Diagonal Block Method for Stiff Van der Pol Equation

Nooraini Zainuddin, Zarina Bibi Ibrahim, and Iskandar Shah Mohd Zawawi

Abstract—**Stiff equation is known for its rapid and slow varying time component, for which the method dedicated for this system must be capable on changing the step size depending on the varying component of the interval. This is to make sure that the computational cost can be reduced while the accuracy is preserved. In this paper, the diagonal block method derived from the family of backward differentiation formula is proposed for the direct solution of stiff Van der Pol equation. The method is implemented by varying the step size in the fixed ratios of 1, 2 and 10/19 which corresponds to constant, by halving and increasing the step respectively. The method is derived in block forms to compute the approximate solutions at two points simultaneously. By controlling the constants in its linear difference operator, the consistency of the derived method is verified. The Newton iteration technique which is derived in the block matrix form is also presented in this paper. The robustness of the proposed method is validated by solving the stiff Van der Pol equation directly and compared with the ode15s from MATLAB. Numerical results demonstrate the capability of the proposed method in solving the stiff ODEs directly.**

Index Terms—**BDF, block method, stiff, Van der Pol equation**

I. Introduction

Van der Pol (VDP) equation is a second order differential equation in the form of:

$$y''(x) = \mu(1 - y^2(x))y'(x)) - y(x),\ \mu \geq 0, \qquad (1)$$

where parameter μ indicates the degree of stiffness for (1).

Equation (1) was first introduced by Van der Pol in 1926 during his investigation on the triode circuit. He found that for a large value of μ, such equation exhibited a relaxation oscillation and these oscillations were of a limit cycle. Since then, equation (1) has been used as a basic model for oscillatory systems in the fields of physic, electronics, and biology, to mention a few. It is also used in modelling dynamics of elastic excitable media [1] and in macroeconomisc [2].

As closed-form solutions cannot be found analytically, numerical methods of approximating solutions are possible and useful. Numerous studies on various forms of equation (1) that had been conducted, [3] proposed the modified version of Adomian Decomposition Method to solve the forced and unforced VDP equation with $\mu = 1$. In the study done by [4], the variable order fractional VDP was treated by the method of Adams Bashforth Moulton with $\mu = 2.5$. [5] successfully used the Homotopy analysis method to deal with the fractional order VDP equation.

In the paper of [6], the author noted that for VDP equation, at large μ, the equation was very stiff and exhibited a relaxation oscillator where it produced fast and slow states in a limit cycle. The concept of stiff ODEs was first introduced by [7]. [8] stated that the stiff problems had some steady and transient solutions where all solutions became steady after a short time (after the transient phase had finished) while [9] expressed stiffness as the solution to be computed was slowly varying, perturbations that existed were rapidly damped. These properties of stiff problems indicated that the method dedicated for solving the stiff problems should be able to solve the fast and slow states effectively. The transient reactions have the rapid change in solution and therefore, the method used for solving the stiff problem is expected to provide good solution for this transient phase.

The widely used codes when dealing with stiff differential equations are based on backward differentiation formulas (BDFs) [10]. [11] was the first to design the codes based on BDFs, known as DIFSUB. Later on, [12] and [13] had made improvements for this code which are known as GEAR and EPISODE respectively. Several attempts to increase the accuracy and computational time of BDFs were made, including the implementation of BDFs in block scheme [14], [15], [16], [17], [18]. The r-point block BDF, r=2, 3 methods introduced by [19] gave two and three solutions simultaneously. [16] derived the hybrid 3 point block BDF for solving the stiff chemical kinetics problems. Other solvers based on the block BDF for solving stiff ODEs can be found from these literatures [20], [21], [22].

A popular technique for solving (1) is by reducing it to a system of first order ODEs and then solving it with methods that suit such systems. However, solving (1) directly is favorable [23], [24], [25], [26] since the advantages of this approach are clear in saving the storage space [27], and thus reducing the computational work [23], [24], [28]. In contrast, reducing (1) into the first order ODEs double the number of equations which therefore leads to higher computational work. This drawback has attracted researchers to propose methods for solving general form of (1) directly [29], [30], [31], [32], [33], [34].

This paper aims to solve the stiff VDP equation directly by using the diagonal block backward differentiation formula with a variable step size approach. It provides two approximation solutions for each successful step. The derivation of the method by controlling its constants is given in Section 2. In Section 3, the consistency, zero-stability, convergence and linear stability properties of the method are analyzed. Section 4 further discusses the algorithm on the implementation of the proposed diagonal block method. Numerical performance

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of the method on dealing with stiff VDP problems is demonstrated in Section 5 and finally the conclusion is given in Section 6.

II. Formulation of the Method

In this section, the 2-point diagonal block backward differentiation formula (2DBBBDF) is derived. As the proposed method is in a block form, the change in the step size from the current block to the previous block is differentiated with the introduction of r, where r is the step size ratio. This ratio represents the distance of the preceding $2rh$ step size and the current $2h$ step size block as shown in Figure 1.

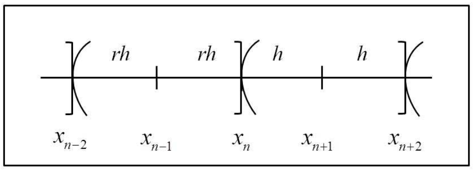


Fig. 1. 2-point diagonal block backward differentiation formula (2DBBBDF).

The 2DBBBDF interpolates three previous values, (x_{n-2}, y_{n-2}), $(x_{n-1},\ y_{n-1})$ and (x_n, y_n) for each successful integration step to produce solution at the points (x_{n+1}, y_{n+1}) and (x_{n+2}, y_{n+2}) simultaneously. For each point, two corrector formulas are derived which are y and y'. Therefore, the 2DBBBDF has four corrector formulas, $y_{n+1}, y'_{n+1}, y_{n+2}$ and y'_{n+2} which are implemented together in a matrix form to produce four solutions simultaneously.

The derivation of the corrector formulas at x_{n+1} starts by giving the general form of the formulas as:

$$hy'_{n+1} = \alpha_{-2,1}y_{n-2} + \alpha_{-1,1}y_{n-1} + \alpha_{0,1}y_n + \alpha_{1,1}y_{n+1}, \quad (2)$$

$$y_{n+1} = \theta_{-2,1}y_{n-2} + \theta_{-1,1}y_{n-1} + \theta_{0,1}y_n + h^2\beta_{1,1}f_{n+1}. \quad (3)$$

The associate linear difference operators for (2) and (3) are given as the following, respectively.

$$\begin{aligned} L_{1,1}[y(x_n);h] =& hy'_{n+1} - (\alpha_{-2,1}y_{n-2} + \alpha_{-1,1}y_{n-1} + \\ & \alpha_{0,1}y_n + \alpha_{1,1}y_{n+1}), \end{aligned} \quad (4)$$

$$\begin{aligned} L_{2,1}[y(x_n);h] =& y_{n+1} - (\theta_{-2,1}y_{n-2} + \theta_{-1,1}y_{n-1} + \\ & \theta_{0,1}y_n + h^2\beta_{1,1}f_{n+1}). \end{aligned} \quad (5)$$

Referring to Figure 1 and by defining $f_{n+1} = y''_{n+1}$, equations (4) and (5) can be written respectively as:

$$\begin{aligned} L_{1,1}[y(x_n);h] =& hy'(x_n+h) - (\alpha_{-2,1}y(x_n-2rh) + \\ & \alpha_{-1,1}y(x_n-rh) + \alpha_{0,1}y(x_n) + \\ & \alpha_{1,1}y(x_n+h)), \end{aligned} \quad (6)$$

$$\begin{aligned} L_{2,1}[y(x_n);h] =& y(x_n+h) - (\theta_{-2,1}y(x_n-2rh) + \\ & \theta_{-1,1}y(x_n-rh) + \theta_{0,1}y(x_n) + \\ & h^2\beta_{1,1}y''(x_n+h)). \end{aligned} \quad (7)$$

The test functions $y(x_n-2rh), y(x_n-rh), y(x_n), y(x_n+h), y'(x_n+h)$, and $y''(x_n+h)$ are expanded as Taylor series about x_n. By collecting the terms of derivative y as in (6) and (7) gives:

$$L_{1,1}[y(x_n);h] = C_0y(x_n) + C_1y'(x_n) + C_2y''(x_n) + \cdots, \quad (8)$$

$$L_{2,1}[y(x_n);h] = D_0y(x_n) + D_1y'(x_n) + D_2y''(x_n) + \cdots. \quad (9)$$

The constants for C_q equal to:

$$\begin{aligned} C_0 =& 1 - (\alpha_{-2,1} + \alpha_{-1,1} + \alpha_{0,1} + \alpha_{1,1}), \\ C_1 =& 1 - ((-2r)\alpha_{-2,1} + (-r)\alpha_{-1,1} + (0)\alpha_{0,1} + \alpha_{1,1}), \\ C_q =& 1 - \left(\frac{(-2r)^q}{q!}\alpha_{-2,1} + \frac{(-r)^q}{q!}\alpha_{-1,1} + \frac{(0)^q}{q!}\alpha_{0,1} \right. \\ & \left. + \frac{(1)}{(q-1)!}\alpha_{1,1}\right), q = 2, 3, \cdots, \end{aligned} \quad (10)$$

and the constants for D_q are given as:

$$\begin{aligned} D_0 =& 1 - (\theta_{-2,1} + \theta_{-1,1} + \theta_{0,1}), \\ D_1 =& 1 - ((-2r)\theta_{-2,1} + (-r)\theta_{-1,1} + (0)\theta_{0,1}), \\ D_q =& 1 - \left(\frac{(-2r)^q}{q!}\theta_{-2,1} + \frac{(-r)^q}{q!}\theta_{-1,1} + \frac{(0)^q}{q!}\theta_{0,1} \right. \\ & \left. - \frac{(1)}{(q-2)!}\beta_{1,1}\right),\ q = 2, 3, \cdots. \end{aligned} \quad (11)$$

The four coefficients in (2) are determined by solving $C_0 = C_1 = C_2 = C_3 = 0$ simultaneously and are given as:

$$\begin{aligned} \alpha_{-2,1} &= -\frac{1+r}{2r^2(1+2r)},\ \alpha_{-1,1} = \frac{1+2r}{r^2(1+r)}, \\ \alpha_{0,1} &= -\frac{(1+r)(1+2r)}{2r^2},\ \alpha_{1,1} = \frac{3+6r+2r^2}{(1+r)(1+2r)}. \end{aligned} \quad (12)$$

Meanwhile, the four coefficients in (3) are derived by solving $D_0 = D_1 = D_2 = D_3 = 0$ concurrently and are equivalent to:

$$\begin{aligned} \theta_{-2,1} &= \frac{2+r}{6r^2},\ \theta_{-1,1} = -\frac{2+4r}{3r^2}, \\ \theta_{0,1} &= \frac{(2+7r+6r^2)}{6r^2},\ \beta_{1,1} = \frac{(1+2r)}{6}. \end{aligned} \quad (13)$$

The corrector formulas at x_{n+2} are derived by using the same strategy for x_{n+1} and these take the following forms:

$$\begin{aligned} hy'_{n+2} =& \alpha_{-2,2}y_{n-2} + \alpha_{-1,2}y_{n-1} + \alpha_{0,2}y_n + \alpha_{1,2}y_{n+1} \\ & + \alpha_{2,2}y_{n+2}, \end{aligned} \quad (14)$$

$$\begin{aligned} y_{n+2} =& \theta_{-2,2}y_{n-2} + \theta_{-1,2}y_{n-1} + \theta_{0,2}y_n + \theta_{1,2}y_{n+1} + \\ & h^2\beta_{2,2}f_{n+2}. \end{aligned} \quad (15)$$

The coefficients in (14) and (15) are determined by taking the values of constants C_0, C_1, C_2, C_3, C_4 and D_0, D_1, D_2, D_3, D_4 equal to 0. The formulas for coefficient of hy'_{n+2} and y_{n+2} are respectively given as below:

$$\begin{aligned} \alpha_{-2,2} &= \frac{2+r}{2r^2(1+3r+2r^2)},\ \alpha_{-1,2} = -\frac{4}{r^2(2+r)}, \\ \alpha_{0,2} &= \frac{(1+r)(2+r)}{2r^2},\ \alpha_{1,2} = -\frac{4(2+r)}{1+2r}, \\ \alpha_{2,2} &= \frac{10+12r+3r^2}{2(1+r)(2+r)}, \end{aligned} \quad (16)$$

and

$$\begin{aligned}\theta_{-2,2} &= -\frac{16+14r+3r^2}{r^2(18+51r+32r^2+4r^3)},\\ \theta_{-1,2} &= \frac{8(4+3r)}{r^2(18+15r+2r^2)},\\ \theta_{0,2} &= -\frac{(16+42r+39r^2+15r^3+2r^4)}{r^2(18+15r+2r^2)},\\ \theta_{1,2} &= \frac{8(12+18r+8r^2+r^3)}{18+51r+32r^2+4r^3},\\ \beta_{2,2} &= \frac{2(3+3r+r^2)}{18+15r+2r^2}.\end{aligned} \tag{17}$$

The 2DBBDF for three different values of r are tabulated as in Table I.

TABLE I
COEFFICIENTS OF 2DBBDF FOR $r = 1, 2$, AND $r = 10/19$

r		y_{n-2}	y_{n-1}	y_n	y_{n+1}	y_{n+2}	f_{n+1}	f_{n+2}
	y_{n+1}	$-\frac{1}{2}$	2	$-\frac{5}{2}$	0	0	$\frac{1}{2}$	0
1	y'_{n+1}	$-\frac{1}{3}$	$\frac{3}{2}$	-3	$\frac{11}{6}$	0	0	0
	y_{n+2}	$\frac{11}{35}$	$-\frac{8}{5}$	$\frac{114}{35}$	$-\frac{104}{35}$	0	0	$\frac{12}{35}$
	y'_{n+2}	$\frac{1}{4}$	$-\frac{4}{3}$	3	-4	$\frac{25}{12}$	0	0
	y_{n+1}	$-\frac{1}{6}$	$\frac{5}{6}$	$-\frac{5}{3}$	0	0	$\frac{5}{6}$	0
2	y'_{n+1}	$-\frac{3}{40}$	$\frac{5}{12}$	$-\frac{15}{8}$	$\frac{23}{15}$	0	0	0
	y_{n+2}	$\frac{1}{20}$	$-\frac{5}{14}$	$\frac{51}{28}$	$-\frac{88}{35}$	0	0	$\frac{3}{7}$
	y'_{n+2}	$\frac{1}{30}$	$-\frac{1}{4}$	$\frac{3}{2}$	$-\frac{16}{5}$	$\frac{23}{12}$	0	0
	y_{n+1}	$-\frac{38}{25}$	$\frac{247}{50}$	$-\frac{221}{50}$	0	0	$\frac{13}{38}$	0
$\frac{10}{19}$	y'_{n+1}	$-\frac{10469}{7800}$	$\frac{14079}{2900}$	$-\frac{1131}{200}$	$\frac{2423}{1131}$	0	0	0
	y_{n+2}	$\frac{13718}{8525}$	$-\frac{363527}{59675}$	$\frac{417426}{59675}$	$-\frac{8384}{2387}$	0	0	$\frac{696}{2387}$
	y'_{n+2}	$\frac{13718}{9425}$	$-\frac{6859}{1200}$	$\frac{174}{25}$	$-\frac{64}{13}$	$\frac{3095}{1392}$	0	0

III. ANALYSIS OF THE METHOD

This section discusses the basic properties of the proposed method which comprises the consistency, zero-stability, convergence and linear stability. The discussion is applied to 2DBBDF with a fixed step size *i.e.* $r = 1$.

The method derived in the previous section can be expressed in the standard form of block method as follows:

$$A_0Y_{m-2} + A_1Y_{m-1} + A_2Y_m = h^2B_2F_m,\ \ n = 2m, \tag{18}$$

where,

$$A_0 = \begin{bmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{11}{35} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & -\frac{5}{2} & 0 & 2 \\ 0 & 3 & 0 & -\frac{3}{2} \\ 0 & \frac{114}{35} & 0 & -\frac{8}{5} \\ 0 & -3 & 0 & \frac{4}{3} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & h & -\frac{11}{6} \\ 0 & 1 & 0 & -\frac{104}{35} \\ h & -\frac{25}{12} & 0 & 4 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{12}{25} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Y_{m-2} = \begin{bmatrix} y'_{n-2} \\ y_{n-2} \\ y'_{n-3} \\ y_{n-3} \end{bmatrix} = \begin{bmatrix} y'_{2m-2} \\ y_{2m-2} \\ y'_{2m-3} \\ y_{2m-3} \end{bmatrix} = \begin{bmatrix} y'_{2(m-2)+2} \\ y_{2(m-2)+2} \\ y'_{2(m-2)+1} \\ y_{2(m-2)+1} \end{bmatrix},$$

$$Y_{m-1} = \begin{bmatrix} y'_{n} \\ y_{n} \\ y'_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} y'_{2m} \\ y_{2m} \\ y'_{2m-1} \\ y_{2m-1} \end{bmatrix} = \begin{bmatrix} y'_{2(m-1)+2} \\ y_{2(m-1)+2} \\ y'_{2(m-1)+1} \\ y_{2(m-1)+1} \end{bmatrix},$$

$$Y_{m} = \begin{bmatrix} y'_{n+2} \\ y_{n+2} \\ y'_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} y'_{2(m)+2} \\ y_{2(m)+2} \\ y'_{2(m)+1} \\ y_{2(m)+1} \end{bmatrix}, \text{and}$$

$$F_{m} = \begin{bmatrix} f'_{n+2} \\ f_{n+2} \\ f'_{n+1} \\ f_{n+1} \end{bmatrix} = \begin{bmatrix} f'_{2(m)+2} \\ f_{2(m)+2} \\ f'_{2(m)+1} \\ f_{2(m)+1} \end{bmatrix}.$$

According to [35], the direct method has the order of p if $D_0 = D_1 = \cdots = D_{p+1} = 0$, $D_{p+2} \neq 0$ and by following [36], the block method (18) is the order of p provided $D_{p+2} \neq 0$. D_{p+2} is the error constant and the principal local truncation error at the point x_n is given by $D_{p+2}h^{p+2}y^{(p+2)}(x_n)$. In deriving the coefficients at the point (x_{n+1}, y_{n+1}), the constant must be $D_4 = -\frac{11}{24} \neq 0$. Therefore, the 2DBBDF method is of order 2.

The consistency, zero-stability, convergence and linear stability for the 2DBBDF method are verified by applying the following definitions:

Definition 3.1: The block method (18) is said to be consistent if it has order $p \geq 1$.

Definition 3.2: The block method (18) is zero-stable provided the roots R_j of its first characteristic polynomial satisfies $|R_j| \leq 1, j = 1(1)k$ and for those roots with $|R_j| = 1$,the multiplicity must not exceed 2 [36].

Definition 3.3: The linear multistep method is said to be absolutely stable if the roots of the characteristic equation are in moduli less than one for all values of the step length h.

To verify this property, the linear test equation $y'' = \theta y' + \mu y$, where θ and μ are real numbers, is applied to the block method (18) for $r = 1$. The terms in (18) are rearranged to obtain the following matrix equation,

$$A_0Y_{m-2} + A_1Y_{m-1} + (A_2 - h^2B_2)Y_m = 0. \tag{19}$$

The stability polynomial $L(R, h, \theta, \mu)$ is determined by evaluating the determinant of $A_0 + A_1R + (A_2 - h^2B_2)R^2 = 0$, which is equivalent to,

$$\begin{aligned} L(R, H_1, H_2) = &\frac{1}{420}R^5(72 - 37H_2 + 5H_2^2) + \\ &\frac{1}{140}R^6(-188 + 9H_2^2 + 21H_2 + H_1(22 - 6H_2)) + \\ &\frac{1}{140}R^7(304 - 108H_1(-3 + H_2) - 237H_2 + 81H_2^2) + \\ &\frac{1}{420}R^8(-420 + 354H_1 - 72H_1^2 + 685H_2 - \\ &282H_1H_2 - 275H_2^2) = 0, \end{aligned} \tag{20}$$

where $H_1 = h^2\mu$ and $H_2 = h\theta$. As $h \to 0$, the coefficients $H_1,\ H_2 \to 0$. Thus, the first characteristic polynomial is attained as follows:

$$\frac{6}{35}R^5 - \frac{47}{35}R^6 + \frac{76}{35}R^7 - R^8 = 0. \tag{21}$$

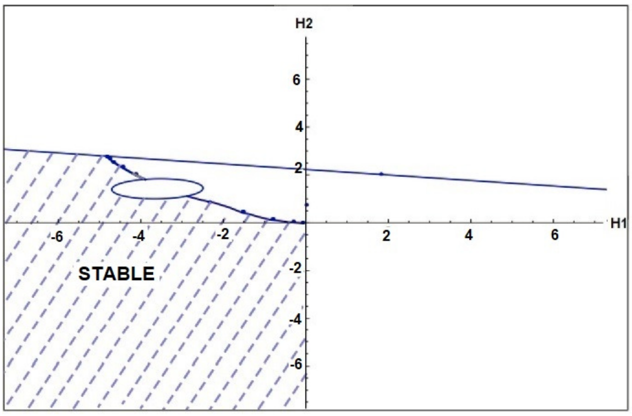


Fig. 2. Stability Region of the 2DBBDF method

Solving (21) for R gives,

$$R_{1,2,3,4,5} = 0,\ R_6 = 0.171429,\ R_{7,8} = 1.$$

From the Definition 3.2, the block method (18) is zero-stable.

The 2DBBDF is consistent since it is of order $p = 2 \geq 1$ and it is proven to be zero-stable. Referring to [35], the 2DBBDF converges.

The stability region for the proposed method is plotted in Figure 2. The region is defined by $L(R, H_1, H_2) = 0$ for $|R_j| < 1$ in $H_1 - H_2$-plane. The boundary of the region is determined by setting $R = 1, -1$ and $e^{i\theta} = \cos\theta + i\sin\theta,\ 0 < \theta < 2\pi$ and the region is equivalent to $H_1, H_2 < 0$ in $H_1 - H_2$-plane [25].

IV. Implementation of Method

In this section, the modified Newton iteration technique is used for the implementation purposes. To facilitate the iteration process, the 2DBBDF method is rewritten as follows:

$$\begin{aligned} y_{n+1} &= \beta_{1,1}h^2 f_{n+1} + W_1, \\ y_{n+2} &= \beta_{2,2}h^2 f_{n+2} + \theta_{1,2}y_{n+1} + W_2, \\ hy'_{n+1} &= \alpha_{1,1}y_{n+1} + V_1, \\ hy'_{n+2} &= \alpha_{1,2}y_{n+1} + \alpha_{2,2}y_{n+2} + V_2, \end{aligned} \quad (22)$$

where W_1, W_2, V_1, V_2 are the back values. The difference between i and $i+1$ iterations for $y_{n+1}, y_{n+2}, y'_{n+1}$ and y'_{n+2} are given as,

$$\begin{aligned} e_{n+s}^{(i+1)} &= y_{n+s}^{(i+1)} - y_{n+s}^{(i)}, \\ e\prime_{n+s}^{(i+1)} &= y\prime_{n+s}^{(i+1)} - y\prime_{n+s}^{(i)},\ s = 1, 2. \end{aligned} \quad (23)$$

Following the same iteration process as given by [32], the following matrices are obtained and subsequently solved using *LU* decomposition.

For $e_{n+s}^{(i+1)}, s = 1, 2$.

$$AE = B, \quad (24)$$

where $A =$

$$\begin{bmatrix} 1 - \beta_{1,1}h^2 J - \beta_{1,1}\alpha_{1,1}hJ' & 0 \\ -\theta_{1,2} - \beta_{2,2}\alpha_{1,2}hK' & 1 - \beta_{2,2}h^2 K - \beta_{2,2}\alpha_{2,2}hK' \end{bmatrix}$$

$$E = \begin{bmatrix} e_{n+1} \\ e_{n+2} \end{bmatrix}^{(i+1)}$$

and,

$$B = \begin{bmatrix} -y_{n+1}^{(i)} + \beta_{1,1}h^2 f_{n+1}^{(i)} + W_1 \\ -y_{n+2}^{(i)} + \beta_{2,2}h^2 f_{n+2}^{(i)} + \theta_{1,2}y_{n+1}^{(i)} + W_2 \end{bmatrix}.$$

For $e\prime_{n+s}^{(i+1)}, s = 1, 2$.

$$\begin{bmatrix} e'_{n+1} \\ e'_{n+2} \end{bmatrix}^{(i+1)} = h \begin{bmatrix} \alpha_{1,1} & 0 \\ \alpha_{1,2} & \alpha_{2,2} \end{bmatrix} \begin{bmatrix} e_{n+1} \\ e_{n+2} \end{bmatrix}^{(i+1)}. \quad (25)$$

J and J' are the Jacobian of f_{n+1} with respect to y_{n+1} and y'_{n+1} respectively. While K and K' are the Jacobian of f_{n+2} with respect to y_{n+2} and y'_{n+2} respectively. The iteration process is started by finding the required preliminary values over sub interval $[x_{n-2}, x_n]$. The direct Euler method is used for this purpose. Two-stage of modified Newton iteration where $i = 0, 1$ is applied throughout the iteration process. The structure of the algorithm used for the 2DBBDF is described briefly as follows:

Step 1 : <u>Predictor Estimation</u>
- P: estimation of predicted values $y_{n+1}^{(0)}$, $y_{n+2}^{(0)}$, $y\prime_{n+1}^{(0)}$ and $y\prime_{n+2}^{(0)}$
- E: evaluation of $f_{n+1}^{(0)}$ and $f_{n+2}^{(0)}$.

Step 2 : <u>Two Stage of Newton Iteration</u>

for $i = 0, 1$, do
- C:
  a) computation of $e_{n+s}^{(i+1)}$ and $e\prime_{n+s}^{(i+2)}$ by solving the matrices (24) and (25).
  b) calculation of the corrected values $y_{n+1}^{(i+1)}$, $y_{n+2}^{(i+1)}$, $y\prime_{n+1}^{(i+1)}$ and $y\prime_{n+2}^{(i+1)}$.
- E: evaluation of $f_{n+1}^{(i+1)}$ and $f_{n+2}^{(i+1)}$.

end for

Step 3 : <u>Convergence Test</u>

if $LTE \leq (0.1 \times TOL)$

if constant for at least two blocks

$h_{acc} = sf \times h_{old} \times \left(\frac{TOL}{LTE}\right)^{\frac{1}{p+1}}$

if $h_{acc} > 1.9 \times h_{old}$

$h_{new} = 1.9 \times h_{old}$

else

$h_{new} = h_{old}$

* Repeat Step 1 - 3 for next block

else

$h_{new} = 0.5 \times h_{old}$

* Repeat Step 1 - 3 for current block

The *LTE* is calculated by employing the following equation:

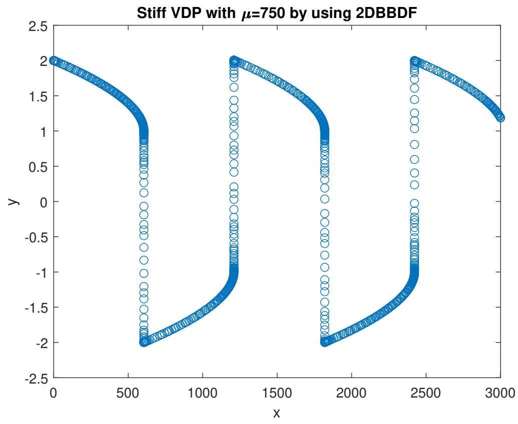
$$LTE = \left| y_{n+2}^{(p)} - y_{n+2}^{(p-1)} \right|, \quad (26)$$

where p is the order of the method, *sf* is the safety factor and is fixed to 0.8 in order to reduce the number of failure steps.

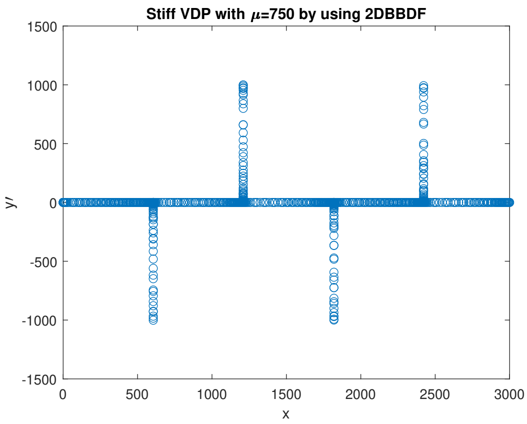
V. Numerical Results

Numerical experiments on the various values of μ are conducted in order to illustrate the performance of the 2DBBDF in solving stiff VDP. The values of μ used are 750, 1000, and 1500. For the complete oscillation of solutions, the equation is solved for interval up to $x = 3000$, to allow complete relaxation oscillation to occur. All the experiments used initial conditions $y(0) = 2$ and $y'(0) = 0$.The 2DBBDF code is written in Microsoft Visual Studio C++ 2010. All the plots for 2DBBDF used tolerance 10^{-4}. Meanwhile, the plots for ode15s used tolerance 5^{-14}, which is considered as the exact solution for the VDP equation.

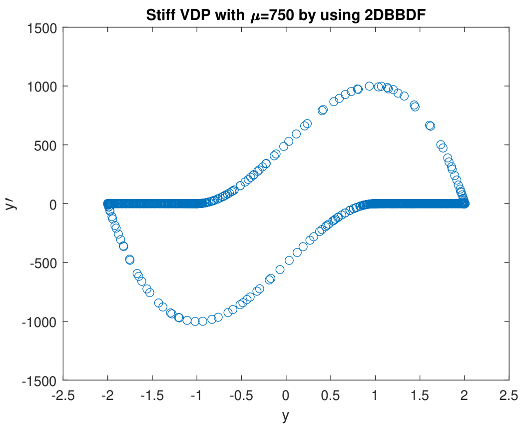
A. $\mu = 750$



(a) Plot of x against y.



(b) Plot of x against y'.



(c) Plot of y against y'.

Fig. 3. Plots of approximation given by 2DBBDF method for $\mu = 750$.

Figure 3 shows the numerical plotting by the 2DBBDF for $\mu = 750$. Two complete oscillation can be seen from figures 3a and 3b . There are four fast reactions and the plotting in figure 4 confirms that the solution given by the 2DBBDF conforms with the exact solution as given by ode15 from MATLAB. This shows the capability of 2DBBDF in solving stiff VDP problem, especially in dealing with the transient phase.

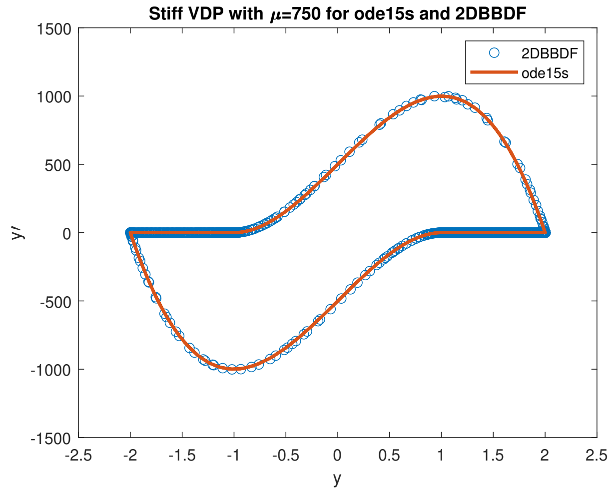


Fig. 4. Plots of y against y' given by ode15s and 2DBBDF for $\mu = 750$.

TABLE II
PERCENTAGE OF RELATIVE ERROR AT ENDPOINT, $x = 3000$ FOR y.

μ	2DBBDF	ode15s
750	0.84949	0.60975
1000	0.33870	0.34158
1500	0.20167	0.10924

B. $\mu = 1000$

The value of μ is increased to $\mu = 1000$, where only one complete oscillation is found (figures 5a and 5b). Three fast reaction occurred for this value of μ. Values of $y\prime$ dropped to almost to -1400 and increased to approximately 1400 in the fast phase. These fast states happened in the short interval of x. The numerical solution from figure 6 shows that the given solution from 2DBBDF in line with the exact solution from ode15s.

C. $\mu = 1500$

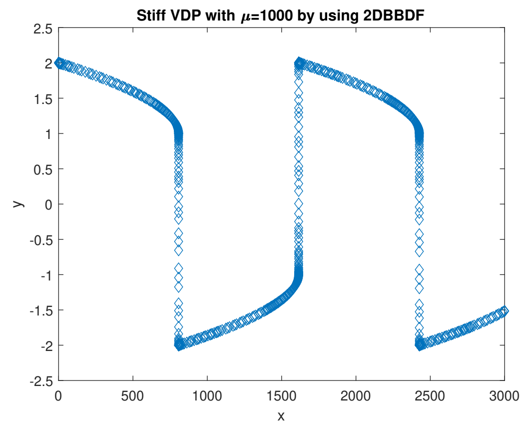
The value of μ is increased to $\mu = 1500$ and the numerical plotting is given (figure 7). Only one complete oscillation occurred for the interval $x \in [0, 3000]$. Two fast states are found and the solutions also confirmed the exact solution from ode15s as given in figure 8.

Figure 9 shows the phase portrait for all the values of μ when the VDP is solved with 2DBBDF. The evolution of the limit cycle in the phase plane is plotted. It is clear from the figure that, as the value of μ increases, the limit cycle becomes increasingly sharp. This is an example of a relaxation oscillator.

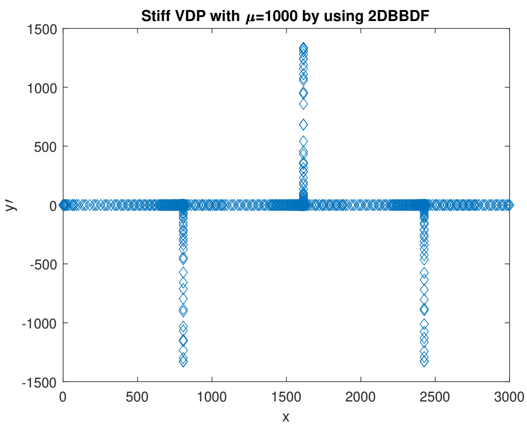
Tables II and III give the percentage of relative error for the solutions of y and y' respectively. The errors are

TABLE III
PERCENTAGE OF RELATIVE ERROR AT ENDPOINT, $x = 3000$ FOR y'.

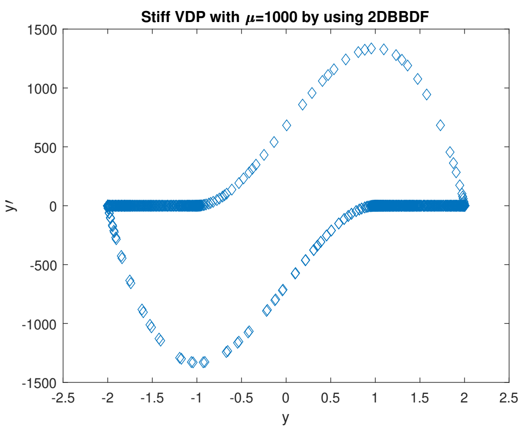
μ	2DBBDF	ode15s
750	5.04577	3.45585
1000	0.85881	0.36243
1500	0.41475	0.19457



(a) Plot of x against y.



(b) Plot of x against y'.



(c) Plot of y against y'.

Fig. 5. Plots of approximation given by 2DBBDF method for $\mu = 1000$.

calculated for the solutions at tolerance 10^{-4}. The solutions of these two methods are compared with the exact solution, which is assumed given by the ode15s at the tolerance 5^{-14}. From these two tables, the errors generated by the 2DBBDF are slightly higher than the ode15s for all μ. However, these errors decrease as the μ increases. This shows that the 2DBBDF gives accurate result as stiffness increases. Therefore, the 2DBBDF method is capable in solving the stiff second order ODEs.

The number of steps taken by the 2DBBDF and ode15s when solving the VDP at tolerance 10^{-4} are tabulated in table IV. The steps taken for the 2DBBDF are higher than

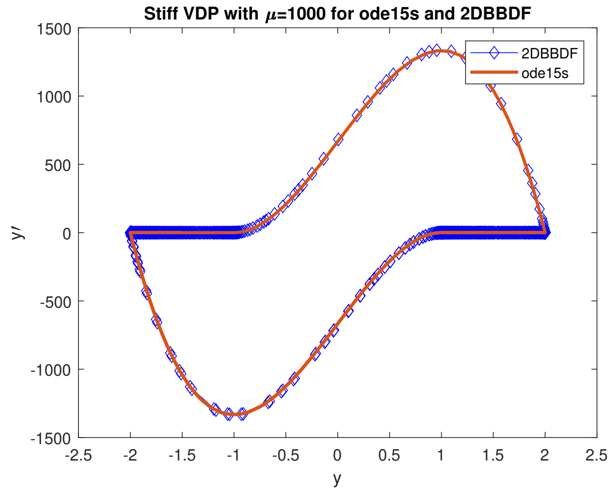


Fig. 6. Plots of y against y' given by ode15s and 2DBBDF for $\mu = 1000$.

ode15s as the μ increases. This is due to the step size restriction, in which the 2DBBDF is only allowed to increase the step size after applying a constant step size for at least two blocks, and an increase of step size is only allowed to increase by the factor of 1.9. Nevertheless, this number of steps is in par with the steps taken by the ode15s.

TABLE IV
TOTAL NUMBER OF STEPS.

μ	2DBBDF	ode15s
750	1081	1128
1000	857	844
1500	636	595

VI. Conclusion

In this paper, the 2DBBDF is developed for solving the problem of second order ODEs of stiff VDP. The convergence criterion and the stability analysis prove that the proposed method is suitable for solving stiff ODEs. Numerical plotting of the tested VDP for different values of μ demonstrating the accuracy of the 2DBBDF compared with the stiff solver ode15s. These figures demonstrate that the proposed method is well suited for stiff VDP since the solutions produced by the 2DBBDF coincide with the well-known ode15s code of MATLAB. Therefore, it can be concluded that the 2DBBDF is capable in solving the stiff ODEs and this can be one option in solving nonlinear stiff second order ODE directly especially stiff VDP.

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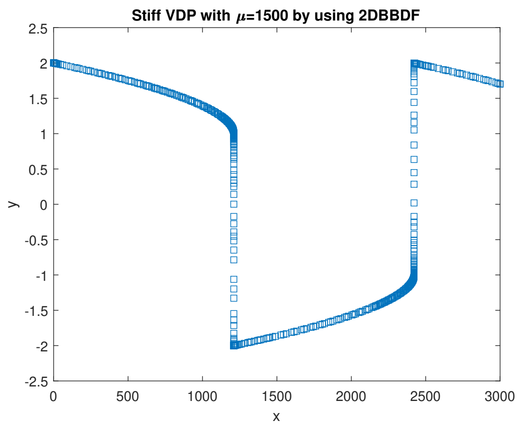
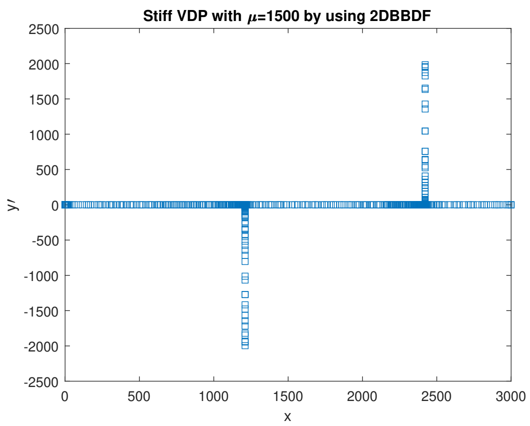
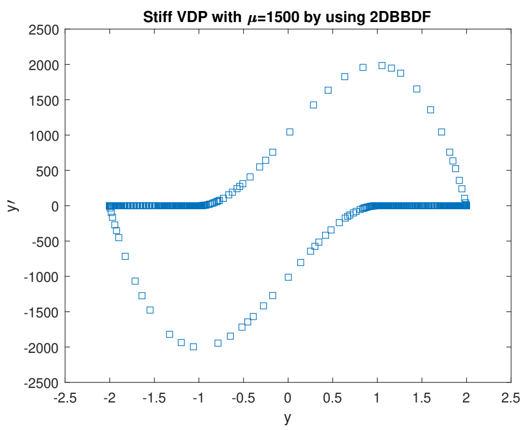


(a) Plot of x against y.



(b) Plot of x against y'.



(c) Plot of y against y'.

Fig. 7. Plots of approximation given by 2DBBDF method for $\mu = 1500$.

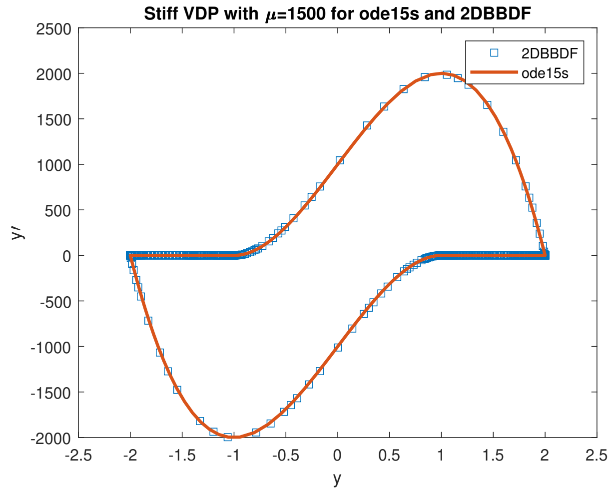


Fig. 8. Plots of y against y' given by ode15s and 2DBBDF for $\mu = 1500$.

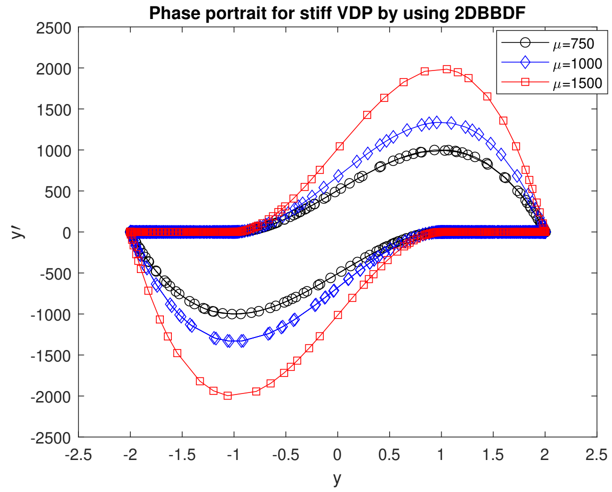


Fig. 9. Plots of y against y' given by 2DBBDF for $\mu = 750, 1000$ and 1500.

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