Homogenization of the Helmholtz Problem with Layered Viscoelastic Media Including Finite Size Effect

Rachid BELEMOU, *Member, IAENG,* Amine SBITTI, Jean-Jacques MARIGO, and Hassan EL AMRI

***Abstract*—We propose a homogenization method based on a matched asymptotic expansion technique to obtain the effective behavior of a two-dimensional linear viscoelastic periodically stratified slab, which accounts for the finite size of the slab. The problem is investigated for shear waves, and the wave equation in the harmonic regime is considered. The obtained effective behavior is that of a homogeneous anisotropic slab associated with jump conditions, for the displacement and the normal stress at the boundaries of the slab. These jump conditions are implemented in a numerical scheme in the case of layers associated with Neumann boundary conditions and compared to the results of the direct problem.**

***Index Terms*—Homogenization; Matched asymptotic expansion; Effective jump conditions; Viscoelastic; Stratified media; Finite size effects.**

I. INTRODUCTION

IN complex geometries, it is not always possible to solve a wave propagation problem analytically, and this requires a numerical resolution. The heterogeneity of the domain is considered to be one of the sources of complexity in numerical calculations, since the existence of scales of very different lengths has a great impact on wave behavior. This requires that the mesh must be able to resolve the rapid variations associated with a small scale, while the propagation domain must be sized to the largest scale. Therefore, if the heterogeneity of the propagation medium is that of wavelength, a well-adapted numerical method is sufficient to choose. Otherwise, the model can be simplified by using the classical homogenization technique, which derives an equivalent homogenized problem. For example, if the entire propagation medium (or a large part) has a micro-structuring whose smallest scale is under the wavelength [1]-[2]. In other situations, where only a small or thin region contains a micro-structuring; they are originally developed in the context of acoustics [3]-[4]. Interface homogenization has been studied at certain problems, particularly in electromagnetism [5]-[6] and solid mechanics [4]-[7]-[8].

In this present paper, we extend a study of shear wave scattering by a slab that is periodically stratified in elasticity[9]. This time, the periodic layers are considered to be viscoelastic and satisfy $\varepsilon = K_R h \ll 1$, where K_R is the real part of the complex shear wavenumber K^*, and h is the periodicity of the structure. We noticed that the wave equation of the real problem takes the same form in the case of elastic media. Except in our case, the coefficients of physical parameters entering the equation are complex, which did not change the homogenization procedure or even the form of the homogenized wave equations obtained at different orders. We have shown that this problem can be replaced by an equivalent slab whose scattering parameters accurately describe those of the actual structure. Furthermore, we have established that, in addition to the mass parameters entering the effective wave equation, homogenization reveals interface parameters, which enter the conditions of jump at the boundaries of the equivalent slab (Figure 1).

The paper is organized as follows. In Sect.II, we summarize the results of the asymptotic analysis in the case of two-dimensional linear viscoelastic periodically stratified slab which accounts for the finite size of the slab, whose main steps of derivation are given in the Appendix A. The resulting system (7) represents the homogenized problem associated with jump conditions of the displacement and of the normal stress across an equivalent homogeneous anisotropic slab. In Sect.III, we discuss the discrepancy between the homogenized and the actual solution, by representing the transmission coefficients as a function of the frequency $K_R h$, the thickness of stratified medium e/h and the reciprocal quality factor Q^{-1} of viscoelastic media. We finish the study in Sect.IV with concluding remarks and perspectives.

II. THE PHYSICAL AND THE HOMOGENIZED PROBLEMS

A. Position of the physical problem and notations

We start with the viscoelastic wave equation for the scalar displacement field $U(\mathbf{X})$ written in the harmonic regime (and U is anti-plane), with $\mathbf{X} \in \Omega$ the spatial coordinates and $\Omega = \{(X_1, X_2) \in \mathbb{R} \times (-H/2, H/2)\}$ (Figure 1).

$$\operatorname{div}(M\boldsymbol{\nabla} U) + \rho\omega^2 U = 0 \tag{1}$$

With M and ρ being the complex shear modulus and the mass density respectively, and ω the frequency. Equation (1) can be written in terms of the non-dimensional parameters.

$$\alpha^*(\mathbf{X}) \equiv \frac{M(\mathbf{X},\omega)}{M_m} \quad \text{and} \quad \beta(\mathbf{X}) \equiv \frac{\rho(\mathbf{X})}{\rho_m}$$

With M_m the complex shear modulus and ρ_m the mass density of the substrate surrounding the stratified medium

Manuscript received August 28, 2022; revised Janaury 11, 2023.

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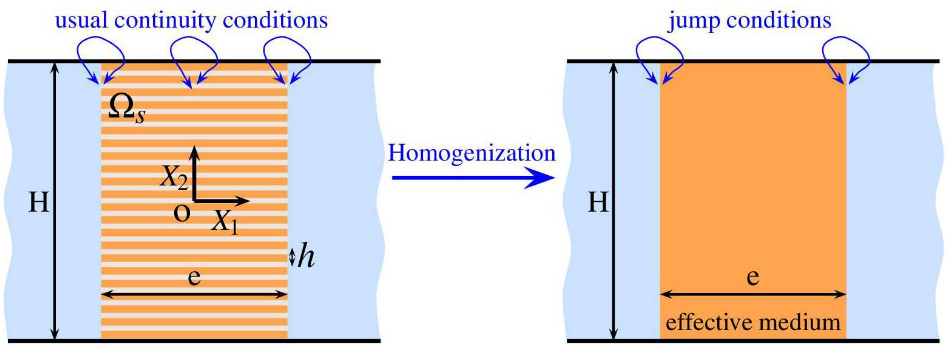

Fig. 1. On the left, the actual configuration of a viscoelastic body (in blue) with a viscoelastic stratified medium $\Omega_s = \{(X_1, X_2) \in (-e/2, e/2) \times (-H/2, H/2)\}$. On the right, the homogenized configuration where the stratified medium is replaced by an equivalent homogeneous anisotropic slab, which associated with jump conditions apply at the boundaries of the slab.

occupying the region Ω_s; with $K^* = \omega\sqrt{\rho_m/M_m}$ the complex wavenumber in the substrate Ω/Ω_s, we get

$$\operatorname{div}(\alpha^*\nabla U) + \beta {K^*}^2 U = 0 \tag{2}$$

This allows us to write the Helmholtz equation in the substrate as follows

$$\Delta U + {K^*}^2 U = 0$$

In the harmonic regime, we consider viscoelastic waves with a minimum wavelength $2\pi/K_R$ larger than the typical periodicity of the stratified structure h (K_R being the real part of the complex shear wavenumber K^*), such that

$$\varepsilon = K_R h \ll 1$$

To be consistent, we shall work in dimensionless coordinate and on a problem simplified regarding that in (Figure 1) in the sense that we consider a single interface, which means that we focus on a region near the boundary of the stratified medium at $X_1 = e/2$; and to achieve that, we define it as follows:

$$x_1 = K_R(X_1 - e/2), x_2 = K_R X_2$$

We shall assume that the stratified medium occupies the region $x1 < 0$. Doing so, we assume implicitly that the wave passing through the stratified slab in the configuration of (Figure 2) feels the boundaries and the bulk of the stratified medium. This means that the slab is thick enough, and thickness means that the evanescent fields at both boundaries of the slab do not interact. We shall define the actual problem for $\mathbf{x} = (x_1, x_2) \in \mathbb{R} \times (-K_R H/2, K_R H/2)$, and we denote it as follows:

$$a^{*\varepsilon}(\mathbf{x}) \equiv \alpha^*(\mathbf{X}), \quad b^{*\varepsilon}(\mathbf{x}) \equiv \beta(\mathbf{X})(\tfrac{K^*}{K_R})^2 \quad ;$$

$$u^\varepsilon(\mathbf{x}) \equiv U(\mathbf{X}), \quad \sigma^\varepsilon(\mathbf{x}) \equiv {K_R}^{-1}\alpha^*(\mathbf{X})\nabla U(\mathbf{X}).$$

Where the functions a^* and b^* are 1-periodic and piecewise complex constant, such that

$$a^{*\epsilon}(\mathbf{x}) = \begin{cases} 1, & x_1 > 0 \\ a^*\left(\frac{x_2}{\varepsilon}\right), & x_1 < 0 \end{cases}$$
$$b^{*\varepsilon}(\mathbf{x}) = \begin{cases} (\frac{K^*}{K_R})^2, & x_1 > 0 \\ b^*\left(\frac{x_2}{\varepsilon}\right), & x_1 < 0 \end{cases} \tag{3}$$

Moreover, we explicitly indicated the dependence of $(u^\varepsilon, \sigma^\varepsilon)$ on ε being the periodicity of the stratified medium in dimensionless form. Now, (2) reads the following

$$\begin{cases} \operatorname{div}\sigma^\varepsilon(\mathbf{x}) + b^{*\varepsilon}(\mathbf{x})u^\varepsilon(\mathbf{x}) = 0 & \text{(4a)} \\ \sigma^\varepsilon(\mathbf{x}) = a^{*\varepsilon}(\mathbf{x})\nabla u^\varepsilon(\mathbf{x}) & \text{(4b)} \end{cases}$$

With $\mathbf{x} \in \mathbb{R} \times (-K_R H/2, K_R H/2)$. At the boundaries between two layers within the stratified medium and at the boundaries between the layers and the substrate at $x_1 = 0$ (Figure 2), the continuity of the displacement, and the continuity of the normal stress $\sigma^\varepsilon.\mathbf{n}$ apply. Finally, appropriate boundary conditions at $|\mathbf{x}| \to +\infty$ and $x_2 = \pm K_R H/2$, often referred to as radiation conditions, apply once the wave source has been defined. For the moment, we do not need to specify their form.

B. The homogenized problem

Below we summarize the main results of the analysis developed in the Appendix. A and which provides the so-called (homogenized problem) where the stratified medium is replaced by an equivalent anisotropic slab associated with the jump conditions for the displacement and the normal stress across two interfaces at $X_1 = \pm e$ (Figure 1). The homogenized problem of a welded boundary between infinite layers and substrate (4a-4b), is done by defining the fields (u^h, σ^h) satisfying the following homogenized problem:

$$\begin{cases} \operatorname{div}\sigma^h + \langle b^*\rangle u^h = 0, & x_1 < 0 \\ \sigma^h = \begin{pmatrix} \langle a^*\rangle & 0 \\ 0 & \langle 1/a^*\rangle^{-1} \end{pmatrix} \nabla u^h & \\ \operatorname{div}\sigma^h + (\frac{K^*}{K_R})^2 u^h = 0, \quad \sigma^h = \nabla u^h, & x_1 > 0 \\ [\![u^h]\!] = \frac{\varepsilon\mathcal{B}}{2}\left[\sigma_1^h\left(0^-, x_2\right) + \sigma_1^h\left(0^+, x_2\right)\right] & \\ [\![\sigma_1^h]\!] = -\frac{\varepsilon\mathcal{C}}{2}\left[\frac{\partial^2 u^h}{\partial x_2^2}\left(0^-, x_2\right) + \frac{\partial^2 u^h}{\partial x_2^2}\left(0^+, x_2\right)\right] & \end{cases} \tag{5}$$

Where $(\mathcal{B}, \mathcal{C})$ the interface parameters (39-41) and we defined

$$[\![f]\!] \equiv f\left(0^+, x_2, \tau\right) - f\left(0^-, x_2, \tau\right), \tag{6}$$

For any outer terms, f being discontinuous across an equivalent interface at $x_1 = 0$, with $(f^-; f^+)$ its values on both sides. And the average over $y_2 \in Y$ for any function f, is defined by

$$\langle f\rangle(\mathbf{x}) \equiv \int_Y \mathrm{d}y_2 f\left(\mathbf{x}, y_2\right)$$

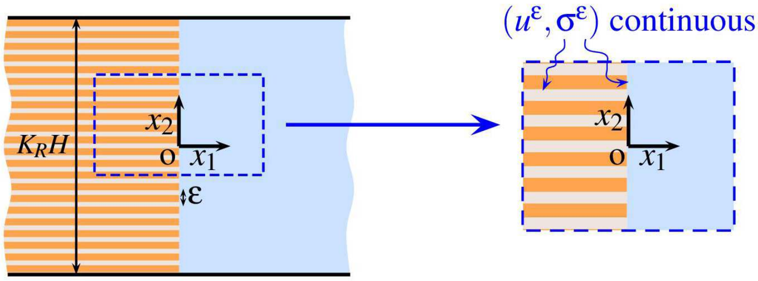

Fig. 2. Single interface between the stratified medium occupying the region $x_1 < 0$ and the substrate occupying the region $x_1 > 0$. The usual continuity conditions apply at the boundaries between the layers $\left(u^\varepsilon, \sigma_2^\varepsilon\right)$ and at the boundaries between the layers and the substrate $\left(u^\varepsilon, \sigma_1^\varepsilon\right)$ at $x_1 = 0$.

Finally, from (5) coming back to the real space with a welded boundary is considered at $X_1 = e$, in the $\mathbf{X} = \mathbf{x}/K_R$ coordinate and with $U^h(\mathbf{X}) = u^h(\mathbf{x}), \Sigma^h(\mathbf{X}) = K_R\sigma^h(\mathbf{x})$, we get an effective problem

$$\begin{cases} \operatorname{div}\Sigma^h + \langle b^*\rangle {K_R}^2 U^h = 0, & X_1 < e/2 \\ \Sigma^h = \begin{pmatrix} \langle a^*\rangle & 0 \\ 0 & \langle 1/a^*\rangle^{-1} \end{pmatrix} \nabla U^h & \\ \operatorname{div}\Sigma^h + {K^*}^2 U^h = 0, \quad \Sigma^h = \nabla U^h, & X_1 > e/2 \\ [\![U^h]\!] = \dfrac{hB}{2}\left[\Sigma_1^h\left((e/2)^-, X_2\right) + \Sigma_1^h\left((e/2)^+, X_2\right)\right] & \\ [\![\Sigma_1^h]\!] = & \\ \qquad \dfrac{hC}{2}\left[\dfrac{\partial^2 U^h}{\partial X_2^2}\left((e/2)^-, X_2\right) + \dfrac{\partial^2 U^h}{\partial X_2^2}\left((e/2)^+, X_2\right)\right] & \end{cases}$$

The above problem, written for a single interface at $X_1 = e/2$, correspond to the problem when two interfaces at $X_1 = \pm e/2$ are considered. It reads as follows

$$\begin{cases} \operatorname{div}\Sigma^h + \langle b^*\rangle {K_R}^2 U^h = 0, & X \in \Omega_s \\ \Sigma^h = \begin{pmatrix} \langle a^*\rangle & 0 \\ 0 & \langle 1/a^*\rangle^{-1} \end{pmatrix} \nabla U^h, & \\ \operatorname{div}\Sigma^h + {K^*}^2 U^h = 0, \quad \Sigma^h = \nabla U^h, & X \in \Omega/\Omega_s \\ [\![U^h]\!] = \dfrac{hB}{2}\left(\Sigma^{h-} + \Sigma^{h+}\right)\cdot \mathbf{N} & \\ [\![\Sigma^h]\!]\cdot N = -\dfrac{hC}{2}\left(\dfrac{\partial^2 U^{h-}}{\partial X_2^2} + \dfrac{\partial^2 U^{h+}}{\partial X_2^2}\right) & \end{cases} \tag{7}$$

With the convention $\pm$ refers to the direction of the normal $\mathbf{N}$.

III. Accuracy of the homogenized solution regarding the actual solution

In this section, we inspect the accuracy of the homogenization. To do so, we shall consider the particular scattering problem of the reflection of rectangular voids, free of stresses (with Neumann conditions on their boundaries), periodically spaced in a homogeneous matrix being composed of the same linear viscoelastic material as the substrate. We shall work with complex fields (because the physical fields are the real parts of the computed complex ones); in the harmonic regime, the complex fields (and we shall consider the displacement field U(X)) have a time dependence in $e^{-\mathrm{i}\omega t}$ and it will be omitted in the following. An incident wave arrives from $X_1 < 0$ and hits the array at oblique incidence θ with degree of heterogeneity γ (Figure 3). This type of shear wave is defined as a Type-II S wave [10], of the following form

$$U^{\mathrm{inc}}(\mathbf{X}) = e^{-\mathrm{i}\omega t} e^{i\vec{K}\cdot\vec{r}} = e^{-\mathrm{i}\omega t} e^{-\vec{A}\cdot\vec{r}} e^{i\vec{P}\cdot\vec{r}}. \tag{8}$$

Where $\vec{r} = (X_1, X_2)$ is the position vector, and $\vec{K}$ the complex wave vector is given by

$$\vec{K} = \vec{P} + i\vec{A} = K_S\hat{x_1} + K_{inc}\hat{x_2}$$

And the corresponding propagation and attenuation vectors, are given by

$$\begin{aligned} \vec{P} &= \left|\vec{P}\right|\cos(\theta)\,\hat{x_2} + \left|\vec{P}\right|\sin(\theta)\,\hat{x_2} \\ &= Re\left[K_S\right]\hat{x_1} + Re\left[K_{inc}\right]\hat{x_2} \\ \vec{A} &= \left|\vec{A}\right|\cos(\theta-\gamma)\,\hat{x_1} + \left|\vec{A}\right|\sin(\theta-\gamma)\,\hat{x_2} \\ &= Im\left[K_S\right]\hat{x_1} + Im\left[K_{inc}\right]\hat{x_2} \end{aligned}.$$

With $(\hat{x_1}, \hat{x_2})$ are orthogonal real unit vectors for a Cartesian coordinate system, k_{inc} the complex wave number for the assumed general SII wave, and $K_S = \sqrt{{K^*}^2 - {K_{inc}}^2}$, where "$\sqrt{\ }$" is understood to indicate the principal value of the square root of a complex number $z = z_R + iz_I$ defined in terms of the positive square root of real numbers by

$$\sqrt{z} = \sqrt{\frac{|z|+z_R}{2}} + i\,\mathrm{sign}\left[z_I\right]\sqrt{\frac{|z|-z_R}{2}}$$

with

$$\mathrm{sign}\left[z_I\right] \equiv \begin{Bmatrix} 1 & \text{if} & z_I \geq 0 \\ -1 & \text{if} & z_I < 0 \end{Bmatrix}$$

Hence, the complex wave numbers K_{inc} and K_S reads

$$K_{inc} = \left|\vec{P}\right|\sin(\theta) + i\left|\vec{A}\right|\sin(\theta-\gamma)$$

$$K_S = \left|\vec{P}\right|\cos(\theta) + i\left|\vec{A}\right|\cos(\theta-\gamma).$$

Where the magnitudes of the propagation and attenuation are specified in terms of the given material parameters, the complex wave number K^* or wave speed $(v_m = \omega/K_R)$ and the reciprocal quality factors $({Q_m}^{-1} = M_{mI}/M_{mR})$, and the given degree of heterogeneity γ [10].

A. Solutions of the actual problem

The actual problem is solved numerically by using the same multimodal method [9], which reduces to the determination of a set of scattering coefficients R^{num} and T^{num} for $|X_1| > e/2$. The reference numerical solution U^{num} is sought in the substrate where the Helmholtz equation applies

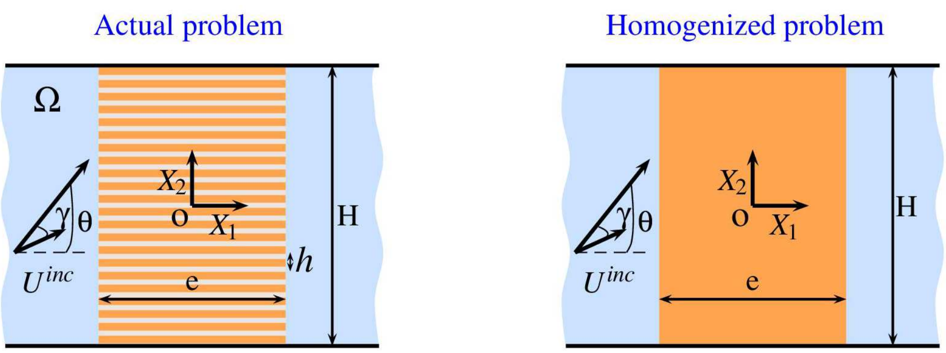

Fig. 3. Left: Actual problem of the scattering of a plane wave at oblique incidence θ of an array of rectangular voids, with degree of heterogeneity γ. Right : The homogenized problem involves a slab of same thickness e filled with a homogeneous anisotropic material, which associated with jump conditions and apply at $X_1 = \pm e/2$.

with Neumann boundary conditions on the boundaries of the voids occupying the subdomains $\Omega_i,\ i = 1, \ldots, N_v$, and $\Omega_v = \cup_i \Omega_i$. Such a problem is solved in $\Omega \backslash \Omega_v$ with $\Omega = \{(X_1, X_2) \in \mathbb{R} \times (-H/2, H/2)\}$ and reads

$$\begin{cases} \Delta U + K^{*2} U = 0, & \text{in } \Omega \backslash \Omega_v \\ \nabla U \cdot \mathrm{n} = 0, & \text{on } \partial\Omega_i, i = 1,..,N_v \\ \lim\limits_{X_1 \to \pm\infty} \left[\dfrac{\partial}{\partial X_1} \left(U - U^{\mathrm{inc}}\right) \mp \mathrm{i} K_S \left(U - U^{\mathrm{inc}}\right) \right] = 0 & \\ U\left(X_1, \dfrac{H}{2}\right) = e^{\mathrm{i} K_{inc} H} U\left(X_1, -\dfrac{H}{2}\right), & X_1 \in \mathbb{R} \\ \dfrac{\partial U}{\partial X_2}\left(X_1, \dfrac{H}{2}\right) = e^{i K_{inc} H} \dfrac{\partial U}{\partial X_2}\left(X_1, -\dfrac{H}{2}\right), & X_1 \in \mathbb{R} \end{cases} \tag{9}$$

Where the scattered waves $\left(U - U^{\mathrm{inc}}\right)$ are the radiation condition at $X_1 \to \pm\infty$ [11], and are considered in the low frequency regime [12]. The last condition is referred to as the Floquet condition (pseudo-periodicity) [13], which applies in the case where $H = nh$, with n an integer, for the incident wave and for the total field.

B. Solutions of homogenized problem

To numerically validate the homogenized problem, we treat the case of scattering by an array of rectangular voids. In acoustics, this corresponds to an array of sound hard material in a fluid; in electromagnetism, it corresponds to a perfect conducting metallic array in a dielectric or in the air. For the first validation, we consider rectangular voids spaced periodically in a homogeneous matrix composed of the same viscoelastic material as the substrate. In the second case, the stratified medium is considered to be elastic and the substrate to be viscoelastic.

1) The case of the substrate and layers is the same viscoelastic media: The homogenized problem can be solved exactly in the limiting case of voids with $a = b = 0$ (leading to the Neumann boundary condition at the boundary with any other material). Hereafter, we consider $a^* = 1$, $b^* = (K^*/K_R)^2$ in the substrate, and φ the filling fraction of the substrate in the layers. The bulk parameters in the equivalent medium become $\langle a^* \rangle = \varphi$, $\langle b^* \rangle = (\frac{K^*}{K_R})^2 \varphi$ and $\langle 1/a^* \rangle^{-1} = 0$, where the homogenized wave equation (30) reads

$$\begin{cases} \operatorname{div} \boldsymbol{\Sigma}^{\boldsymbol{h}} + \varphi K^{*2} U^h = 0, \\ \boldsymbol{\Sigma}^{\boldsymbol{h}} = \begin{pmatrix} \varphi & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{\nabla} U^h \end{cases}$$

It follows that the homogenized problems read

$$\begin{cases} \dfrac{\partial^2 U^h}{\partial X_1^2} + K^{*2} U^h = 0, & |X_1| < \dfrac{e}{2} \\ \Delta U^h + K^{*2} U^h = 0, & |X_1| > \dfrac{e}{2} \\ \text{jump conditions (7)}, & X_1 = \pm\dfrac{e}{2} \\ \lim\limits_{X_1 \to \pm\infty} \left[\dfrac{\partial}{\partial X_1} \left(U^h - U^{\mathrm{inc}}\right) \mp i K_S \left(U^h - U^{inc}\right) \right] = 0 & \\ U^h\left(X_1, \dfrac{H}{2}\right) = e^{i K_{inc}} U^h\left(X_1, -\dfrac{H}{2}\right), & X_1 \in \mathbb{R} \\ \dfrac{\partial U^h}{\partial X_2}\left(X_1, \dfrac{H}{2}\right) = e^{i K_{inc}} \dfrac{\partial U^h}{\partial X_2}\left(X_1, -\dfrac{H}{2}\right), & X_1 \in \mathbb{R} \end{cases} \tag{10}$$

To obtain the effective parameters $(\mathcal{B}, \mathcal{C})$ entering the jump conditions (7), we use the same method based on the modal methods [9] for solving numerically the elementary problems (37) and (38) in the case of an array of rectangular voids. The solution of (10) with (8) is of the form

$$\begin{cases} U(\mathbf{X}) = e^{i K_S (X_1 + e/2)} e^{i K_{inc} X_2} & \\ \qquad\qquad + R e^{-i K_S (X_1 + e/2)} e^{i K_{inc} X_2}, & X_1 < -\dfrac{e}{2} \\ U(\mathbf{X}) = \left[a e^{i K^* X_1} + b e^{-i K^* X_1} \right] e^{i K_{inc} X_2}, & |X_1| < \dfrac{e}{2} \\ U(\mathbf{X}) = T e^{i K_S (X_1 - e/2) + i K_{inc} X_2}, & X_1 > \dfrac{e}{2} \end{cases} \tag{11}$$

With (R, T, a, b) are given by using jump conditions (7) with (11). In particular, the scattering coefficients (R, T) read

$$\begin{cases} R = -\dfrac{z_1 \mathrm{e}^{-i K^* e} z_4 - \mathrm{e}^{i K^* e} z_3 z_2}{\mathrm{e}^{-i K^* e} {z_4}^2 - \mathrm{e}^{i K^* e} {z_3}^2} \\ T = \dfrac{-z_1 z_3 + z_2 z_4}{\mathrm{e}^{-i K^* e} {z_4}^2 - \mathrm{e}^{i K^* e} {z_3}^2} \end{cases} \tag{12}$$

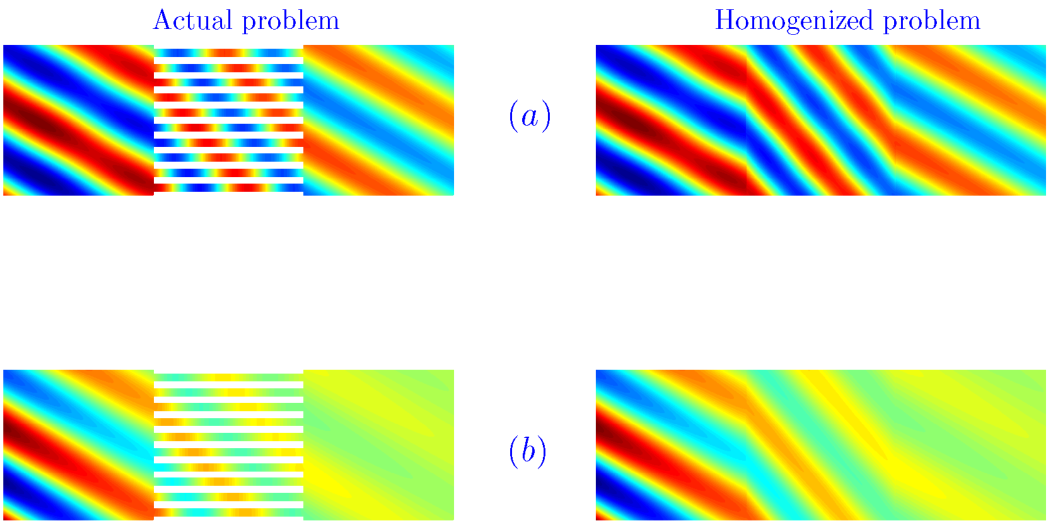

Fig. 4. (a) The numerical solution U^{num} in the actual problem of low-loss viscoelastic media ($Q^{-1} = 0.05$), for an oblique incident plane wave $\theta = \pi/3$ with degree of heterogeneity $\gamma = \pi/6$ and $K_R h = 1$ on an array made of rectangular voids ($e/h = 10$ and $\varphi = 0.5$); the right shows the homogenized fields U. (b) Same representation as (a) with $Q^{-1} = 0.2$ in the case of low-loss viscoelastic media.

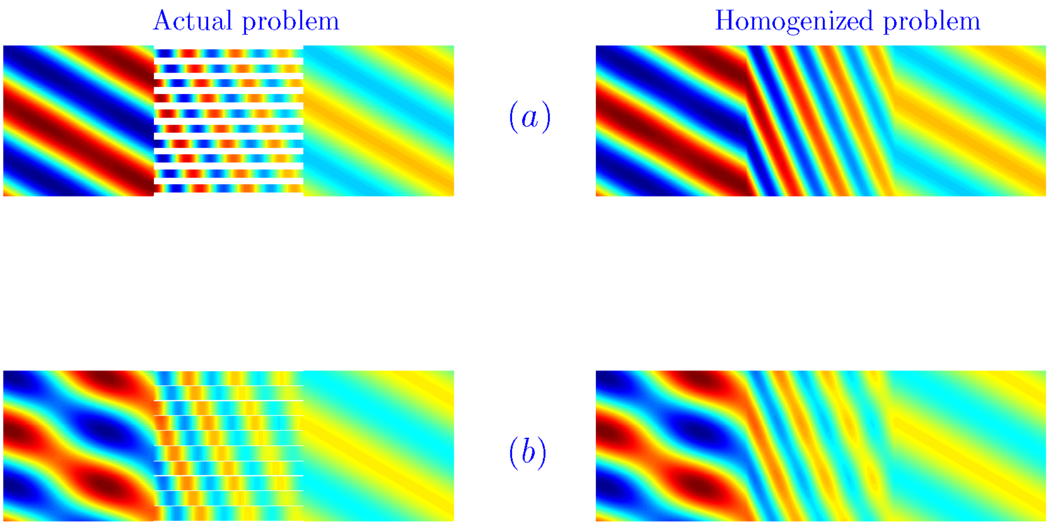

Fig. 5. (a) Same representation as in Figure 4 with the reciprocal quality factor $Q_{in}^{-1} = 0.1$ for viscoelastic layers of rectangular voids Ω_s and $Q_m^{-1} = 0$ for elastic substrate. (b) Same representation as (a) with $\varphi = 0.9$.

With

$$\left\{\begin{aligned} z_1 &\equiv \left(1 - \frac{K_S}{K^*\varphi}\right) + ih\left(\mathcal{B}K_S + \mathcal{C}\frac{K_{inc}}{K^*\varphi}\right) \\ &\quad + (K_{inc}h)^2 \frac{\mathcal{BC}}{4}\left(1 - \frac{K_S}{K^*\varphi}\right) \\ z_2 &\equiv \left(1 - \frac{K_S}{K^*\varphi}\right) - ih\left(\mathcal{B}K_S + \mathcal{C}\frac{K_{inc}}{K^*\varphi}\right) \\ &\quad + (K_{inc}h)^2 \frac{\mathcal{BC}}{4}\left(1 - \frac{K_S}{K^*\varphi}\right) \\ z_3 &\equiv \left(1 + \frac{K_S}{K^*\varphi}\right) + ih\left(\mathcal{B}K_S + \mathcal{C}\frac{K_{inc}}{K^*\varphi}\right) \\ &\quad + (K_{inc}h)^2 \frac{\mathcal{BC}}{4}\left(1 + \frac{K_S}{K^*\varphi}\right) \\ z_4 &\equiv \left(1 + \frac{K_S}{K^*\varphi}\right) - ih\left(\mathcal{B}K_S + \mathcal{C}\frac{K_{inc}}{K^*\varphi}\right) \\ &\quad + (K_{inc}h)^2 \frac{\mathcal{BC}}{4}\left(1 + \frac{K_S}{K^*\varphi}\right) \end{aligned}\right. \tag{13}$$

2) The case of an elastic substrate with viscoelastic layers: In this case, we consider $a^* = 1$, $b^* = 1$ in the elastic substrate with reciprocal quality factor $Q_{in}^{-1} = 0$, and φ the filling fraction of the viscoelastic media in the layers with reciprocal quality factor $Q_m^{-1} = 0.1$. The bulk parameters in the equivalent medium become $\langle a^* \rangle = \xi\varphi$ with ($\xi = \frac{M_i}{M_m}$); $\langle b^* \rangle = (\frac{\rho_i}{\rho_m})(\frac{K^*}{K_R})^2\varphi$ and $\langle 1/a^* \rangle^{-1} = 0$, where the homogenized wave equation (30) read

$$\left\{\begin{aligned} &\operatorname{div} \boldsymbol{\Sigma^h} + \varphi K_i^{*2} U^h = 0, \\ &\boldsymbol{\Sigma^h} = \begin{pmatrix} \xi\varphi & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{\nabla} U^h \end{aligned}\right.$$

With M_i the complex shear modulus and ρ_i the mass density of the stratified medium occupying the region $\Omega_s = \{(X_1, X_2) \in (-e/2, e/2) \times (-H/2, H/2)\}$; with $K_i^* = \omega\sqrt{\rho_i/M_i}$ the complex wavenumber in layers of rectangular

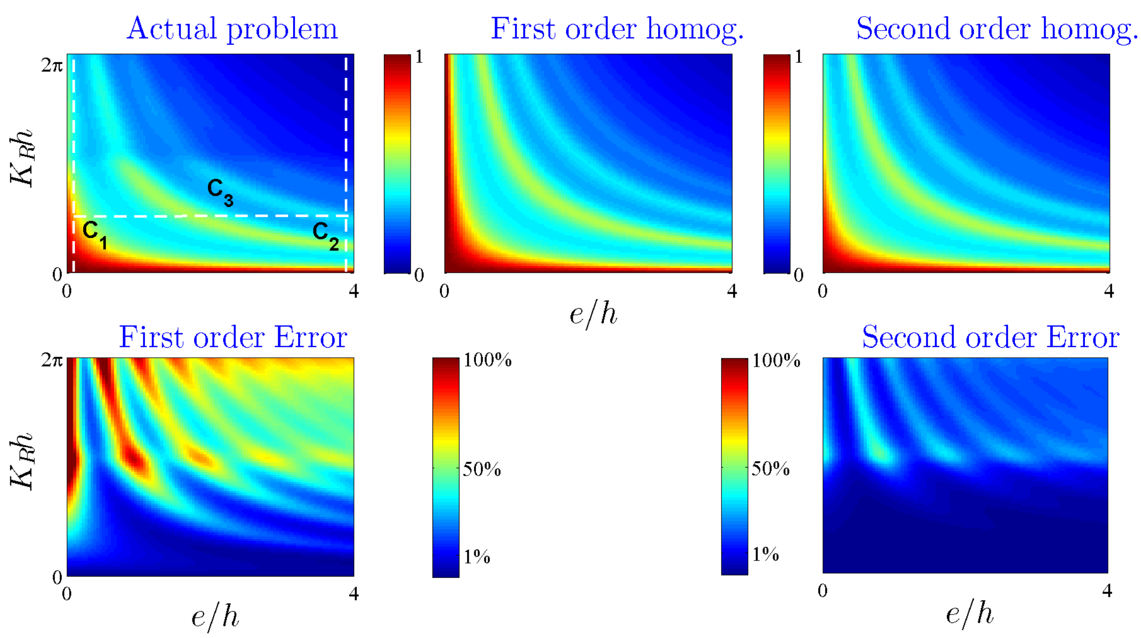

Fig. 6. Up: Transmission coefficients in actual problem $|T^{num}|$ and homogenized $|T|$ at first order ($\mathcal{B} = \mathcal{C} = 0$), and at second order as a function of e/h and of the frequency $K_R h$; ($Q^{-1} = 0.2$, $\varphi = 0.1$, $\theta = \pi/3$ and $\gamma = \pi/6$) have been considered. Down: Errors ΔT on the transmission coefficient, which are calculated numerically. Errors smaller than 1% appear in dark blue, and errors greater than 100% appear in dark red.

voids Ω_s, It follows that the homogenized problems read

$$\begin{cases} \dfrac{\partial^2 U^h}{\partial X_1^2} + {K_i^*}^2 U^h = 0, & |X_1| < \dfrac{e}{2} \\ \Delta U^h + {K_R}^2 U^h = 0, & |X_1| > \dfrac{e}{2} \\ \text{jump conditions (7)}, & X_1 = \pm\dfrac{e}{2} \\ \lim\limits_{X_1 \to \pm\infty} \dfrac{\partial}{\partial X_1}\left(U^h - U^{\text{inc}}\right) = \mp i K_R \cos\theta \left(U^h - U^{inc}\right) & \\ U^h\left(X_1, \dfrac{H}{2}\right) = e^{iK_R \sin\theta} U^h\left(X_1, -\dfrac{H}{2}\right), & X_1 \in \mathbb{R} \\ \dfrac{\partial U^h}{\partial X_2}\left(X_1, \dfrac{H}{2}\right) = e^{iK_R \sin\theta} \dfrac{\partial U^h}{\partial X_2}\left(X_1, -\dfrac{H}{2}\right), & X_1 \in \mathbb{R} \end{cases} \tag{14}$$

The solution of (14) with (8) is of the form

$$\begin{cases} U(\mathbf{X}) = e^{iK_R \cos\theta (X_1 + e/2)} e^{iK_R \sin\theta X_2} & \\ \qquad + R e^{-iK_R \cos\theta (X_1 + e/2)} e^{iK_R \sin\theta X_2}, & X_1 < -\dfrac{e}{2} \\ U(\mathbf{X}) = \left[a e^{iK_i^* X_1} + b e^{-iK_i^* X_1}\right] e^{iK_R \sin\theta X_2}, & |X_1| < \dfrac{e}{2} \\ U(\mathbf{X}) = T e^{iK_R \cos\theta (X_1 - e/2) + iK_R \sin\theta X_2}, & X_1 > \dfrac{e}{2} \end{cases} \tag{15}$$

with (R, T, a, b) are given by using jump conditions (7) with (15). In particular, the scattering coefficients (R, T) reads

$$\begin{cases} R = -\dfrac{z_1 \mathrm{e}^{-iK_i^* e} z_4 - \mathrm{e}^{iK_i^* e} z_3 z_2}{\mathrm{e}^{-iK_i^* e} {z_4}^2 - \mathrm{e}^{iK_i^* e} {z_3}^2} \\ T = \dfrac{-z_1 z_3 + z_2 z_4}{\mathrm{e}^{-iK_i^* e} {z_4}^2 - \mathrm{e}^{iK_i^* e} {z_3}^2} \end{cases} \tag{16}$$

With

$$\begin{cases} z_1 \equiv \left(1 - \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) + ih\left(\mathcal{B} K_R \cos\theta + \mathcal{C} \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) \\ \qquad + (h K_R \sin\theta)^2 \dfrac{\mathcal{BC}}{4}\left(1 - \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) \\ z_2 \equiv \left(1 - \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) - ih\left(\mathcal{B} K_R \cos\theta + \mathcal{C} \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) \\ \qquad + (h K_R \sin\theta)^2 \dfrac{\mathcal{BC}}{4}\left(1 - \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) \\ z_3 \equiv \left(1 + \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) + ih\left(\mathcal{B} K_R \cos\theta + \mathcal{C} \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) \\ \qquad + (h K_R \sin\theta)^2 \dfrac{\mathcal{BC}}{4}\left(1 + \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) \\ z_4 \equiv \left(1 + \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) - ih\left(\mathcal{B} K_R \cos\theta + \mathcal{C} \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) \\ \qquad + (h K_R \sin\theta)^2 \dfrac{\mathcal{BC}}{4}\left(1 + \dfrac{K_R \cos\theta}{K_i^* \xi \varphi}\right) \end{cases} \tag{17}$$

C. *Numerical validation of the homogenized solutions*

We shall inspect the validity of the homogenized solution for an oblique incident wave. First, we report the fields U^{num} calculated numerically and the fields U of the homogenized solutions (11) with (12)-(13), for the low-loss viscoelastic media $Q^{-1} \ll 1$ and for no low-loss media $Q^{-1} = 0.2$ (Figure 4). For the second case of the elastic substrate with viscoelastic layers problem (15) with (16)-(17), the reciprocal quality factor $Q_{in}^{-1} = 0.1$ is considered for viscoelastic layers of rectangular voids Ω_s, and $Q_m^{-1} = 0$ for elastic substrate (Figure 5).
Defining $\Delta U \equiv |U - U^{\text{num}}| / |U^{\text{num}}|$ (for $|X_1| > e/2$ and with $\|.\|$ the L^2 norm), we get a discrepancy of 0.5% ($Q^{-1} =$

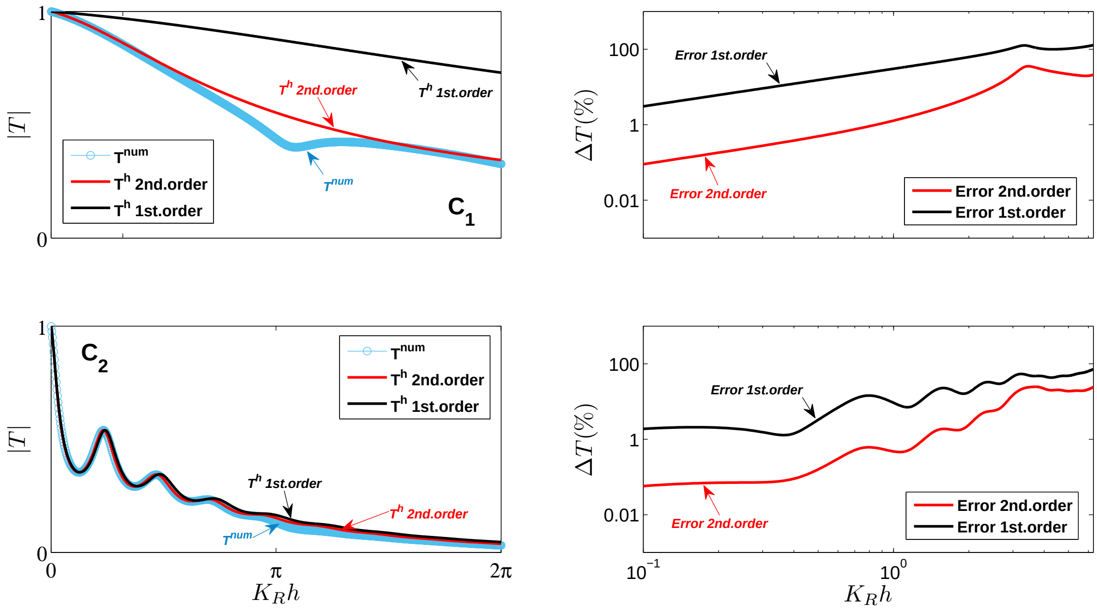

Fig. 7. Left: Transmission coefficients $|T^{\mathrm{num}}|$ and $|T|$ as a function of $K_R h$ for $e/h = 0.05$ and $e/h = 4$ ($\mathrm{C}_1, \mathrm{C}_2$ profiles from Figure 6), with $|T^{num}|$: blue symbols and $|T|$: black lines at first order and red lines at second order. Right: The corresponding error ΔT of the homogenized predictions, which are shown in percent (black lines at first order and red lines at second order).

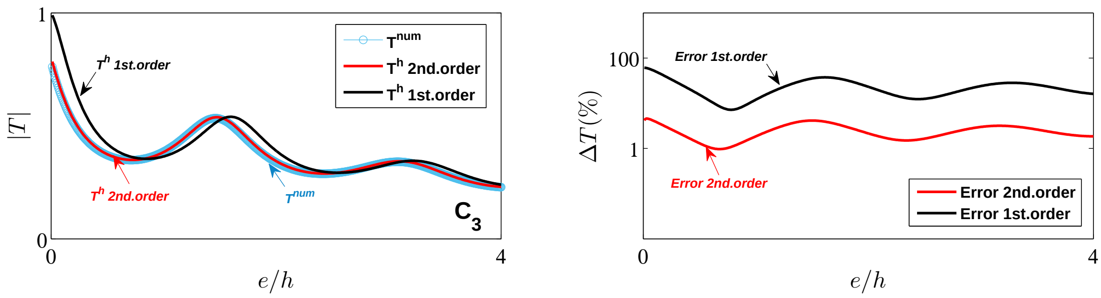

Fig. 8. Transmission coefficients $|T^{\mathrm{num}}|$ and $|T|$ and errors ΔT as a function of e/h for $K_R h = 0.6\pi$ (C_3 profile from Figure 6). Same representation as in Figure 7.

0.05) and 0.6% ($Q^{-1} = 0.2$) for the first case, and almost the same discrepancy 0.5% ($\varphi = 0.5$) and 0.7% ($\varphi = 0.9$) for second case. It is worth noting that a small error was found even at this relatively large $kh = 1$ value.

1) The error of the transmission as a function of frequencies : We first inspect for $Q^{-1} = 0.2$ (Figure 6), for which we report the spectra of the transmission as a function of kh and e/h, and the corresponding errors $\Delta T \equiv |T^{\mathrm{num}} - T| / |T^{\mathrm{num}}|$. We considered $K_R h \in [0, 2\pi]$, the frequency range includes $K_R h > 2\pi/(1 + \sin\theta) \simeq 1.07\pi$, corresponding to the Wood anomaly (cut-off frequency) [14]. This range is outside the range of validity of any homogenization approach, since mode coupling is not possible at an equivalent flat boundary.

In Figure (6), errors smaller than 1% appear in dark blue, and errors greater than 100% appear in dark red. On average, at the intermediate frequencies for $K_R h < \pi/2$ (C_3 profile), the error in the transmission coefficient for $Q^{-1} = 0.2$, is smaller than 1% in the whole range of e/h at second order, and it is of 50% on average at first order ($\mathcal{B} = \mathcal{C} = 0$). On the other hand, the first order homogenization wrongly predicts perfect transmissions for vanishing thicknesses e/h, while including the jump conditions (7) at second order restores the real scattering properties of an array of flat voids. This is corresponding to the result of [15], in which the effective permittivity of electromagnetic waves must depend on the thickness (the effective bulk parameter in our case).

More specifically, we inspect (i) the profiles of $|T^{\mathrm{num}}|$ and its homogenized counterparts $|T|$, with the corresponding errors ΔT as a function of $K_R h$ for $e/h = 0.05$ and $e/h = 4$ (Figure 7). For a small thickness (C_1 profile from Figure 6), the homogenization at second order recovers the actual transmission of the array, while the homogenization at first order largely overestimates the transmission; for a larger array (C_2 profile) the first order homogenization is valid for small $K_R h$; and going up to the second order allows us to increase the range of validity of the homogenized solution. (ii) The variations of $|T^{\mathrm{num}}|$ and $|T|$ (and the corresponding errors ΔT) as a function of e/h for $K_R h = 0.6\pi$ are reported in Figure 8 (C_3 profile from Figure 6). We notice that the great error in the homogenization of the first order seems to be a direct consequence of the disappearance of e/h.

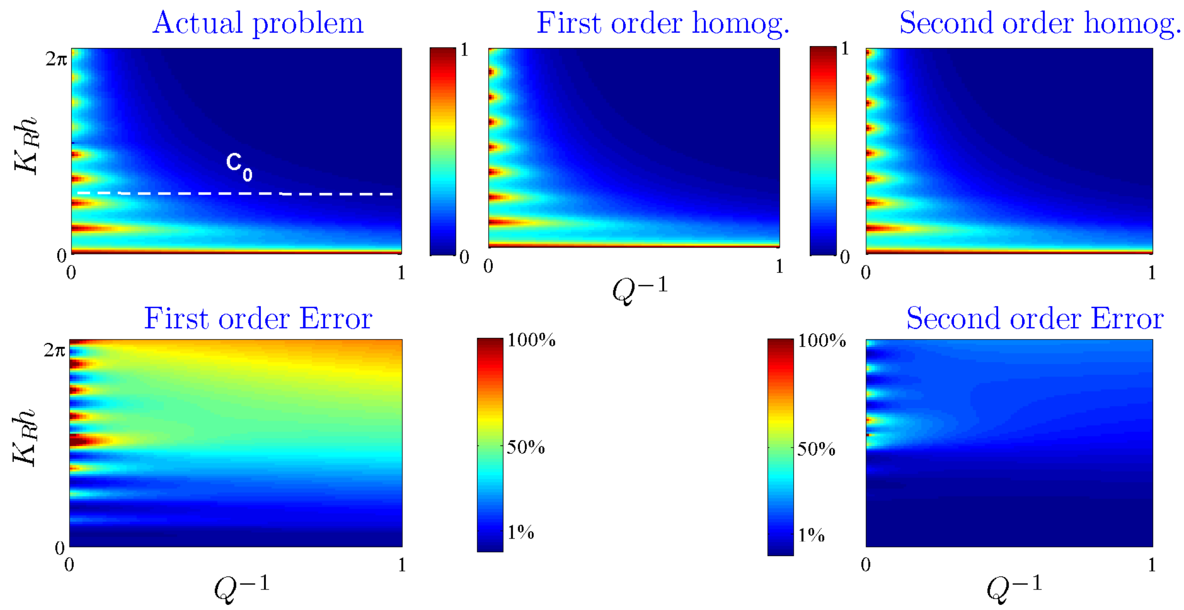

Fig. 9. Up: Transmission coefficients in actual problem $|T^{num}|$ and homogenized $|T|$ at first order ($\mathcal{B} = \mathcal{C} = 0$), and at second order as a function of Q^{-1} and of the frequency $K_R h$; with ($e/h = 4$, $\varphi = 0.1$, $\theta = \pi/3$ and $\gamma = \pi/6$). Down: Errors ΔT on the transmission coefficient, which are calculated numerically. Errors smaller than 1% appear in dark blue, and errors greater than 100% appear in dark red.

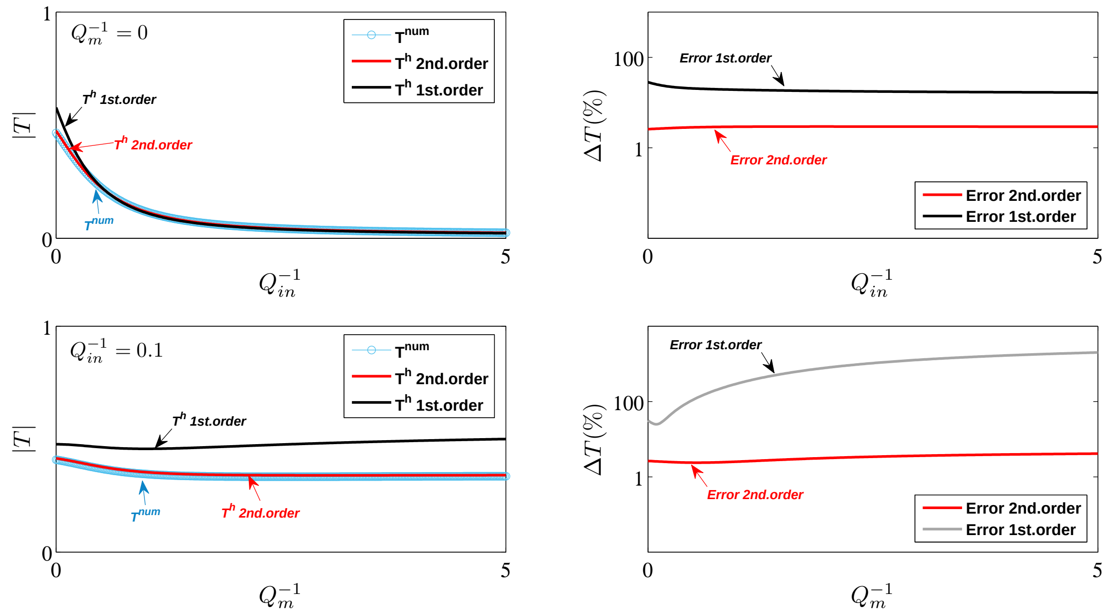

Fig. 10. Up: Transmission coefficients $|T^{num}|$(blue symbol), $|T|$ (black lines at first order and red lines at second order) and errors ΔT(black lines at first order and red liens at second) as a function of Q_{in}^{-1} for elastic substrate ($Q_m^{-1} = 0, K_R h = 0.6\pi$), with ($e/h = 4$, $\varphi = 0.1$, $\theta = \pi/3$). Down: Same representation as a function of Q_m^{-1} for viscoelastic substrate ($Q_{in}^{-1} = 0.1, K_R h = 0.6\pi$), with ($e/h = 4$, $\varphi = 0.1$, $\theta = \pi/3$ and $\gamma = \pi/6$).

2) The error of the transmission as a function of reciprocal quality factor: Finally, we report the transmission coefficient and the corresponding errors as a function of $K_R h$ and the reciprocal quality factor Q^{-1} (Figure 9). We considered $K_R h \in [0, 2\pi]$, $K_R h \simeq 1.07\pi$ corresponding to the cut-off frequency that exists in the actual problem.

In Figure 9, errors smaller than 1% appear in dark blue, and errors greater than 100% appear in dark red. On average, at the intermediate frequencies for $K_R h < \pi/2$ (C_0 profile), the error in the transmission coefficient is smaller than 1% in the whole range of Q^{-1}; it is of 40% on average at first order.

More specifically, we inspect the variations of $|T^{\text{num}}|$ and its homogenized counterparts $|T|$, with the corresponding error ΔT as a function of Q_{in}^{-1} and Q_m^{-1} (Figure 10). In both cases, the homogenized solution at second order is valid for $K_R h = 0.6\pi$ by noting that the ΔT error is less than 2.5% in the whole range of Q_{in}^{-1} or Q_m^{-1}. In general, the second order homogenized solution is more significant than the first order because we recover almost the same results as those observed in the other cases.

IV. Conclusion

In this paper, we have studied a homogenized problem that can replace the actual problem of the scattering of shear waves at a periodically stratified viscoelastic media including finite size e/h. The parameters characteristic of an equivalent anisotropic slab enter into the jump conditions for displacement and normal stress at the boundaries of the slab. They are given by the resolution of elementary problems written in the static limit, and they are therefore wave independent by construction. A significant simplicity of the presented approach is the derivation of effective bulk parameters, which are simply averages of the bulk parameters in each layer. These effective bulk parameters enter into the homogenized wave equation. We have validated this model in the simple case of rectangular voids with Neumann conditions on their boundaries and for a plane wave at oblique incidence on the stratified medium. Note that this simplicity would be lost if periodic media had a more complex unit cell. Finally, explicit expressions of the transmission coefficients deduced from the effective interface parameters have been shown to be accurate for the low-loss viscoelastic media and no low-loss media, with a range of validity being $K_R h < \pi/2$. The present model can be extended to a large class of wave problems, including those in acoustics and electromagnetism.

Appendix A
Determination of the homogenized problem

From the position of a physical problem, we have noticed that the wave equation is identical to the one homogenized in[9], except that in our case, the coefficients of physical parameters entering the equation are complex; which did not change the homogenization procedure and even the form of the homogenized wave equations obtained at different orders. For this reason, we will quote, in this paper, the main steps of the results found.

A. The matched asymptotic expansion

1) Inner and outer expansions: We shall apply the same asymptotic expansion technique as in[9] by spearing the space into three regions. The inner region contains the boundary between the stratified medium and the substrate (Figure 11). The two outer regions for $x_1 > 0$ and $x_1 < 0$ are the regions far enough from the interface, where the evanescent field can be neglected. Next, the inner region and the outer regions are connected using so-called matching conditions, which will constitute boundary conditions for the outer solutions. Owing to this approach, the expansions read

$$\begin{cases} \text{outer region } x_1 > 0, & u^\varepsilon = u^0(\mathbf{x}) + \varepsilon u^1(\mathbf{x}) + \cdots \\ & \sigma^\varepsilon = \sigma^0(\mathbf{x}) + \varepsilon \sigma^1(\mathbf{x}) + \cdots \\ \text{outer region } x_1 < 0, & u^\varepsilon = u^0(\mathbf{x}, y_2) + \varepsilon u^1(\mathbf{x}, y_2) + \cdots \\ & \sigma^\varepsilon = \sigma^0(\mathbf{x}, y_2) + \varepsilon \sigma^1(\mathbf{x}, y_2) + \cdots \\ \text{inner region,} & u^\varepsilon = v^0(x_2, \mathbf{y}) + \varepsilon v^1(x_2, \mathbf{y}) + \cdots \\ & \sigma^\varepsilon = \tau^0(x_2, \mathbf{y}) + \varepsilon \tau^1(x_2, \mathbf{y}) + \cdots \end{cases} \tag{18}$$

With the outer terms (u^n, σ^n) for $x_1 < 0$ and the inner terms (v^n, τ^n) being Y_2 periodic with $Y_2 = (-1/2, 1/2)$; and now, (4a)and (4b) can be written in the inner and in the outer regions, owing to the expressions of the following differential operator

$$\begin{cases} \text{in the outer region, } \nabla \to \nabla_{\mathbf{x}}, & x_1 > 0 \\ \qquad \nabla \to \nabla_{\mathbf{x}} + \dfrac{1}{\varepsilon}\dfrac{\partial}{\partial y_2}\mathbf{e}_2, & x_1 < 0 \\ \text{in the inner region, } \nabla \to \dfrac{\partial}{\partial x_2}\mathrm{e}_2 + \dfrac{1}{\varepsilon}\nabla_{\mathbf{y}} \end{cases} \tag{19}$$

Such as ∇_x and ∇_y means gradient with respect to x and y respectively. Thus, in the inner region, the rapid variations of $(u^\varepsilon, \sigma^\varepsilon)$ are accounted introducing a new system of coordinates $\mathbf{y} = \mathbf{x}/\varepsilon$ such that $(|\nabla U| \sim U/h$ gives $|\nabla_y u| \sim u)$ accounts for the rapid variations of the evanescent field of u^ε; on the other hand, the slow variations along x_2 are accounted for by keeping x_2 as additional coordinate. In the outer region for $x_1 > 0$; there is no rapid variations due to the evanescent field, and the natural coordinates $\mathbf{x} \equiv (x_1, x_2)$ are adapted to describe the propagating field with $(|\nabla U| \sim K_R U$ and thus $\nabla_x u \sim u)$. But for $x_1 < 0$; there is also experiences rapid variations across the layers, and this is accounted for by keeping the coordinate y_2. Finally, from (3), $(a^{*\varepsilon}, b^{*\varepsilon})$ can be specified in the outer regions as

$$\begin{cases} a^{*\varepsilon}(\mathbf{x}) = 1, \quad b^{*\varepsilon}(\mathbf{x}) = (\dfrac{K^*}{K_R})^2, & x_1 > 0 \\ a^{*\varepsilon}(\mathbf{x}) = a^*\left(\dfrac{x_2}{\varepsilon}\right), \quad b^{*\varepsilon}(\mathbf{x}) = b^*\left(\dfrac{x_2}{\varepsilon}\right), & x_1 < 0 \end{cases} \tag{20}$$

and in the inner region as $a^{*\varepsilon}(\mathbf{x}) = \tilde{a^*}(\mathbf{x}/\varepsilon)$ and $b^{*\epsilon}(\mathbf{x}) = \tilde{b^*}(\mathbf{x}/\varepsilon)$ with

$$\tilde{a^*}(\mathbf{y}) = \begin{cases} a^*(y_2), & y_1 < 0 \\ 1, & y_1 > 0 \end{cases} ; \ \tilde{b^*}(\mathbf{y}) = \begin{cases} b^*(y_2), & y_1 < 0 \\ (\dfrac{K^*}{K_R})^2, & y_1 > 0 \end{cases} \tag{21}$$

with $a^*(y_2), b^*(y_2)$ 1-periodic and piecewise complex constant.

2) Boundary conditions and matching conditions: Due to the separation of the space into two regions, something must be said regarding the boundary conditions. For the inner solution, the continuities of the displacement and of the normal stress apply at the boundaries between two layers within the stratified medium and at the boundaries between the layers and the substrate at $y_1 = 0$, whence

$$u^n, \tau^n \cdot \mathbf{n} \quad \text{are continuous everywhere,} \quad n = 0, 1 \ldots \tag{22}$$

But boundary conditions at $|y_1| \to +\infty$ are missing. Reversely, the outer solution is submitted to the radiation conditions at $|x_1| \to +\infty$, but the boundary conditions for $x_1 \to 0^\pm$ are unknown a priori. In fact, these boundary conditions that will provide the jump conditions. The missing conditions for the inner and outer terms are given simultaneously by so-called matching conditions, which indicate that the two solutions have to match in some intermediate region. Following[16] the matching is written for $x_1 \to 0^\pm$ corresponding to $y_1 \to \pm\infty$ (and we denote $f(0^\pm)$ the limit values of f for $x_1 \to 0^\pm$). To do so, we use the Taylor expansions of $u^0(x_1, x_2) = u^0(0^\pm, x_2) + x_1 \partial_{x_1} u^0(0^\pm, x_2) + \cdots = u^0(0^\pm, x_2) + \varepsilon y_1 \partial_{x_1} u^0(0^\pm, x_2) + \cdots$, same for σ^0. Identifying the terms in $\varepsilon^n, n = 0, 1$ in the inner and outer

Fig. 11. Left: configuration in the x coordinate; the periodicity along x_2 is $\varepsilon \equiv K_R h$; the inner region corresponds to the neighborhood of the boundary between the stratified medium $(x_1 < 0)$ and the substrate being a homogeneous medium $(x_1 > 0)$. Right: the unit cell (inner region) in the y coordinate, with $\mathbf{y} = \mathbf{x}/\varepsilon$, and $\mathbf{y} \in \mathbb{R} \times Y_2$, with $Y_2 = (-1/2, 1/2)$.

expansions (18), we get, for $n = 0$

$$\begin{cases} u^0\left(0^-, x_2, y_2\right) = \lim\limits_{y_1 \to -\infty} v^0\left(x_2, y\right) & \text{(23a)} \\ u^0\left(0^+, x_2\right) = \lim\limits_{y_1 \to +\infty} v^0\left(x_2, y\right) & \text{(23b)} \\ \sigma^0\left(0^-, x_2, y_2\right) = \lim\limits_{y_1 \to -\infty} \tau^0\left(x_2, y\right) & \text{(23c)} \\ \sigma^0\left(0^+, x_2\right) = \lim\limits_{y_1 \to +\infty} \tau^0\left(x_2, y\right) & \text{(23d)} \end{cases}$$

and for $n = 1$

$$\begin{cases} u^1\left(0^-, x_2, y_2\right) = \\ \qquad \lim\limits_{y_1 \to -\infty} \left[v^1\left(x_2, \mathbf{y}\right) - y_1 \dfrac{\partial u^0}{\partial x_1}\left(0^-, x_2, y_2\right) \right] \\ u^1\left(0^+, x_2\right) = \\ \qquad \lim\limits_{y_1 \to +\infty} \left[v^1\left(x_2, \mathbf{y}\right) - y_1 \dfrac{\partial u^0}{\partial x_1}\left(0^+, x_2\right) \right] \\ \sigma^1\left(0^-, x_2, y_2\right) = \\ \qquad \lim\limits_{y_1 \to -\infty} \left[\tau^1\left(x_2, \mathbf{y}\right) - y_1 \dfrac{\partial \sigma^0}{\partial x_1}\left(0^-, x_2, y_2\right) \right] \\ \sigma^1\left(0^+, x_2\right) = \\ \qquad \lim\limits_{y_1 \to +\infty} \left[\tau^1\left(x_2, \mathbf{y}\right) - y_1 \dfrac{\partial \sigma^0}{\partial x_1}\left(0^+, x_2\right) \right] \end{cases} \tag{24}$$

B. The homogenized wave equations

We shall directly start by reporting the outer and inner solution at first and second orders, which will be needed to generate the wave equation, up to second order, satisfied by the mean fields $(\bar{u}(\mathbf{x}), \bar{\sigma}(\mathbf{x}))$ with

$$\bar{u} \equiv \left\langle u^0 \right\rangle + \varepsilon \left\langle u^1 \right\rangle, \quad \bar{\sigma} \equiv \left\langle \sigma^0 \right\rangle + \varepsilon \left\langle \sigma^1 \right\rangle. \tag{25}$$

and the average over $y_2 \in Y$ for any function f, is defined by

$$\langle f \rangle(\mathbf{x}) \equiv \int_Y \mathrm{d}y_2 f\left(\mathbf{x}, y_2\right)$$

1) Solutions at first and second order: We note that if f does not depend on $y_2, \langle f \rangle = f$, and using (18) to (21) in actual wave equations (4a)-(4b). We obtain the homogenized wave equations at first and second orders in the following forms : For outer solution $x_1 > 0$, pour $(n = 0, 1)$

$$\begin{cases} \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}^n + \left(\dfrac{K^*}{K_R}\right)^2 u^n = 0 \\ \boldsymbol{\sigma}^n = \boldsymbol{\nabla}_{\mathbf{x}} u^n \end{cases} \tag{26}$$

For inner solution $x_1 < 0$, reads at order ε^{-1} :

$$\begin{cases} u^0\left(\mathbf{x}, y_2\right) = u^0(\mathbf{x}) \\ \sigma_2^0\left(\mathbf{x}, y_2\right) = \sigma_2^0(\mathbf{x}) \end{cases} \tag{27}$$

at order ε^0 :

$$\begin{cases} \operatorname{div}_{\mathbf{x}} \left\langle \sigma^0 \right\rangle + \langle b^* \rangle u^0 = 0 \\ \left\langle \sigma^0 \right\rangle(\mathbf{x}) = \langle a^* \rangle \dfrac{\partial u^0}{\partial x_1}(\mathbf{x}) \mathbf{e}_1 + \langle 1/a^* \rangle^{-1} \dfrac{\partial u^0}{\partial x_2}(\mathbf{x}) \mathbf{e}_2 \end{cases} \tag{28}$$

at order ε^1 :

$$\begin{cases} \operatorname{div}_{\mathbf{x}} \left\langle \boldsymbol{\sigma}^1 \right\rangle(\mathbf{x}) + \langle b^* \rangle \left\langle u^1 \right\rangle(\mathbf{x}) = 0 \\ \left\langle \sigma_1^1 \right\rangle(\mathbf{x}) = \langle a^* \rangle \dfrac{\partial \left\langle u^1 \right\rangle}{\partial x_1}(\mathbf{x}) \quad ; \\ \left\langle \sigma_2^1 \right\rangle(\mathbf{x}) = \langle 1/a^* \rangle^{-1} \dfrac{\partial \left\langle u^1 \right\rangle}{\partial x_2}(\mathbf{x}) \end{cases} \tag{29}$$

We got (29) at order ε^1 thanks to the two following relations demonstrated in [9] (see subsection 2.2.2)

$$\left\langle f(\cdot) u^1(\mathbf{x}, \cdot) \right\rangle = \langle f \rangle \left\langle u^1 \right\rangle(\mathbf{x})$$

and

$$\left\langle f(\cdot) \sigma_2^1(\mathbf{x}, \cdot) \right\rangle = \langle f \rangle \left\langle \sigma_2^1 \right\rangle(\mathbf{x}) \quad \text{for any even f}$$

Both relations use the same property: consider a piecewise differentiable function $g(y)$, with $g'(y)$ even; then $(g - \langle g \rangle)$ is odd, and for any function $f(y)$ being even, $f(g - \langle g \rangle)$ is odd.

2) Up to second order solution: Finally, to determine the homogenized wave equation up to second order for $(\bar{u}(\mathbf{x}), \bar{\sigma}(\mathbf{x}))$, it is enough to apply (26), (28) and (29) in (25)

$$\begin{cases} \operatorname{div} \bar{\sigma} + \langle b^* \rangle \bar{u} = 0, & \text{for } x_1 < 0 \\ \bar{\sigma} = \begin{pmatrix} \langle a^* \rangle & 0 \\ 0 & \langle 1/a^* \rangle^{-1} \end{pmatrix} \nabla \bar{u} \end{cases} \tag{30}$$

Next, (26) to (29) with the boundary conditions and the matching conditions will be used to find the conditions to be applied on an equivalent interface at $x_1 = 0$, so-called jump condition.

C. The jump conditions and determination of the interface parameters

We start with the jump conditions at the first order $[\![v^0]\!]$ and $[\![\sigma_1^0]\!]$, where we defined

$$[\![f]\!] \equiv f\left(0^+, x_2, \tau\right) - f\left(0^-, x_2, \tau\right) \tag{31}$$

For any outer terms f being discontinuous across an equivalent interface at $x_1 = 0$, with $(f^-; f^+)$ its values on both sides.

1) Jump conditions at first order: The actual wave equations (4a)-(4b) for the inner problem at the leading order in ε^{-1} give

$$\nabla_{\mathbf{y}} v^0 = 0, \quad div_y \tau^0 = 0,$$

from which we deduce that v^0 does not depend on $\mathbf{y}$. With (27), $u^0(\mathbf{x})$ does not depend too on y_2, thus

$$u^0\left(0^-, x_2\right) = u^0\left(0^+, x_2\right) = v^0\left(x_2\right) \tag{32}$$

Next, integrating $div_y \tau^0 = 0$ over $\mathbb{R} \times Y_2$ (Figure 12), and using (i) the continuity of $\tau^0 \cdot \mathbf{n}$ between the layers along y_2 , and (ii) the periodicity of τ^0 with respect to y_2, we get

$$\int_Y \mathrm{d}y_2 \left[\tau_1^0\left(x_2, +\infty, y_2\right) - \tau_1^0\left(x_2, -\infty, y_2\right)\right] = 0$$

Finally, integrating the matching condition (23c) and (23d) over Y, we get

$$\left\langle\sigma_1^0\right\rangle\left(0^-, x_2\right) = \sigma_1^0\left(0^+, x_2\right) \tag{33}$$

By using (31), we deduce from (32)-(33) the jump conditions at the first order

$$\llbracket u^0 \rrbracket = \llbracket\left\langle\sigma_1^0\right\rangle\rrbracket = 0. \tag{34}$$

From(34), we note that the normal displacement and stress are continued, which requires us to go up to the second order to capture the effect of boundary layers at $x_1 = 0$. To obtain the jump conditions at the second order, we need to find the solutions of the elementary problems.

2) The elementary problems: From actual wave equations (4a) at order ε^{-1} and (4b) at order ε^0, the matching conditions (23c)-(23d), it follows that the system satisfied by $v^1\left(x_2, \mathbf{y}\right)$ can be written as follows :

$$\begin{cases} \operatorname{div}_{\mathbf{y}} \tau^0 = 0 \quad \text{with} \\ \tau^0 = \tilde{a^*}(\mathbf{y})\left[\dfrac{\partial u^0}{\partial x_2}\left(0, x_2\right) \mathrm{e}_2 + \nabla_{\mathbf{y}} v^1\left(x_2, \mathbf{y}\right)\right] \\ v^1 \text{ and } \tau^0 \cdot \mathbf{n} \text{ continuous,} \\ \displaystyle\lim_{y_1 \to -\infty} \nabla_{\mathbf{y}} v^1\left(x_2, \mathbf{y}\right) = \left\langle a^*\right\rangle^{-1}\left\langle\sigma_1^0\right\rangle\left(0, x_2\right) \mathbf{e}_1 \\ \qquad + \dfrac{1/a^*\left(y_2\right) - \left\langle 1/a^*\right\rangle}{\left\langle 1/a^*\right\rangle} \dfrac{\partial u^0}{\partial x_2}\left(0, x_2\right) \mathbf{e}_2 \\ \displaystyle\lim_{y_1 \to +\infty} \nabla_{\mathbf{y}} v^1\left(x_2, \mathbf{y}\right) = \left\langle\sigma_1^0\right\rangle\left(0, x_2\right) \mathbf{e}_1 \end{cases} \tag{35}$$

with v^1 and τ^0 periodic with respect to y_2, and we have used (22). The system (35) is linear with respect to $\left\langle\sigma_1^0\right\rangle\left(0, x_2\right)$ and $\partial_{x_2} u^0\left(0, x_2\right)$. Thus, we define $V^{(1)}(\mathbf{y})$ and $V^{(2)}(\mathbf{y})$ such that

$$\begin{cases} v^1\left(x_2, \mathbf{y}\right) = \left\langle\sigma_1^0\right\rangle\left(0, x_2\right) V^{(1)}(\mathbf{y}) \\ \qquad + \dfrac{\partial u^0}{\partial x_2}\left(0, x_2\right)\left[A^*\left(y_2\right) + V^{(2)}(\mathbf{y})\right] + \hat{v}\left(x_2\right) \\ \tau^0\left(x_2, \mathbf{y}\right) = \left\langle\sigma_1^0\right\rangle\left(0, x_2\right) \mathbf{T}^{(1)}(\mathbf{y}) \\ \qquad + \dfrac{\partial u^0}{\partial x_2}\left(0, x_2\right)\left[\dfrac{\tilde{a^*}(\mathbf{y})/a^*\left(y_2\right)}{\left\langle 1/a^*\right\rangle} \mathbf{e}_2 + \mathbf{T}^{(2)}(\mathbf{y})\right] \end{cases} \tag{36}$$

with

$$A^*\left(y_2\right) \equiv \int_{-1/2}^{y_2} \mathrm{d}y \frac{1/a^*(y) - \left\langle 1/a^*\right\rangle}{\left\langle 1/a^*\right\rangle}$$

And

$$\mathbf{T}^{(1)}(\mathbf{y}) \equiv \tilde{a^*}(\mathbf{y}) \nabla V^{(1)}(\mathbf{y}) \quad ; \quad \mathbf{T}^{(2)}(\mathbf{y}) \equiv \tilde{a^*}(\mathbf{y}) \nabla V^{(2)}(\mathbf{y})$$

We notice that the field v^1 in (35) is defined up to a function of x_2, and it is denoted $\hat{v}\left(x_2\right)$ in (36); we shall see that the determination of $\hat{v}\left(x_2\right)$ is not needed. It is easy to see that if $\left(V^{(1)}, \mathbf{T}^{(1)}\right)$ satisfy the elementary problems.

$$\begin{cases} \operatorname{div} \mathbf{T}^{(1)} = 0 \quad \text{with } \mathbf{T}^{(1)}(\mathbf{y}) = \tilde{a^*}(\mathbf{y}) \nabla V^{(1)}(\mathbf{y}) \\ V^{(1)} \text{ and } \mathbf{T}^{(1)} \cdot \mathbf{n} \text{ continuous} \\ V^{(1)}, \mathbf{T}^{(1)} \quad \text{periodic with respect to } y_2 \\ \displaystyle\lim_{y_1 \to -\infty} \nabla V^{(1)}(\mathbf{y}) = \frac{\mathbf{e}_1}{\left\langle a^*\right\rangle}, \quad \lim_{y_1 \to +\infty} \nabla V^{(1)}(\mathbf{y}) = \mathbf{e}_1 \end{cases} \tag{37}$$

and

$$\begin{cases} \operatorname{div}\left[\mathbf{T}^{(2)} + \dfrac{\tilde{a^*}(\mathbf{y})/a^*\left(y_2\right)}{\left\langle 1/a^*\right\rangle} \mathbf{e}_2\right] = 0 \quad \text{with} \\ \mathbf{T}^{(2)}(\mathbf{y}) = \tilde{a^*}(\mathbf{y}) \nabla V^{(2)}(\mathbf{y}) \\ V^{(2)} \text{ and } \left[\mathbf{T}^{(2)} + \dfrac{\tilde{a^*}(\mathbf{y})/a^*\left(y_2\right)}{\left\langle 1/a^*\right\rangle} \mathrm{e}_2\right] \cdot \mathbf{n} \text{ continuous,} \\ V^{(2)}, \mathbf{T}^{(2)} \text{ periodic with respect to } y_2 \\ \displaystyle\lim_{y_1 \to -\infty} \nabla V^{(2)}(\mathbf{y}) = 0, \\ \displaystyle\lim_{y_1 \to +\infty} \nabla V^{(2)}(\mathbf{y}) = -\frac{1/a^*\left(y_2\right) - \left\langle 1/a^*\right\rangle}{\left\langle 1/a^*\right\rangle} \mathrm{e}_2 \end{cases} \tag{38}$$

then $v^1\left(x_2, \mathbf{y}\right)$ satisfies (35). Next, by integrating the limits of $\nabla V^{(i)}, (i = 1, 2)$, with $V^{(\mathrm{i})}$ are defined up to the constants in 37 and 38, we can write the following

$$\begin{cases} \displaystyle\lim_{y_1 \to -\infty}\left[V^{(1)} - \frac{y_1}{\left\langle a^*\right\rangle}\right] = -\mathcal{B} \\ \displaystyle\lim_{y_1 \to +\infty}\left[V^{(1)} - y_1\right] = 0 \end{cases} \begin{cases} \displaystyle\lim_{y_1 \to -\infty} V^{(2)} = -\mathcal{B}' \\ \displaystyle\lim_{y_1 \to +\infty} V^{(2)} = -A^*\left(y_2\right) \end{cases} \tag{39}$$

For instance, we denoted by $-\mathcal{B}$ (it is the first interface parameter) and $-\mathcal{B}'$ the constants at $y_1 \to -\infty$ for $V^{(1)}$ and $V^{(2)}$ respectively. Next, $V^{(2)}$ being odd with respect to y_2, we have $\mathcal{B}' = 0$. Finally, since the unknown constants being a priori different at $y_1 \to \pm\infty$, we can set these constants equal zero at $y_1 \to +\infty$ for $V^{(1)}$ and $V^{(2)}$.

3) Jump conditions at second order: In order to find $\left\langle u^1\right\rangle$ and $\left\langle\sigma_1^1\right\rangle$, one can use the same steps followed in [9] (see subsection 2.3.3) by using the matching conditions (24). We finally obtain the following results:

$$\llbracket\left\langle u^1\right\rangle\rrbracket = \mathcal{B}\left\langle\sigma_1^0\right\rangle\left(0, x_2\right) \tag{40}$$

And

$$\llbracket\left\langle\sigma_1^1\right\rangle\rrbracket = -\mathcal{C} \frac{\partial^2 u^0}{\partial x_2^2}\left(0, x_2\right) \tag{41}$$

With $\mathcal{C} \equiv \int_Y \mathrm{d}\mathbf{y} T_2^{(2)}(\mathbf{y})$ is the second interface parameter.

4) Up to second order jump conditions: Finlay, we have used (25) and (31) to obtain the final jumps on $\bar{u}$ and $\bar{\sigma}$, as follows

$$\llbracket\bar{u}\rrbracket = \llbracket\left\langle u^0\right\rangle\rrbracket + \varepsilon\llbracket\left\langle u^1\right\rangle\rrbracket, \quad \llbracket\overline{\sigma_1}\rrbracket = \llbracket\left\langle\sigma_1^0\right\rangle\rrbracket + \varepsilon\llbracket\left\langle\sigma_1^1\right\rangle\rrbracket \tag{42}$$

Then, we deduce from 34,(40) and (41) the following:

$$\llbracket\bar{u}\rrbracket = \varepsilon\mathcal{B}\left\langle\sigma_1^0\right\rangle\left(0, x_2\right), \quad \llbracket\bar{\sigma}_1\rrbracket = -\varepsilon\mathcal{C} \frac{\partial^2 u^0}{\partial x_2^2}\left(0, x_2\right) \tag{43}$$

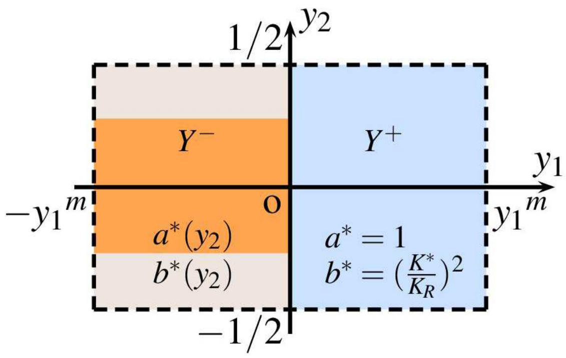

Fig. 12. The domain $\mathbf{Y} = Y^- \cup Y^+$, with $Y^- = \left(-y_1^m, 0\right) \times Y_2, Y^+ = \left(0, +y_1^m\right) \times Y_2$. $\tilde{a^*}(\mathbf{y}) = a^*\left(y_2\right)$ and $\tilde{b^*}(\mathbf{y}) = b^*\left(y_2\right)$ in Y^-, and $a = 1, b = (\frac{K^*}{K_R})^2$ in Y^+

D. The final homogenized problem

The equations in the substrate 26 and the associated jump conditions 34 could be used to solve the homogenized problem iteratively: first compute $\left(u^0, \left\langle\sigma^0\right\rangle\right)$ (also compute $\mathcal{B}$ and $\mathcal{C}$) and use the results to get the right hand-side term in (43); then, compute $\left(u^1, \sigma^1\right)$; finally, we obtain $(\bar{u}, \bar{\sigma})$ in (25) which approximate $\left(u^\varepsilon, \sigma^\varepsilon\right)$ up to $O\left(\varepsilon^2\right)$. As discussed in [17], it is preferable to handle a unique problem, and this is done by defining the fields $\left(u^h, \sigma^h\right)$ satisfying the following homogenized problem:

$$\begin{cases} \operatorname{div} \sigma^h + \langle b^* \rangle u^h = 0, & x_1 < 0 \\ \sigma^h = \begin{pmatrix} \langle a^* \rangle & 0 \\ 0 & \langle 1/a^* \rangle^{-1} \end{pmatrix} \nabla u^h & \\ \operatorname{div} \sigma^h + (\dfrac{K^*}{K_R})^2 u^h = 0, & x_1 > 0 \\ \sigma^h = \nabla u^h & \\ [\![u^h]\!] = \dfrac{\varepsilon \mathcal{B}}{2} \left[\sigma_1^h \left(0^-, x_2\right) + \sigma_1^h \left(0^+, x_2\right) \right] & \\ [\![\sigma_1^h]\!] = -\dfrac{\varepsilon \mathcal{C}}{2} \left[\dfrac{\partial^2 u^h}{\partial x_2^2} \left(0^-, x_2\right) + \dfrac{\partial^2 u^h}{\partial x_2^2} \left(0^+, x_2\right) \right] & \end{cases} \tag{44}$$

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