Common Fixed Point Theorem in Non-Archimedean Menger PM-Spaces Using CLR Property with Application to Functional Equations

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Abstract—**In this paper, we prove the common fixed point theorems for weakly compatible mappings in non-Archimedean Menger PM-spaces, that use the common limit range property. In addition, we give some examples of these results. Then we will extend our main result to four finite families of self-mappings by the notion of pairwise commuting. Finally, we will give applications for our main theorem.**

Index Terms—**Parabolic equation, Non-linear equations, Integral condition, Existence, Uniqueness, Fadeo-Galarkin method. blow-up.**

I. Introduction

THE non-Archimedean probabilistic metric spaces (briefly, N.A.PM-spaces) and some of their topological properties were first studied by Istrătescu and Criv ăt [1] in 1974. Istrătescu [2] obtained some fixed point theorems on N.A. Menger PM-spaces and generalized the results of Sehgal and Bharicha Reid [3]. Further, Hadžić [4] improved the results of Istrătescu [2]. The theory of probabilistic metric spaces is mainly used in probabilistic functional analysis since it has many applications in random differential equations and integral equations [2]. Many mathematicians studied the weaker forms of commutativity in N.A. Menger PM-spaces, e.g. Singh and Pant [5] studied the notion of weakly commuting mappings (introduced by Sessa [6] in metric space), Cho et al. [7] studied the notion of compatible mappings (introduced by Jungck [8] in metric space). Besides, Rao and Ramudu [9] studied the notion of weakly compatible mappings (introduced by Jungck and Rhoades [10] in metric space) and proved several fixed point results in this direction. The concept of weakly compatible mappings is the most general among all the commutativity concepts, as each pair of weakly commuting self mappings is compatible, and each pair of compatible self mappings is weakly compatible. But the converse is not true. In 2002, Aamri and Moutawakil [11] defined the notion of property (E.A) which contained the class of non-compatible mappings. They observed that the property (E.A) requires the completeness (or closedness) of the subspaces for the existence of a common fixed point. As a further generalization, a new notion of CLR_g property given by Sintunavarat and Kuman [12], does not impose such conditions. The importance of CLR_g property is to ensure that one does not require the closedness of the range of subspaces. This concept was used by Singh et al. [13] who proved a common fixed point theorem for a pair of weakly compatible self mappings in an N.A. Menger PM-space employing common limit range property. Recently, Imdad et al. [14] extended the notion of common limit range property to two pairs of self mappings which further relaxes the requirement on closedness of the subspaces. Since then, a number of fixed point theorems have been established by several researchers in different ways under the common limit range property. For more information about such topic, we refer the reader to [15]–[35] and references therein. We underline the fact that the theory of fixed points in N.A. Menger PM-spaces is an active area of mathematical research, for example, Dimri and Pant, studied the application of N.A. Menger PM-spaces to product spaces. Thus far, several authors studied common fixed point theorems in N.A. Menger PM-spaces which include [9], [13]. In this paper, we prove the common fixed point theorems for weakly compatible mappings in N.A. Menger PM-spaces that satisfy the common limit range property and give some examples to illustrate our results. We also extend our main result to four finite families of self-mappings. Our results substantially improve and generalize several known existing in the literature. We conclude with an application to some system of functional equations arising in dynamic programming.

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II. Preliminaries

In this section, we will define some definitions and mathematical preliminaries that we will use later.

Definition 1: ([2]) *A triangular norm (briefly a t-norm)*

Δ is a binary operation on the unit interval $[0,1]$ such that for all $a,b,c,d \in [0,1]$ the following conditions are satisfied:

1) $\Delta(a,1) = a$;
2) $\Delta(a,b) = \Delta(b,a)$;
3) $\Delta(a,b) \geq \Delta(c,d)$, whenever $a \geq c, b \geq d$;
4) $\Delta(\Delta(a,b),c) = \Delta(a,\Delta(b,c))$.

Some examples of t-norm are $\Delta_M(a,b) = \min\{a,b\}$, Δ_P $(a,b) = ab$ and $\Delta(a,b) = \max\{a+b-1,0\}$.

Definition 2: ([2]) A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is a distribution function if it is non-decreasing and left continuous with the $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and the $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

We denote $\Im$ as the set of all distribution functions, and denote H the specific distribution function defined by:

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

If X is a non-empty set, $\digamma : X \times X \to \Im$ is called a probabilistic distance on X. In addition $\digamma(x,y)$ is denoted by $F_{x,y}$.

Definition 3: ([2]) The ordered pair $(X,\digamma)$ is an N.A.PM-space, if X is a non-empty set and $\digamma$ is a probabilistic distance satisfying the flowing conditions: for all $x,y,z \in X$ and $t,t_1,t_2 > 0$,

1) $F_{x,y}(t) = 1 \iff x = y$;
2) $F_{x,y}(t) = F_{y,x}(t)$;
3) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$, then $F_{x,z}(\max\{t_1,t_2\}) = 1$. The ordered triplet $(X,\digamma,\Delta)$ is called an N.A. Menger PM-space if $(X,\digamma)$ is an N.A.PM-space, Δ is a t-norm and the following inequality holds:

$$F_{x,z}(\max\{t_1,t_2\}) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2)),$$

for all $x,y,z \in X$ and $t_1,t_2 > 0$.

The concept of neighborhoods in Menger PM-spaces was introduced by Schweizer and Sklar [36]. If $x \in X, \epsilon > 0$ and $\lambda \in (0,1)$, then an (ϵ,λ)-neighbourhood of $x, U_x(\epsilon,\lambda)$ is defined by:

$$U_x(\varepsilon,\lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1-\lambda\}.$$

If the t-norm Δ be continuous and strictly increasing, then $(X,\digamma,\Delta)$ is a Hausdorff space in the topology induced by the family $\{U_x(\varepsilon,\lambda) : x \in X, \varepsilon > 0, \lambda \in (0,1)\}$ of neighbourhoods [5].

Example 1: Let X be any set with at least two elements. If we define $F_{x,x}(t) = 1$ for all $x \in X, t > 0$ and

$$F_{x,y}(t) = \begin{cases} 0 & \text{if} \quad t \leq 1, \\ 1 & \text{if} \quad t > 1, \end{cases}$$

where $x,y \in X, x \neq y$, then $(X,\digamma,\Delta)$ is an N.A. Menger PM-space with $\Delta(a,b) = \min\{a,b\}$ or (ab) for all $a,b \in [0,1]$.

Example 2: Let $X = \mathbb{R}$ be the set of real numbers equipped with the metric defined by $d(x,y) = |x-y|$ and

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|} & \text{if} \quad t > 0; \\ 0, & \text{if} \quad t = 0. \end{cases}$$

Then $(X,\digamma,\Delta)$ is an N.A. Menger PM-space with Δ as continuous t-norm satisfying $\Delta(a,b) = \min\{a,b\}$ or (ab) for all $a,b \in [0,1]$.

Next, we let $\Omega = \{g \,/ g : [0,1] \to [0,\infty)\}$ be a continuous, strictly, decreasing function such that $g(1) = 0$ and $g(0) < \infty$.

Definition 4: ([7]) Let $g \in \Omega$. An N.A. Menger PM-space $(X,\digamma,\Delta)$ is said to be of type $(C)_g$ if

$$g(F_{x,z}(t)) \leq g(F_{x,y}(t)) + g(F_{y,z}(t))$$

for all $x,y,z \in X, t > 0$.

Definition 5: ([7]) Let $g \in \Omega$. An N.A. Menger PM-space $(X,\digamma,\Delta)$ is said to be of type $(D)_g$ if

$$g(\Delta(t_1,t_2)) \leq g(t_1) + g(t_2),$$

for all $t_1,t_2 \in [0,1]$.

Remark 1: ([2]) If an N.A. Menger PM-space $(X,\digamma,\Delta)$ is of type $(D)_g$, then:

1) it is of type $(C)_g$,
2) it is metrizable, where the metric d on X is defined by:

$$d(x,y) = \int_0^1 g(F_{x,y}(t))\,dt,$$

for all $x,y \in X$.

We denote the $(X,\digamma,\Delta)$ as N.A. Menger PM-space with a continuous strictly increasing t-norm Δ.

Definition 6: ([2]) Two self mappings A and S of an N.A. Menger PM-space $(X,\digamma,\Delta)$ are said to be compatible if $\lim_{n\to\infty} g(F_{ASx_n,SAx_n}(t)) = 0$ for all $t > 0$ and $g \in \Omega$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

Definition 7: A pair (A,S) of self mappings of an N.A. Menger PM-space $(X,\digamma,\Delta)$ is said to satisfy $(E.A)$ property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in X$.

Definition 8: ([2]) A pair (A,S) of self mappings of a non-empty set X is said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $Az = Sz$ for some $z \in X$, then $ASz = SAz$. If two self-mappings A and S of an N.A. Menger PM-space $(X,\digamma,\Delta)$ are compatible, then they are weakly compatible, but the converse need not be true. It can be noticed that the notions of weak compatibility and property $(E.A)$ are independent of each other.

Definition 9: Two pairs (A,S) and (B,T) of self mappings of an N.A. Menger PM-space $(X,\digamma,\Delta)$ are said to satisfy the common property $(E.A)$, if there exist two sequences $\{x_n\}\, and\, \{y_n\}$ in X for some z in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z.$$

Definition 10: A pair (A,S) of self mappings of an N.A. Menger PM-space $(X,\digamma,\Delta)$ is said to satisfy the common limit range property with respect to mapping S, denoted by (CLR_S), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

for some $z \in S(X)$.

Definition 11: Two pairs (A,S) and (B,T) of self mappings of an N.A. Menger PM-space $(X,\digamma,\Delta)$ are said to satisfy the common limit range property with respect to mappings S and T, denoted by (CLR_{ST}), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Definition 12: ([14]) Two families of self mappings $\{A_i\}$ and $\{S_j\}$ are said to be pairwise commuting if they satisfy:

1) $A_iA_j = A_jA_i, i,j \in \{1,2,...,m\}$,
2) $S_kS_l = S_lS_k, k,l \in \{1,2,...,n\}$,
3) $A_iS_k = S_kA_i\ , i \in \{1,2,...,m\}, k \in \{1,2,...,n\}$.

III. Main results

In this section, we prove the common fixed point theorem for the compatible mappings in Non-Archimedean Menger PM-spaces by denoting Φ as the collection of all functions $\varphi : [0,\infty) \to [0,\infty)$ which are upper semicontinuous from the right and satisfy $\varphi(t) < t$, for all $t > 0$.

Lemma 1: ([7]) If a function $\phi : [0,\infty) \to [0,\infty)$ belongs to the class Φ, then we have:

1) for all $t \geq 0$, $\lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is the n^{th}-iteration of $\phi(t)$;
2) if $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n)$ where $n = 1,2,...$, then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$ for each $t \geq 0$, then $t = 0$.

Now we state and prove our first main result.

Theorem 1: Let A, B, S and T be four self-mappings of an N.A. Menger PM-space (X, F, Δ), where Δ is a continuous t-norm, satisfying

$$g(F_{Ax,By}(t)) \leq \phi\left(\max\left\{\begin{array}{c} g(F_{Sx,Ty}(t)), g(F_{Sx,Ax}(t)), \\ g(F_{Ty,By}(t)) \\ \frac{1}{2}[g(F_{Sx,By}(t)) + g(F_{Ty,Ax}(t))], \\ \min\{g(F_{Sx,By}(t)), g(F_{Ty,Ax}(t))\} \\ \sqrt{g(F_{Sx,Ty}(t)).g(F_{Ty,Ax}(t))}, \\ \frac{g(F_{Sx,Ax}(t)).g(F_{Sx,By}(t))}{g(F_{Ax,By}(t))} \end{array}\right\}\right),$$

for all $x, y \in X, t > 0$, where $g \in \Omega$ and $\phi \in \Phi$. If the pairs (A,S) and (B,T) have the (CLR_{ST}) property, then (A,S) and (B,T) have a coincidence point each. Moreover, A, B, S, and T have a unique common fixed point provided both pairs (A,S) and (B,T) are weakly compatible.

Proof: In view of the fact that the pairs (A,S) and (B,T) have the (CLR_{ST}) property, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. As $z \in S(X)$, there exists a point $v \in X$ such that $Sv = z$. First we assert that $Av = Sv$. Then by using the inequality (1) with $x = v and y = y_n$, we get:

$$g(F_{Av,By_n}(t)) \leq \phi\left(\max\left\{\begin{array}{c} g(F_{Sv,Ty_n}(t)), g(F_{Sv,Av}(t)), \\ g(F_{Ty_n,By_n}(t)) \\ \frac{1}{2}[g(F_{Sv,By_n}(t)) + g(F_{Ty_n,Av}(t))], \\ \min\{g(F_{Sv,By_n}(t)), g(F_{Ty_n,Av}(t))\} \\ \sqrt{g(F_{Ty_n,By_n}(t)).g(F_{Ty_n,Av}(t))}, \\ \frac{g(F_{Sv,Av}(t)).g(F_{Sv,By_n}(t))}{g(F_{Av,By_n}(t))} \end{array}\right\}\right)$$

Passing to the limit as $n \to \infty$, this reduces to

$$\begin{aligned} g(F_{Av,z}(t)) &\leq \phi\left(\max\left\{\begin{array}{c} g(F_{z,z}(t)), g(F_{z,Av}(t)), \\ g(F_{z,z}(t)) \\ \frac{1}{2}[g(F_{z,z}(t)) + g(F_{z,Av}(t))], \\ \min\{g(F_{z,z}(t)), g(F_{z,Av}(t))\} \\ \sqrt{g(F_{z,z}(t)).g(F_{z,Av}(t))}, \\ \frac{g(F_{z,Az}(t)).g(F_{z,z}(t))}{g(F_{Av,z}(t))} \end{array}\right\}\right) \\ &= \phi\left(\max\left\{\begin{array}{c} g(1), g(F_{z,Av}(t)), g(1) \\ \frac{1}{2}[g(1) + g(F_{z,Av}(t))], \min\{g(1), \\ g(F_{z,Av}(t))\} \\ \sqrt{g(1).g(F_{z,Av}(t))}, g(1) \end{array}\right\}\right) \\ &= \phi\left(\max\left\{0, g(F_{z,Av}(t)), 0, \frac{1}{2}g(F_{z,Av}(t)), 0, 0, 0\right\}\right) \\ &= \phi(g(F_{z,Av}(t))). \end{aligned}$$

Making use of Lemma 1, we get $Av = Sv = z$, which shows v is a coincidence point of the pair (A,S). As $z \in T(X)$, there exists a point $u \in X$ such that $Tu = z$. We show that $Bu = Tu$. Using the inequality (1) with $x = v and y = u$, we get:

$$g(F_{Av,Bu}(t)) \leq \phi\left(\max\left\{\begin{array}{c} g(F_{Sv,Tu}(t)), g(F_{Sv,Av}(t)), \\ g(F_{Tu,Bu}(t)) \\ \frac{1}{2}[g(F_{Sv,Bu}(t)) + g(F_{Tu,Av}(t))], \\ \min\{g(F_{Sv,Bu}(t)), g(F_{Tu,Av}(t))\} \\ \sqrt{g(F_{Tu,Bu}(t)).g(F_{Tu,Av}(t))}, \\ \frac{g(F_{Sv,Av}(t)).g(F_{Sv,Bu}(t))}{g(F_{Av,Bu}(t))} \end{array}\right\}\right),$$

that is,

$$\begin{aligned} g(F_{z,Bu}(t)) &\leq \phi\left(\max\left\{\begin{array}{c} g(F_{z,Tu}(t)), g(F_{z,z}(t)), \\ g(F_{z,Bu}(t)) \\ \frac{1}{2}[g(F_{z,Bu}(t)) + g(F_{z,z}(t))], \\ \min\{g(F_{z,Bu}(t)), g(F_{z,z}(t))\} \\ \sqrt{g(F_{z,Bu}(t)).g(F_{Tu,z}(t))}, \\ \frac{g(F_{z,z}(t)).g(F_{z,Bu}(t))}{g(F_{z,Bu}(t))} \end{array}\right\}\right) \\ &= \phi\Big(\max\Big\{g(1), g(1), g(F_{z,Bu}(t)), \frac{1}{2}g(F_{z,Bu}(t)), \\ &\quad g(1), 0, g(1)\Big\}\Big) = \phi(g(F_{z,Bu}(t))). \end{aligned}$$

Hence, by Lemma 1, we have $Bu = Tu = z$, which shows that u is a coincidence point of the pair (B,T). In the case when the pair (A,S) is weakly compatible, $Av = Sv$, which implies that $Az = ASv = SAv = Sz$. Now, we can show that z is a common fixed point of the pair (A,S). By putting $x = z$ and $y = v$ in the inequality (1), we have

$$g(F_{Az,Bu}(t)) \leq \phi\left(\max\left\{\begin{array}{c} g(F_{Sz,Tu}(t)), g(F_{Sz,Az}(t)), \\ g(F_{Tu,Bu}(t)) \\ \frac{1}{2}[g(F_{Sz,Bu}(t)) + g(F_{Tu,Az}(t))], \\ \min\{g(F_{Sz,Bu}(t)), g(F_{Tu,Az}(t))\} \\ \sqrt{g(F_{Tu,Bu}(t)).g(F_{Tu,Az}(t))}, \\ \frac{g(F_{Sz,Az}(t)).g(F_{Sz,Bu}(t))}{g(F_{Az,Bu}(t))} \end{array}\right\}\right),$$

implying that

$$g\left(F_{Az,z}\left(t\right)\right) \leq \phi\left(\max\left\{\begin{array}{c} g\left(F_{Az,z}\left(t\right)\right), g\left(F_{Az,Az}\left(t\right)\right), \\ g\left(F_{z,z}\left(t\right)\right) \\ \frac{1}{2}\left[g\left(F_{Az,z}\left(t\right)\right)+g\left(F_{z,Az}\left(t\right)\right)\right], \\ \min\left\{g\left(F_{Az,z}\left(t\right)\right), g\left(F_{z,Az}\left(t\right)\right)\right\} \\ \sqrt{g\left(F_{z,z}\left(t\right)\right).g\left(F_{z,Az}\left(t\right)\right)}, \\ \frac{g\left(F_{Az,Az}(t)\right).g\left(F_{Az,z}(t)\right)}{g\left(F_{Az,z}(t)\right)} \end{array}\right\}\right)$$

$$= \phi\big(\max\left\{g\left(F_{Az,z}\left(t\right)\right), g\left(1\right), g\left(1\right), g\left(F_{Az,z}\left(t\right)\right), g\left(1\right), 0, g\left(1\right)\right\}\big) = \phi\left(g\left(F_{Az,z}\left(t\right)\right)\right).$$

By Lemma 1, we have $Az = z = Sz$ which shows that z is a common fixed point of the pair (A, S). Again, when the pair (B, T) is weakly compatible, $Bu = Tu$, which implies that $Bz = BTu = TBu = Tz$. Now, we show that z is a common fixed point of the pair (B, T). By putting $x = v$ and $y = z$ in the inequality (1), we have

$$g\left(F_{Av,Bz}\left(t\right)\right) \leq \phi\left(\max\left\{\begin{array}{c} g\left(F_{Sv,Tz}\left(t\right)\right), g\left(F_{Sv,Av}\left(t\right)\right), \\ g\left(F_{Tz,Bz}\left(t\right)\right) \\ \frac{1}{2}\left[g\left(F_{Sv,Bz}\left(t\right)\right)+g\left(F_{Tz,Av}\left(t\right)\right)\right], \\ \min\left\{g\left(F_{Sv,Bz}\left(t\right)\right), g\left(F_{Tz,Av}\left(t\right)\right)\right\} \\ \sqrt{g\left(F_{Tz,Bz}\left(t\right)\right).g\left(F_{Tz,Av}\left(t\right)\right)}, \\ \frac{g\left(F_{Sv,Av}(t)\right).g\left(F_{Sv,Bz}(t)\right)}{g\left(F_{Av,Bz}(t)\right)} \end{array}\right\}\right),$$

that is

$$g\left(F_{z,Bz}\left(t\right)\right) \leq \phi\left(\max\left\{\begin{array}{c} g\left(F_{z,Bz}\left(t\right)\right), g\left(F_{z,z}\left(t\right)\right), \\ g\left(F_{Bz,Bz}\left(t\right)\right) \\ \frac{1}{2}\left[g\left(F_{z,Bz}\left(t\right)\right)+g\left(F_{Bz,z}\left(t\right)\right)\right], \\ \min\left\{g\left(F_{z,Bz}\left(t\right)\right), g\left(F_{Bz,z}\left(t\right)\right)\right\} \\ \sqrt{g\left(F_{Bz,Bz}\left(t\right)\right).g\left(F_{Bz,z}\left(t\right)\right)}, \\ \frac{g\left(F_{z,z}(t)\right).g\left(F_{z,Bz}(t)\right)}{g\left(F_{z,Bz}(t)\right)} \end{array}\right\}\right)$$

$$= \phi\big(\max\left\{g\left(F_{z,Bz}\left(t\right)\right), g\left(1\right), g\left(1\right), g\left(F_{z,Bz}\left(t\right)\right), g\left(1\right), 0, g\left(1\right)\right\}\big) = \phi\left(g\left(F_{z,Bz}\left(t\right)\right)\right).$$

By Lemma 1, we have $Bz = z = Tz$ which shows that z is a common fixed point of the pair (B, T) and z is a common fixed point of the pairs (A, S) and (B, T). The uniqueness of the common fixed point is an easy consequence of the inequality (1) by Lemma 1. This concludes the proof.

The following proposition will help us to get further results.

Proposition 1: Let A, B, S and T be four self mappings of an N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm. Suppose that

1) the pair (A, S) satisfies the (CLR_S) property (or the pair (B, T) satisfies the (CLR_T) property),
2) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$),
3) $T(X)$ (or $S(X)$) is a closed subset of X,
4) $\{By_n\}$ converges for every sequence $\{y_n\}$ in X whenever $\{Ty_n\}$ converges (or $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in X whenever $\{Sx_n\}$ converges),
5) the mappings A, B, S and T satisfy inequality (1) of Theorem 10.

Then the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property.

Proof: If the pair (A, S) satisfies the (CLR_S) property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X)$. Since $A(X) \subset T(X)$, hence for each $\{x_n\} \subset X$ there corresponds a sequence $\{y_n\} \subset X$ such that $Ax_n = Ty_n$. Therefore, by the closedness of $T(X)$,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z,$$

where $z \in S(X) \cap T(X)$. Thus in all, we have $Ax_n \to z, Sx_n \to z$ and $Ty_n \to z$ as $n \to \infty$. By part (4), the sequence $\{By_n\}$ converges and we just need to show that $By_n \to z$ as $n \to \infty$. By putting $x = x_n$ and $y = y_n$ in the inequality (1), we get:

$$g\left(F_{Ax_n,By_n}\left(t\right)\right) \leq \phi\left(\max\left\{\begin{array}{c} g\left(F_{Sx_n,Ty_n}\left(t\right)\right), g\left(F_{Sx_n,Ax_n}\left(t\right)\right), \\ g\left(F_{Ty_n,By_n}\left(t\right)\right) \\ \frac{1}{2}\left[g\left(F_{Sx_n,By_n}\left(t\right)\right)+g\left(F_{Ty_n,Ax_n}\left(t\right)\right)\right], \\ \min\left\{g\left(F_{Sx_n,By_n}\left(t\right)\right), g\left(F_{Ty_n,Ax_n}\left(t\right)\right)\right\} \\ \sqrt{g\left(F_{Ty_n,By_n}\left(t\right)\right).g\left(F_{Ty_n,Ax_n}\left(t\right)\right)}, \\ \frac{g\left(F_{Sx_n,Ax_n}(t)\right).g\left(F_{Sx_n,By_n}(t)\right)}{g\left(F_{Ax_n,By_n}(t)\right)} \end{array}\right\}\right)$$

Let $By_n \to l(\neq z)$ as $n \to \infty$. Then, passing to the limit as $n \to \infty$, we get

$$g\left(F_{z,l}\left(t\right)\right) \leq \phi\left(\max\left\{\begin{array}{c} g\left(F_{z,z}\left(t\right)\right), g\left(F_{z,z}\left(t\right)\right), \\ g\left(F_{z,l}\left(t\right)\right) \\ \frac{1}{2}\left[g\left(F_{z,l}\left(t\right)\right)+g\left(F_{z,l}\left(t\right)\right)\right], \\ \min\left\{g\left(F_{z,l}\left(t\right)\right), g\left(F_{z,z}\left(t\right)\right)\right\} \\ \sqrt{g\left(F_{z,l}\left(t\right)\right).g\left(F_{z,z}\left(t\right)\right)}, \\ \frac{g\left(F_{z,z}(t)\right).g\left(F_{z,l}(t)\right)}{g\left(F_{z,l}(t)\right)} \end{array}\right\}\right)$$

$$= \phi\big(\max\left\{g\left(1\right), g\left(1\right), g\left(F_{z,l}\left(t\right)\right), g\left(F_{z,l}\left(t\right)\right), g\left(1\right), 0, g\left(1\right)\right\}\big) = \phi\left(g\left(F_{z,l}\left(t\right)\right)\right).$$

So, by Lemma 1, we have $z = l$. Hence the pairs (A, S) and (B, T) share the (CLR_{ST}) property. The converse of proposition 1 is not true. For a counterexample, see Example 3.5 in [14].

Theorem 2: Let A, B, S and T be four self-mappings of an N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm, satisfying all the hypotheses of Proposition 1. Then A, B, S and T have a unique common fixed point, Where the both pairs (A, S) and (B, T) are weakly compatible.

Proof This follows by combining Theorem 1 with proposition 1.

It is clear that, if the pairs (A, S) and (B, T) satisfy the common property (E.A), and, at the same time, $S(X)$ and $T(X)$ are closed subsets of X, then the pairs (A, S) and (B, T) share the (CLR_{ST}) property. Hence, we have the following variant of Theorem 1.

Theorem 3: Let A, B, S and T be four self-mappings of an N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm, satisfying inequality (1) and the following hypotheses holds:

1) the pairs (A, S) and (B, T) satisfy the common property (E.A);
2) $S(X)$ and $T(X)$ are closed subsets of X.

Then (A, S) and (B, T) have a coincidence point each. Moreover A, B, S and T have a unique common fixed point provided both pairs (A, S) and (B, T) are weakly compatible.

Next, we state two more variants of our results, which can be proved by Theorems (2 and 3).

Corollary 1: The conclusions of Theorem 3 remain true if condition (2) is replaced by the following condition:

$$\overline{A(X)} \subset T(X) \text{ and } \overline{B(X)} \subset S(X),$$

where $\overline{A(X)}$ and $\overline{B(X)}$ denote the closure of ranges of the mappings A and B.

Corollary 2: The conclusions of Theorem 3 remain true if condition (2) is replaced by the following condition:

$$A(X) \text{ and } B(X) \text{ are closed subsets of } X,$$

and

$$A(X) \subset T(X), B(X) \subset S(X).$$

By choosing A, B, S and T suitably in Theorem 1, we can deduce some corollaries for a pair as well as for a triple of self mappings. Since the formulations of these results are similar to those in [14], [15], we omit the details here. Now we utilize this notion for six self-mappings in an N.A. Menger PM-space.

Theorem 4: Let A, B, R, S, H and T be six self mappings of an N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm. Suppose that

1) the pairs (A, SR) and (B, TH) satisfy the $(CLR_{(SR)(TH)})$ property,
2) the following inequality B hold:

$$g(F_{Ax,By}(t)) \leq \phi\left(\max\left\{\begin{array}{c} g(F_{SRx,THy}(t)), g(F_{SRx,Ax}(t)), \\ g(F_{THy,By}(t)) \\ \frac{1}{2}[g(F_{SRx,By}(t)) + g(F_{THy,Ax}(t))], \\ \min\{g(F_{SRx,By}(t)), g(F_{THy,Ax}(t))\} \\ \sqrt{g(F_{SRx,THy}(t)).g(F_{THy,Ax}(t))}, \\ \frac{g(F_{SRx,Ax}(t)).g(F_{SRx,By}(t))}{g(F_{Ax,By}(t))} \end{array}\right\}\right)$$

for all $x, y \in X, t > 0$, where $g \in \Omega$ and $\phi \in \Phi$. Then (A, SR) and (B, TH) have a coincidence point each. Moreover A, B, H, R, S and T have a unique common fixed point provided $AS = SA, AR = RA, SR = RS, BT = TB, BH = HB$ and $TH = HT$.

Proof: By Theorem 1, A, B, SR and TH have a unique common fixed point z in X. We show that z is a unique common fixed point of the self mappings A, B, R, S, H and T. By putting $x = Rz$ and $y = z$ in the inequality (2), we get:

$$g(F_{A(Rz),Bz}(t)) \leq \phi \max\left\{\begin{array}{c} g(F_{SR(Rz),THz}(t)), g(F_{SR(Rz),A(Rz)}(t)), \\ g(F_{THz,Bz}(t)) \\ \frac{1}{2}[g(F_{SR(Rz),Bz}(t)) + g(F_{THz,A(Rz)}(t))], \\ \min\{g(F_{SR(Rz),Bz}(t)), g(F_{THz,A(Rz)}(t))\} \\ \sqrt{g(F_{SR(Rz),THz}(t)).g(F_{THz,A(Rz)}(t))}, \\ \frac{g(F_{SR(Rz),A(Rz)}(t)).g(F_{SR(Rz),Bz}(t))}{g(F_{A(Rz),Bz}(t))} \end{array}\right\},$$

and

$$\begin{aligned} &g(F_{Rz,z}(t)) \\ &\leq \phi\left(\max\left\{\begin{array}{c} g(F_{Rz,z}(t)), g(F_{Rz,Rz}(t)), \\ g(F_{z,z}(t)) \\ \frac{1}{2}[g(F_{Rz,z}(t)) + g(F_{z,Rz}(t))], \\ \min\{g(F_{Rz,z}(t)), g(F_{z,Rz}(t))\} \\ \sqrt{g(F_{Rz,z}(t)).g(F_{z,Rz}(t))}, \\ \frac{g(F_{Rz,Rz}(t)).g(F_{Rz,z}(t))}{g(F_{Rz,z}(t))} \end{array}\right\}\right) \\ &= \phi\big(\max\{g(F_{Rz,z}(t)), g(1), g(1), g(F_{Rz,z}(t)), \\ &g(F_{Rz,z}(t)), g(F_{Rz,z}(t)), g(1)\}\big) \\ &= \phi\big(\max\{g(F_{Rz,z}(t)), 0, 0, g(F_{Rz,z}(t)), \\ &g(F_{Rz,z}(t)), g(F_{Rz,z}(t)), 0\}\big) \\ &= \phi(g(F_{Rz,z}(t))). \end{aligned}$$

Using Lemma 1, we have $z = Rz$. Hence $Sz = S(Rz) = z$. Therefore, we have $z = Az = Sz = Rz$. Now we assert that z is a common fixed point of B, T and H. To accomplish this, we use the inequality (2) with $x = z and y = Hz$ we obtain:

$$g(F_{Az,B(Hz)}(t)) \leq \phi \max\left\{\begin{array}{c} g(F_{SRz,TH(Hz)}(t)), g(F_{SRz,Az}(t)), \\ g(F_{TH(Hz),B(Hz)}(t)) \\ \frac{1}{2}[g(F_{SRz,B(Hz)}(t)) + g(F_{TH(Hz),Az}(t))], \\ \min\{g(F_{SRz,B(Hz)}(t)), g(F_{TH(Hz),Az}(t))\} \\ \sqrt{g(F_{SRz,TH(Hz)}(t)).g(F_{TH(Hz),Az}(t))}, \\ \frac{g(F_{SRz,Az}(t)).g(F_{SRz,B(Hz)}(t))}{g(F_{Az,B(Hz)}(t))} \end{array}\right\},$$

that is

$$\begin{aligned} &g(F_{z,Hz}(t)) \\ &\leq \phi\left(\max\left\{\begin{array}{c} g(F_{z,Hz}(t)), g(F_{z,z}(t)), \\ g(F_{Hz,Hz}(t)) \\ \frac{1}{2}[g(F_{z,Hz}(t)) + g(F_{Hz,z}(t))], \\ \min\{g(F_{z,Hz}(t)), g(F_{Hz,z}(t))\} \\ \sqrt{g(F_{z,Hz}(t)).g(F_{Hz,z}(t))}, \\ \frac{g(F_{z,z}(t)).g(F_{z,Hz}(t))}{g(F_{z,Hz}(t))} \end{array}\right\}\right) \\ &= \phi\big(\max\{g(F_{z,Hz}(t)), g(1), g(1), g(F_{z,Hz}(t)), \\ &g(F_{z,Hz}(t)), g(F_{z,Hz}(t)), g(1)\}\big) \\ &= \phi\big(\max\{g(F_{z,Hz}(t)), 0, 0, g(F_{z,Hz}(t)), \\ &g(F_{z,Hz}(t)), g(F_{z,Hz}(t)), 0\}\big) \\ &= \phi(g(F_{z,Hz}(t))). \end{aligned}$$

Thus, by Lemma 1, we have $z = Hz$. Hence $Tz = T(Hz) = z$. Therefore z is a common fixed point of self mappings A, B, R, S, H and T. On the other hand, the uniqueness of the common fixed point is an easy consequence of inequality (2).

In view of Theorem 2, we can derive the fixed point theorem for four finite families of self mappings.

Corollary 3: Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self-mappings of an N.A. Menger space $(X, \mathcal{F}, \Delta)$, where Δ is a continuous t-norm, with $A = A_1A_2....A_m$, $B = B_1B_2....B_n$, $S = S_1S_2....S_p$ and $T = T_1T_2....T_q$ satisfying the inequality (1) of Theorem 1 such that the pairs (A, S) and (B, T) have the (CLR$_{ST}$) property. Then $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$

have a unique common fixed point provided that the pairs of families $(\{A_i\}_{i=1}^m, \{S_k\}_{k=1}^p)$ and $(\{B_r\}_{r=1}^n, \{T_h\}_{h=1}^q)$ commute pairwise.

By setting $A_1 = A_2 = ... = A_m = A$, $B_1 = B_2 = ... = B_n = B$, $S_1 = S_2 = ... = S_p = S$ and $T_1 = T_2 = ... = T_q = T$ in Corollary 1, we get that A, B, S and T have a unique common fixed point provided that the pairs (A^m, S^p) and (B^n, T^q) commute pairwise.

Corollary 4: Let A, B, S and T be self mappings of an N.A. Menger PM-space $(X, \mathcal{F}, \Delta)$ where Δ is a continuous t-norm. Suppose that:

1) The pairs (A^m, S^p) and (B^n, T^q) satisfying the (CRL_{S^p,T^q}) property, where m, n, p, q are fixed positive integers.
2) The following inequality is held:

$$g(F_{A^m x, B^n y}(t)) \le \phi\left(\max\left\{\begin{array}{c} g(F_{S^p x, T^q y}(t)), g(F_{S^p x, A^m x}(t)), \\ g(F_{T^q y, B^n y}(t)) \\ \frac{1}{2}[g(F_{S^p x, B^n y}(t)) + g(F_{T^q y, A^m x}(t))], \\ \min\{g(F_{S^p x, B^n y}(t)), g(F_{T^q y, A^m x}(t))\} \\ \sqrt{g(F_{S^p x, T^q y}(t)).g(F_{T^q y, A^m x}(t))}, \\ \frac{g(F_{S^p x, A^m x}(t)).g(F_{S^p x, B^n y}(t))}{g(F_{A^m x, B^n y}(t))} \end{array}\right\}\right),$$

for all $x, y \in X, t > 0, g \in \Omega$ where $\phi \in \Phi$. Then A, B, S and T have a unique common fixed point provided $AS = SA$ and $BT = TB$.

Remark 2: The conclusions of Theorem 1 remain true if we replace the inequality (1) with the following inequality:

$$g(F_{Ax,By}(t)) \le \phi\left(\max\left\{\begin{array}{c} g(F_{Sx,Ty}(t)), g(F_{Sx,Ax}(t)), \\ g(F_{Ty,By}(t)) \\ \frac{1}{2}[g(F_{Sx,By}(t)) + g(F_{Ty,Ax}(t))], \\ \sqrt{g(F_{Sx,Ty}(t)).g(F_{Ty,Ax}(t))}, \\ \frac{g(F_{Ax,Ty}(t)).g(F_{Sx,Ty}(t))}{g(F_{Sx,Ty}(t))} \end{array}\right\}\right),$$

for all $x, y \in X$ and $t > 0$, where $g \in \omega$ and ϕ belongs to the class Φ.

IV. Illustrative examples

In this section, we give some examples demonstrating the validity of the hypotheses and the degree of generality of our results over some recently established results.

Example 3: Let (X, d) be a metric space with the usual metric d where $X = [3, 12)$ and let $(X, \mathcal{F}, \Delta)$ be the induced N.A. Menger PM-space with $g(t) = 1 - t$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Let A, B, S and T be four mappings from X to itself defined as: $Ax = \begin{cases} 3, & \text{if } x \in \{3\} \cup (6, 12), \\ 9, & \text{if } x \in (3, 6]; \end{cases}$

$$Bx = \begin{cases} 3, & \text{if } x \in \{3\} \cup (6, 12), \\ 5, & \text{if } x \in (3, 6]; \end{cases}$$

$$Sx = \begin{cases} 3, & \text{if } x = 3, \\ 10, & \text{if } x \in (3, 6], \\ \frac{x+3}{3}, & \text{if } x \in (6, 12); \end{cases}$$

$$Tx = \begin{cases} 3, & \text{if } x = 3, \\ 7, & \text{if } x \in (3, 6], \\ x - 3, & \text{if } x \in (6, 12). \end{cases}$$

Then we have $A(X) = \{3, 9\} \nsubseteq [3, 9) = T(X)$ and $B(X) = \{3, 5\} \nsubseteq S(X) = [3, 5) \cup \{10\}$. Taking the sequences $\{x_n\} = \{6 + \frac{1}{n}\}, \{y_n\} = \{3\}$ (or $\{x_n\} = \{3\}, \{y_n\} = \{6 + \frac{1}{n}\}$), the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property, that is,

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 3 \in S(X) \cap T(X).$$

Now, define a function $\phi : [0, +\infty) \to [0, +\infty)$ by:

$$\phi(t) = kt, \text{ with } \frac{6}{7} < k < 1, \text{ for all } t \ge 0.$$

Clearly, $\phi \in \Phi$. By a routine calculation, the first one can easily verify the inequality (1). Thus, all the conditions of Theorem 1 are satisfied, and 3 is a unique common fixed point of the pairs (A, S) and (B, T). It is noted in this example that $S(X)$ and $T(X)$ are not closed subsets of X. Also, all the involved mappings are even discontinuous at their unique common fixed point 3.

In the following illustration, the importance of weakly compatible assumptions for the validity of the result is shown.

Example 4: Let (X, d) be a metric space with the usual metric d where $X = [0, +\infty)$ and let $(X, \mathcal{F}, \Delta)$ be the induced N.A. Menger PM-space with $g(t) = 1 - t$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Consider the mappings $A, B, S, T : X \to X$ given by:

$$Ax = Bx = x + 1 \text{ and } Sx = Tx = 2x.$$

Then the pairs $(A, S)\, and\, (B, T)$ satisfy the (CLR_{ST}) property. Indeed, consider two sequences, $\{x_n\} = \{1 + \frac{1}{n}\}_{n\in\mathbb{N}}\, and\, \{y_n\} = \{1 - \frac{1}{n}\}_{n\in\mathbb{N}}$, then we have:

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 2,$$

where $2 \in S(X) \cap T(X)$. By a simple calculation, taking $\phi(t) = kt$ with a suitable value of k, then the inequality (1) is hold. Thus, all the conditions of the first part of Theorem 1 are satisfied. It can be noted, indeed, that 1 is a coincidence point of (A, S), as well as of (B, T). However, these pairs are not weakly compatible and there is no common fixed point of the pairs (A, S) and (B, T).

Note that Theorem 2 cannot be applied in the case of mappings provided in Example 4, since conditions (2) and (3) of proposition 1 do not hold. Such an example can show when Theorem 2 can be used.

Example 5: In Example 3, replace the mappings S and T with the following:

$$Tx = \begin{cases} 3, & \text{if } x = 3, \\ 9, & \text{if } x \in (3, 6], \\ x - 3, & \text{if } x \in (6, 12); \end{cases}$$

$$Sx = \begin{cases} 3, & \text{if } x = 3, \\ 5, & \text{if } x \in (3, 6], \\ \frac{x-2}{2}, & \text{if } x \in (6, 12). \end{cases}$$

Besides retaining the rest: $\{3, 9\} \nsubseteq [3, 9) = T(X)$ and $B(X) = \{3, 5\} \nsubseteq S(X) = [3, 5) \cup \{10\}$. Then $A(X) = \{3, 9\} \subset [3, 9] = T(X)$ and $B(X) = \{3, 5\} \subset [3, 5] = S(X)$ hold; In fact $S(X)$ and $T(X)$ are closed subsets of X. Thus, all the conditions of Theorem 1 are satisfied, and so 2 is a unique common fixed point of the pairs (A, S) and (B, T).

The next example satisfies condition (1) of theorem 1 such that it is only sufficient and not necessary.

Example 6: Let (X, d) be a metric space with the usual metric d where $X = [1, 10]$ and let (X, F, Δ) be the induced N.A. Menger PM-space with $g(t) = 1 - t$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Consider the mappings $A, B, S, T : X \to X$ given by

$$Ax = Bx = \begin{cases} 1, & \text{if} \quad x = 1, \\ 6, & \text{if} \quad 1 < x \leq 4, \\ 1, & \text{if} \quad 4 < x \leq 10; \end{cases}$$

$$Sx = Tx = \begin{cases} 1, & \text{if} \quad x = 1, \\ 6, & \text{if} \quad 1 < x \leq 4, \\ \frac{x+2}{2}, & \text{if} \quad 4 < x \leq 10. \end{cases}$$

Then the pairs (A, S) and (B, T) satisfy all the conditions of Theorem 1, except the inequality (1) (take, e.g., $x \in (1, 4]$ and $y = 1$). However, these four mappings have a coincidence at $x = 1$ which also remains their common fixed point. This confirms that condition (1) of Theorem 1 is sufficient and not necessary.

Our last example highlights the non-closedness of ranges of S and T in X in corollaries 1 and 2.

Example 7: In Example 3, replace the mappings S and T by the following:

$$Sx = Tx = \begin{cases} 3 & \text{if} \quad x = 3, \\ 13 & \text{if} \quad x \in (3, 6], \\ \frac{3x-12}{2} & \text{if} \quad x \in (6, 12). \end{cases}$$

Then $A(X) = \{3, 9\} \subset [3, 12) \cup \{13\} = T(X)$ and $B(X) = \{3, 5\} \subset [3, 12) \cup \{13\} = S(X)$. Actually, $S(X)$ and $T(X)$ are not closed subspaces of X, but condition (2'), resp, (2") of Corollary 1, resp 2 is satisfied. Again, 2 is a unique common fixed point of A, B, S and T.

V. An application to functional equations

In this section, by using the fixed point results obtained in the previous Section, we study the solvability of the following system of functional equations arising in dynamic programming:

$$q(x) = opt_{y \in D} \{G_i(x, y, q(\tau(x, y)))\}, \tag{3}$$

for $x \in W$ and $i \in \{1, 2, 3, 4\}$, where U and V are Banach spaces, $W \subseteq U$ is a state space, $D \subset V$ is a decision space, while $\tau : W \times D \to W, G_i : W \times D \times \mathbb{R} \to \mathbb{R}$are mappings, $i \in \{1, 2, 3, 4\}$. Denote by X the set of all bounded real-valued functions on W and, for $h \in X$, define $\|h\| = \sup_{x \in W} |h(x)|$. Clearly, $(X, \|.\|)$ is a Banach space, and the convergence in this space is uniform. Therefore, if $\{h_n\}$ is a Cauchy sequence in X, then it converges uniformly to a function $h^* \in X$. The respective metric will be denoted by d. Further, consider operators $A, B, S, T : X \to X$ given by:

$$\begin{cases} Ah(x) & = \sup_{y \in D} \{G_1(x, y, h(\tau(x, y)))\}, \\ Bh(x) & = \sup_{y \in D} \{G_2(x, y, h(\tau(x, y)))\}, \\ Sh(x) & = \sup_{y \in D} \{G_3(x, y, h(\tau(x, y)))\}, \\ Th(x) & = \sup_{y \in D} \{G_4(x, y, h(\tau(x, y)))\}, \end{cases} \tag{4}$$

for $h \in X and x \in W$. These mappings are well-defined if the functions G_i are bounded. From the above discussion, we can provide another theoretical result, reported below for completeness.

Theorem 5: Let $A, B, S, T : X \to X$ given by (4) and suppose that the following hypotheses hold:

(I) The functions $G_i : W \times D \times \mathbb{R} \to \mathbb{R}, i \in \{1, 2, 3, 4\}$, satisfy:

$$e^{\left(-\frac{t}{\sup_{x \in W} \sup_{y \in D} |G_1(x,y,h(x)) - G_2(x,y,k(x))|}\right)}$$
$$\leq \phi\left(\max\left\{\begin{array}{c} g(F_{Sh,Tk}(t)), g(F_{Sh,Ah}(t)), \\ g(F_{Tk,Bk}(t)) \\ \frac{1}{2}[g(F_{Sh,Bk}(t)) + g(F_{Tk,Ah}(t))], \\ \min\{g(F_{Sh,Bk}(t)), g(F_{Tk,Ah}(t))\} \\ \sqrt{g(F_{Sh,Tk}(t)) . g(F_{Tk,Ah}(t))}, \\ \frac{g(F_{Sh,Ah}(t)) . g(F_{Sh,Bk}(t))}{g(F_{Ah,Bk}(t))} \end{array}\right\}\right),$$

for all $h, k \in X$ and $t \in [0, 1]$, where g is given by $g(t) = 1 - t$ for $t \in [0, 1]$;

(II) $G_i : W \times D \times \mathbb{R} \to \mathbb{R}$ are bounded functions, for $i \in \{1, 2, 3, 4\}$;

(III) There exist two sequences $\{h_n\}$ and $\{k_n\}$ in X and $h^* \in X$ such that

$$\lim_{n\to\infty} Ah_n = \lim_{n\to\infty} Bk_n = \lim_{n\to\infty} Sh_n = \lim_{n\to\infty} Tk_n = h^*;$$

(IV) $ASh = SAh$, whenever $Ah = Sh$ for some $h \in X$;

(V) $BTk = TBk$, whenever $Bk = Tk$ for some $k \in X$.

Then the system of functional equations (3) has a unique bounded solution.

Proof: Define

$$F_{h,k}(t) = \begin{cases} 1 - \exp\left(-\frac{t}{d(h,k)}\right) & \text{if } 0 < t < d(h, k), \\ 1 & \text{otherwise,} \end{cases}$$

where $h, k \in X$ such that $h \neq k$. Then $(X, , \Delta)$ is a complete N.A. Menger PM-space (induced by the metric d) with $\Delta(a, b) = \min\{a, b\}$ for $a, b \in [0, 1]$. By hypothesis (III) the pairs (A, S) and (B, T) share the common limit range property with respect to (S, T). Now, let ε be an arbitrary positive number, $x \in W$, and $h, k \in X$. Then there exist $y_1, y_2 \in D$ such that

$$Ah(x) < G_1(x, y_1, h(\tau(x, y_1))) + \varepsilon, \tag{5}$$

$$Ah(x) \geq G_1(x, y_2, h(\tau(x, y_2))), \tag{6}$$

$$Bk(x) < G_2(x, y_2, k(\tau(x, y_2))) + \varepsilon, \tag{7}$$

$$Bk(x) \geq G_2(x, y_1, k(\tau(x, y_1))). \tag{8}$$

Using (5) and (8), we obtain

$$\begin{aligned} &Ah(x) - Bk(x) \\ &< G_1(x, y_1, h(\tau(x, y_1))) - G_2(x, y_1, k(\tau(x, y_1))) + \varepsilon \\ &\leq \left|G_1(x, y_1, h(\tau(x, y_1))) - G_2(x, y_1, k(\tau(x, y_1)))\right| + \varepsilon \\ &\leq \sup_{y \in D} \left|G_1(x, y, h(\tau(x, y))) - G_2(x, y, k(\tau(x, y)))\right| + \varepsilon. \end{aligned}$$

Analogously, by using (6) and (7), we get:

$$\begin{aligned} &Bk(x) - Ah(x) \\ &< \sup_{y \in D} |G_1(x, y, k(\tau(x, y))) - G_2(x, y, h(\tau(x, y)))| + \varepsilon. \end{aligned} \tag{9}$$

From (9), we deduce that

$$\begin{aligned}&|Ah(x)-Bk(x)|\\&<\sup_{y\in D}|G_1(x,y,h(\tau(x,y)))-G_2(x,y,k(\tau(x,y)))|+\varepsilon.\end{aligned}$$

It follows directly that

$$\begin{aligned}&d(Ah,Bk)\\&\leq\sup_{x\in W}\sup_{y\in D}|G_1(x,y,h(\tau(x,y)))-G_2(x,y,k(\tau(x,y)))|\\&\quad+\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was taken arbitrary, we obtain that

$$\begin{aligned}&d(Ah,Bk)\\&\leq\sup_{x\in W}\sup_{y\in D}|G_1(x,y,h(\tau(x,y)))-G_2(x,y,k(\tau(x,y)))|\end{aligned}\tag{10}$$

In view of hypotheses (I) and (10), it follows easily that

$$\begin{aligned}&g(F_{Ah,Bk}(t))\\&\leq\phi\left(\max\left\{\begin{array}{c}g(F_{Sh,Tk}(t)),g(F_{Sh,Ah}(t)),\\g(F_{Tk,Bk}(t))\\\frac{1}{2}[g(F_{Sh,Bk}(t))+g(F_{Tk,Ah}(t))],\\\min\{g(F_{Sh,Bk}(t)),g(F_{Tk,Ah}(t))\}\\\sqrt{g(F_{Sh,Tk}(t)).g(F_{Tk,Ah}(t))},\\\frac{g(F_{Sh,Ah}(t)).g(F_{Sh,Bk}(t))}{g(F_{Ah,Bk}(t))}\end{array}\right\}\right).\end{aligned}$$

Moreover, in view of hypotheses (IV) and (V), the pairs (A,S) and (B,T) are weakly compatible. Hence, Theorem 1 is applicable, and so A,B,S, and T have a unique common fixed point, that is, the system of functional equations (3) has a unique bounded solution.

VI. Conclusion

We prove the common fixed point theorems for weakly compatible mappings in non-Archimedean Menger PM-spaces, and we give some examples. In addition, we extend our main result to four finite families of self-mappings by the notion of pairwise commuting as we show. Also, we introduce some applications for our main theorem.

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