\ &-\Big(\lambda_2\mu_2^\varepsilon - \lambda_1\frac{c_1^u p(M_2+\varepsilon)^{p-1}}{(1+(m_2-\varepsilon))^2}\Big)|x_2(n) - x_2^*(n)| \\ \leq& -\delta(|x_1(n) - x_1^*(n)| + |x_2(n) - x_2^*(n)|).\end{aligned} \tag{3.8}$$

Summing both sides of the above inequality (3.8) from N_1 to k leads to

$$\begin{aligned}&\sum_{n=N_1}^{k}(V(n+1) - V(n)) \\ &\leq -\delta\sum_{n=N_1}^{k}(|x_1(n) - x_1^*(n)| + |x_2(n) - x_2^*(n)|),\end{aligned}$$

which implies

$$V(k+1) + \delta\sum_{n=N_1}^{k}(|x_1(n) - x_1^*(n)| + |x_2(n) - x_2^*(n)|) \leq V(N_1).$$

Thus,

$$\sum_{n=N_1}^{k}(|x_1(n) - x_1^*(n)| + |x_2(n) - x_2^*(n)|) \leq \frac{V(N_1)}{\delta}.$$

Therefore,

$$\sum_{n=N_1}^{+\infty}(|x_1(n) - x_1^*(n)| + |x_2(n) - x_2^*(n)|) \leq \frac{V(N_1)}{\delta} < +\infty,$$

which implies that

$$\lim_{n\to+\infty}(|x_1(n) - x_1^*(n)| + |x_2(n) - x_2^*(n)|) = 0,$$

that is

$$\lim_{n\to+\infty}|x_1(n) - x_1^*(n)| = 0, \quad \lim_{n\to+\infty}|x_2(n) - x_2^*(n)| = 0.$$

This completes the claim.

IV. Numeric simulations

In this part, two examples are given to support our main results.

Example 4.1. Consider the equations as follows

$$\begin{aligned}x_1(n+1) &= x_1(n)\exp\{\frac{2}{1+x_1(n)} - 2.5 - x_1(n) \\ &\quad + \frac{0.5x_2(n)}{1+x_2(n)}\}, \\ x_2(n+1) &= x_2(n)\exp\{1.5 - 0.5x_2(n)\}.\end{aligned} \tag{4.1}$$

Corresponding to system (1.4), one has $M_2 = 2\sqrt{e}$, $\frac{a_{11}^u}{a_{12}^l} - a_{14}^l + \frac{c_1^u M_2^p}{1+M_2^p} = -1 + \frac{\sqrt{e}}{1+2\sqrt{e}} < 0$. Clearly, condition (H_1) is satisfied, from Theorem 2.2, we know that $x_2(n)$ is permanent and $x_1(n)$ is driven to extinction, for all positive solutions $(x_1(n), x_2(n))^T$ of system (4.1). Figure 1 supports the conclusion of Theorem 2.2.

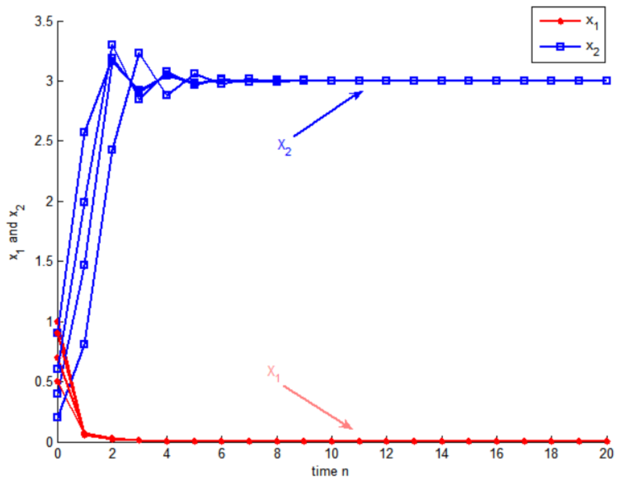

Fig. 1. Numeric simulations of system (41) with initial conditions $(x_1(n), x_2(n))^T = (1, 0.4)^T$, $(0.3, 0.15)^T$, $(0.7, 0.9)^T$, and $(0.9, 0.6)^T$, respectively.

Example 4.2. Consider this model

$$\begin{aligned}x_1(n+1) &= x_1(n)\exp\{\frac{1}{0.5+0.2x_1(n)} \\ &\qquad -2 - x_1(n) + \frac{2x_2(n)}{1+x_2(n)}\}, \\ x_2(n+1) &= x_2(n)\exp\{(0.8 + 0.2cos(n)) \\ &\qquad -(0.7+0.2sin(n))x_2(n)\}.\end{aligned} \tag{4.2}$$

By calculation, one has

$$\begin{aligned}&M_1 = e^{-0.8}, \quad M_2 = 2, \quad m_2 = \tfrac{2}{3}e^{-1.2}, \\ &\frac{a_{11}^l}{a_{12}^u + a_{13}^u M_1} - a_{14}^u + \frac{c_1^l m_2^p}{1+m_2^p} \\ &= \frac{1}{0.5+0.1e^{-0.8}} - 2 + \frac{4}{3e^{-1.2}+2} > 0.\end{aligned}$$

Obviously, condition (H_2) is satisfied and according to Theorem 2.3, we obtain that the model (4.2) is permanent.

One could easily check that there exist positive constants $\lambda_1 = 1$ and $\lambda_2 = 0.05$ such that condition (H_3) is satisfied, hence Theorem 3.1 shows that the system is globally attractive. Figure 2 supports the assertions.

V. Conclusion

In this paper, we incorporated the density dependent birth rate for the first species of a Holling type commensalism model, it extended the model of Wu et al. [8]. In Section 2, we obtained sufficient conditions of the permanence and partial extinction of the system. Theorem 3.1 showed the global attractivity which Chen [10] has not studied, and

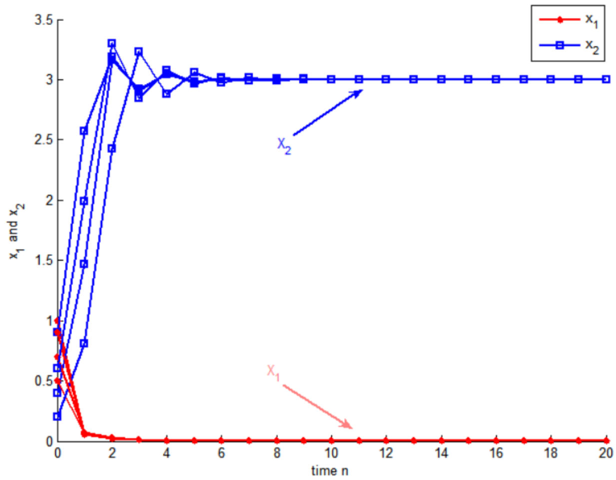

Fig. 2. Numeric simulations of system (42) with initial conditions $(x_1(n), x_2(n)) = (0.9, 0.4)^T$, $(0.2, 0.1)^T$, $(0.6, 0.8)^T$, and $(0.5, 0.5)^T$, respectively.

we can see that the density dependent birth rate plays an important role in the permanence, partial extinction and global attractivity of the species. We also can see that when all the coefficients are positive constants, i.e., system (1.4) in the autonomous case, all the results also held. Numeric simulations validated our analytical results.

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