The Asymptotic Behaviors of Solutions for Higher-order $(m_1, m_2)-$coupled Kirchhoff Models with Nonlinear Strong Damping

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Abstract—**The Kirchhoff model is derived from the vibration problem of stretchable strings. In this paper, we focus on the long-term dynamics of higher-order coupled Kirchhoff systems with nonlinear strong damping. We first proved the existence and uniqueness of their solutions in different spaces through prior estimation and the Faedo-Galerkin method. Subsequently, we proved their family of global attractors using the compactness theorem. In this way, we systematically proposed the definition and proof process of the family of global attractors, thus enriching the related conclusions of higher-order coupled Kirchhoff models and laying a theoretical foundation for future practical applications.**

Index Terms—**Higher-Order Coupled Kirchhoff Models, nonlinear strong damping, global well-posedness, global attractor family.**

I. Introduction

In this study, we consider the dynamic behaviors of the following higher-order coupled Kirchhoff models in a bounded smooth domain $\Omega \subset R^n$:

$$\begin{cases} u_{tt} + N_1(\|\nabla^{m_1}u\|^2)(-\Delta)^{m_1}u_t + \\ M(\|\nabla^{m_1}u\|^2 + \|\nabla^{m_2}v\|^2)(-\Delta)^{m_1}u + g_1(u,v) \\ = f_1(x), \\ v_{tt} + N_2(\|\nabla^{m_2}v\|^2)(-\Delta)^{m_2}v_t + \\ M(\|\nabla^{m_1}u\|^2 + \|\nabla^{m_2}v\|^2)(-\Delta)^{m_2}v + g_2(u,v) \\ = f_2(x), \end{cases} \tag{1}$$

under the following boundary conditions:

$$u(x) = 0, \frac{\partial^i u}{\partial \mathbf{n}^i} = 0, i = 1, \cdots, m_1 - 1, m_1 > 1,$$
$$v(x) = 0, \frac{\partial^j v}{\partial \mathbf{n}^j} = 0, j = 1, \cdots, m_2 - 1, m_2 > 1; \tag{2}$$

and the following initial conditions:

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), v(x,0) = v_0(x),$$
$$v_t(x,0) = v_1(x), x \in \Omega, \tag{3}$$

where Δ is the Laplace operator, N_1, N_2, M_1, and M_2 are scalar functions specified later, g_1 and g_2 are the given source terms, and f_1 and f_2 are the given functions.

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(1) is a set of generalized higher-order quasi-linear wave equations. The proposed equation in this paper originated from the stretchable string vibration problem established by Kirchhoff in 1883:

$$\rho h \frac{\partial^2 u}{\partial t^2} = \{p_0 + \frac{Eh}{2L}\int_0^L (\frac{\partial u}{\partial x})^2 \mathrm{d}x\}\frac{\partial^2 u}{\partial t^2}, \tag{4}$$

where $0 < x < L, t \geq 0, u = u(x,t)$ is the lateral displacement at space coordinate x and time coordinate t, E represents the Young's modulus, ρ represents the mass density, h represents the cross-sectional area, L represents the length, and p_0 represents the axial tension of the accident. In recent decades, the long-term behaviors of Kirchhoff equations in various forms have attracted much academic attention, and abundant research results have been produced [1−8].

For instance, Chueshov [1] studied the well-posedness and long-term dynamic behaviors of the following Kirchhoff equation with a nonlinear strong damping term:

$$u_{tt} + \sigma(\|\nabla u\|^2)\Delta u_t - \phi(\|\nabla u\|^2)\Delta u + f(u) = h(x). \tag{5}$$

Moreover, Lin, Lv, and Lou [2] studied the global dynamics of the following generalized nonlinear Kirchhoff-Boussinesq equations with strong damping:

$$u_{tt} + \alpha u_t - \beta \Delta u_t + \Delta^2 u = div(g(|\nabla u|^2)\nabla u) + \Delta h(u) + f(x). \tag{6}$$

This paper proved that the semi-group conformed to the squeezing property of the system and demonstrated the existence of an exponential attractor. Then, the spectral interval theory proved that the system had an inertial manifold.

Ghisi and Gobbino [3] studied the existence of global and local solutions to the following Kirchhoff model with strong damping:

$$u_{tt}(t) + 2\delta A^{\sigma} u_t(t) + M(|A^{1/2}u(t)|^2)Au(t) = 0. \tag{7}$$

Nakao [4] proved the initial-boundary value problem of a quasi-linear Kirchhoff-type wave equation with standard dissipation u_t:

$$u_{tt} - (1 + \|\nabla u(t)\|_2^2)\Delta u + u_t + g(x,u) = f(x). \tag{8}$$

With the advance of research, scholars began to focus on the dynamics of higher-order Kirchhoff equations. Ye and Tao [9] studied the initial-boundary value problem of the following higher-order Kirchhoff-type equation with a nonlinear dissipation term:

$$u_{tt} + \Phi(\|D^m u\|^2)(-\Delta)^m u + a|u_t|^{q-2}u_t = b|u|^{r-2}u. \tag{9}$$

Lin and Zhu [10] studied the initial-boundary value problems of the following nonlinear nonlocal higher-order Kirchhoff-type equations:

$$u_{tt} + M(\|D^m u\|^2)(-\Delta)^m u + \beta(-\Delta)^m u_t + g(x, u_t) = f(x). \quad (10)$$

In this study, they demonstrated the existence and uniqueness of the solutions and proved the existence of a global attractor family using the compact method, thus obtaining the finite Hausdorff and Fractal dimensions.

Originating from physics, system coupling measures the dependence of two entities on each other. With suitable conditions or parameters, a connected system can be coupled, and its potential energy can enable the generation of new functions by combining the structural functions of different systems. As mathematical equations derived from physics, the Kirchhoff model is naturally considered a coupled system, and Scholars gradually considered the dynamics of coupled Kirchhoff equations. For example, Wang and Zhang [11] studied the long-term dynamics of coupled beam equations with strong damping under nonlinear boundary conditions. Lin and Zhang [12] studied the initial-boundary value problem of the following Kirchhoff coupling group with a source term and strong damping:

$$\begin{cases} u_{tt} - \beta\Delta u_t - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + \\ \qquad g_1(u, v) = f_1(x), \\ v_{tt} - \beta\Delta v_t - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + \\ \qquad g_2(u, v) = f_2(x). \end{cases} \quad (11)$$

The finite Hausdorff dimension of the global attractor was obtained in a previous work [12].

In recent years, Lin et al. [13−15] focused on the dynamics of higher-order coupled Kirchhoff equations and obtained a series of ideal results.

At present, few studies focus on the higher-order coupled Kirchhoff problems, and higher-order $(m_1, m_2)-$coupled Kirchhoff models with nonlinear strong damping have not been studied. The main difficulties lie in the estimation and processing of the harmonic term and the nonlinear damping term. In addition, the nonlinear damping also brings challenges when proving the uniqueness. Therefore, under reasonable assumptions, this paper overcame these difficulties by using Holder's inequality, Young's inequality, Poincare inequality, and Gagliardo-Nirenberg inequality, thus obtaining the global solution and the global attractor family. This study could refine the definition and existence theorem of the global attractor family. The conclusions could fill the gap of the global attractor family for higher-order coupled models (regardless of whether m_1 is equal to m_2) and lay the foundation for subsequent engineering applications.

The rest of this paper is organized as follows. Section II provides the fundamentals for this work, and states the main results. Section III proves the main results. Finally, the Summary and Prospect are presented in Section IV.

II. Preparatory Knowledge and Statement of Main Results

In this section, we introduce some assumptions that we will use, and give main results.

In this paper, $\|\cdot\|$ and $(\cdot,\cdot)$ denote the norm and the inner product in $H = L^2(\Omega)$. Let $H_0^1 = D((-\Delta)^{\frac{1}{2}})$ be the scale of the Hilbert space generated by the Laplacian with Dirichlet boundary condition on H and endowed with standard inner product and norm, respectively, $(\cdot,\cdot)_{H_0^1} = ((-\Delta)^{\frac{1}{2}}\cdot, (-\Delta)^{\frac{1}{2}}\cdot)$ and $\|\cdot\|_{H_0^1} = \|(-\Delta)^{\frac{1}{2}}\cdot\|$. The main goal here is to study the well-posedness and long-term dynamics of problems (1) to (3) under the following set of assumptions:

$(A1).M(s)$ is a continuous function on interval $[0, +\infty)$, $M(s) \in C^1(R^+)$, and
$1)M'(s) \geq 0$,
$2)M(0) \equiv M_0 > 0$.
$(A2)$. For any $u, v \in H$, let
$J(u, v) = \int_\Omega [G_1(u, v) + G_2(u, v)]\mathrm{d}x$,
where $G_1(u, v) = \int_0^u g_1(s, v)\mathrm{d}s, G_2(u, v) = \int_0^v g_2(u, s)\mathrm{d}s$, then for any $\mu \geq 0$, there exists $C_1 \geq 0, C_\mu \geq 0, C'_\mu \geq 0$ that
$G_1(u, v) + G_2(u, v) - C_1 J(u, v) + \mu(\|\nabla^{m_1} u\|^2 + \|\nabla^{m_2} v\|^2) \geq -C_\mu$,
$J(u, v) + \mu(\|\nabla^{m_1} u\|^2 + \|\nabla^{m_2} v\|^2) \geq -C'_\mu$.
$(A3).g_j(u, v)(j = 1, 2) \in C^1(R)$, and
$|g_j(u, v)| \leq C_2(1 + |u|^{p_j} + |v|^{q_j})$;
$|g_{ju}(u, v)| \leq C_3(1 + |u|^{p_j - 1} + |v|^{q_j})$;
$|g_{jv}(u, v)| \leq C_4(1 + |u|^{p_j} + |v|^{q_j - 1})$.
Specifically, when $n = 1, 2$, $1 \leq p_j(q_j)$; when $3 \leq n \leq 2m$, $1 \leq p_j(q_j) \leq \frac{n}{n-2}$; when $2m < n$, $1 \leq p_j(q_j) \leq \frac{n}{n-2m}$, where $m = \min\{m_1, m_2\}$.
$(A4).N_j(s_j) \geq N_{j0}$ and $N_{j0}(j = 1, 2)$ are positive constants, and $\rho > 0$. Thus, $M(s_1 + s_2) - \rho N_1(s_1) - \rho N_2(s_2) > 0$.

Then, the research phase space of this study is obtained:

$$\begin{gathered} V_0 = H, V_k = H^k(\Omega) \cap H_0^1(\Omega), \\ X_{0\times 0} = V_{m_1}(\Omega) \times H(\Omega) \times V_{m_2} \times H(\Omega), \\ X_{k_1\times k_2} = V_{m_1+k_1}(\Omega) \times V_{k_1}(\Omega) \times V_{m_2+k_2} \times V_{k_2}(\Omega), \\ k_1 = 0, 1, 2, \cdots, m_1, k_2 = 0, 1, 2, \cdots, m_2, \end{gathered}$$

and the norms of the corresponding spaces are as follows:

$$\|(u, y_1, v, y_2)\|^2_{X_{k_1\times k_2}} = \|\nabla^{m_1+k_1} u\|^2 + \|\nabla^{k_1} y_1\|^2 + \|\nabla^{m_2+k_2} v\|^2 + \|\nabla^{k_2} y_2\|^2.$$

Meanwhile, the general form of the Poincare inequality is: $\lambda_1\|\nabla^r u\|^2 \leq \|\nabla^{r+1} u\|^2$, where λ_1 is the first eigenvalue of $-\Delta$ with a homogeneous Dirichlet boundary on Ω. In this paper, C_i is a constant, and $C(\cdot)$ is a constant depending on the parameters in parentheses.

Now, we state the main results of this paper.

Theorem 1. Assume that assumptions $(A1) - (A4)$ hold, if $f_1 \in V_{k_1}, f_2 \in V_{k_2}$ and initial data $(u_0, u_1, v_0, v_1) \in X_{k_1\times k_2}, k_1 = 0, 1, 2, \cdots, m_1, k_2 = 0, 1, 2, \cdots, m_2$, then problems (1) to (3) admit a unique solution (u, v) satisfying

$$\begin{gathered} u \in L^\infty(0, \infty; V_{m_1+k_1}); \\ u_t \in L^\infty(0, \infty; H) \cap L^2(0, T; V_{k_1}); \\ v \in L^\infty(0, \infty; V_{m_2+k_2}); \\ v_t \in L^\infty(0, \infty; H) \cap L^2(0, T; V_{k_2}). \end{gathered}$$

Theorem 2. Assume that assumptions $(A1) - (A4)$ hold, if $f_1 \in V_{m_1}, f_2 \in V_{m_2}$ and initial data $(u_0, u_1, v_0, v_1) \in X_{m_1\times m_2}$, then, problems (1) to (3) have a global attractor family $\mathcal{A}$ in $X_{0\times 0}$:

$$\mathcal{A}=\{A_{k_1\times k_2}\}, A_{k_1\times k_2}=\omega(B_{k_1\times k_2,0})=\bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}S(t)B_{k_1\times k_2,0}},$$
$$k_1=1,2,\cdots,m_1, k_2=1,2,\cdots,m_2,$$

where $B_{k_1\times k_2,0}=\{(u,u_t,v,v_t)\in X_{k_1\times k_2}:\|(u,u_t,v,v_t)\|^2_{X_{k_1\times k_2}}=\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{k_1}u_t\|^2+\|\nabla^{m_2+k_2}v\|^2+\|\nabla^{k_2}v_t\|^2\leq C(R_0)+C(R_{k_1\times k_2})\}$ are bounded absorbing sets in $X_{0\times 0}$, $B_{k_1\times k_2,0}$ are compact in $X_{0\times 0}$, $A_{k_1\times k_2}\subset X_{0\times 0}$.
(1) $S(t)A_{k_1\times k_2}=A_{k_1\times k_2}$, (for all $t\geq 0$),
(2) $A_{k_1\times k_2}$ attract all bounded sets in $X_{0\times 0}$, i.e., for all $B_{k_1\times k_2}\subset X_{0\times 0}$ are bounded sets in $X_{0\times 0}$, and

$$dist(S(t)B_{k_1\times k_2},A_{k_1\times k_2})=\sup_{x\in B_{k_1\times k_2}}\inf_{y\in A_{k_1\times k_2}}\|S(t)x-y\|_{X_{0\times 0}}\to 0(t\to\infty),$$

where $\{S(t)\}_{t\geq 0}$ is the solution semi-group generated by problems (1) to (3).

III. Main Steps of Results

In this section, we present the proof process to the existence and uniqueness of the solutions and the family of global attractors to problem (1)-(3).

Let $\varepsilon>0$ be small enough and $\lambda_1^{m_1}N_{10}-2-4\varepsilon-2\varepsilon^2\geq 0, \lambda_1^{m_2}N_{20}-2-4\varepsilon-2\varepsilon^2\geq 0$.

Lemma 1.[16] Let $y:R^+\to R^+$ be an absolutely continuous positive function on $[0,+\infty)$, which satisfies the following differential inequality for some $\delta>0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+2\delta y(t)\leq g(t)y(t)+K, t>0,$$

where $K\geq 0$, and $a\geq 0$ if $t\geq s\geq 0$ so that $\int_s^t g(\tau)d\tau\leq\delta(t-s)+a$. Then,

$$y(t)\leq \mathrm{e}^a y(0)\mathrm{e}^{-\delta t}+\frac{K\mathrm{e}^a}{\delta}, t\geq 0.$$

Lemma 2.[10] Let X be a Banach space, and the continuous operator semi-group $\{S(t)\}_{t\geq 0}$ satisfies the following:
(1) semi-group $\{S(t)\}_{t\geq 0}$ is uniformly bounded in X, i.e., for all $R_0>0$, there exists a positive constant $C_0(R_0)$ that when $\|u\|_X\leq R_0$,

$$\|S(t)u\|_X\leq C_0(R_0), (\text{for all}\quad t\in[0,+\infty));$$

(2) there exists a bounded absorbing set B_0 in X, and for any bounded set $B\subset X$, there exists a moment t_0 that

$$S(t)B\subset B_0(t\geq t_0);$$

(3) if $t>0$, and $S(t)$ is a fully continuous operator,
then semi-group $\{S(t)\}_{t\geq 0}$ has a global attractor A in X, and

$$A=\omega(B_0)=\bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}S(t)B_0}.$$

Lemma 3. Assume that assumptions $(A1)-(A4)$ hold, if $f_j\in H(j=1,2)$ and initial data $(u_0,u_1,v_0,v_1)\in X_{0\times 0}$, then for $R_0>0$, there exist positive constants $C(R_0)$ and t_0 so that when $t\geq t_0$, (u,y_1,v,y_2) determined by problems (1) to (3) satisfies

$$\|(u,y_1,v,y_2)\|^2_{X_{0\times 0}}=\|\nabla^{m_1}u\|^2+\|y_1\|^2+\|\nabla^{m_2}v\|^2+\|y_2\|^2\leq C(R_0), \tag{12}$$

where $y_1=u_t+\varepsilon u, y_2=v_t+\varepsilon v$.

Proof: Multiplying the first equation of (1) by y_1 in H and the second one by y_2 in H, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|y_1\|^2+\|y_2\|^2+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\mathrm{d}\tau\\&+2J(u,v)\Big]+\varepsilon M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)\cdot\\&(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)-\varepsilon(\|y_1\|^2+\|y_2\|^2)+\\&\varepsilon^2((u,y_1)+(v,y_2))+N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1}y_1\|^2+\\&N_2(\|\nabla^{m_2}v\|^2)\|\nabla^{m_2}y_2\|^2-\\&\varepsilon N_1(\|\nabla^{m_1}u\|^2)(\nabla^{m_1}y_1,\nabla^{m_1}u)-\\&\varepsilon N_2(\|\nabla^{m_2}v\|^2)(\nabla^{m_2}y_2,\nabla^{m_2}v)+\varepsilon(g_1(u,v),u)+\\&\varepsilon(g_2(u,v),v)=(f_1,y_1)+(f_2,y_2).\end{aligned}\tag{13}$$

By Holder's inequality, Young's inequality, Poincare inequality, etc., we have

$$\begin{aligned}&-\varepsilon(\|y_1\|^2+\|y_2\|^2)+\varepsilon^2((u,y_1)+(v,y_2))\geq\\&(-\varepsilon-\frac{\varepsilon^2}{2})(\|y_1\|^2+\|y_2\|^2)-\frac{\varepsilon^2}{2}(\|u\|^2+\|v\|^2)\geq\\&(-\varepsilon-\frac{\varepsilon^2}{2})(\|y_1\|^2+\|y_2\|^2)-\frac{\varepsilon^2}{2}\lambda_1^{-m_1}\|\nabla^{m_1}u\|^2-\\&\frac{\varepsilon^2}{2}\lambda_1^{-m_2}\|\nabla^{m_2}v\|^2,\end{aligned}\tag{14}$$

$$\begin{aligned}&N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1}y_1\|^2+N_2(\|\nabla^{m_2}v\|^2)\|\nabla^{m_2}y_2\|^2\\&-\varepsilon N_1(\|\nabla^{m_1}u\|^2)(\nabla^{m_1}y_1,\nabla^{m_1}u)-\\&\varepsilon N_2(\|\nabla^{m_2}v\|^2)(\nabla^{m_2}y_2,\nabla^{m_2}v)\geq\\&\frac{1}{2}N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1}y_1\|^2+\frac{1}{2}N_2(\|\nabla^{m_2}v\|^2)\cdot\\&\|\nabla^{m_2}y_2\|^2-\frac{\varepsilon^2}{2}N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1}u\|^2-\\&\frac{\varepsilon^2}{2}N_2(\|\nabla^{m_2}v\|^2)\|\nabla^{m_2}v\|^2\geq\frac{1}{2}\lambda_1^{m_1}N_1(\|\nabla^{m_1}u\|^2)\cdot\\&\|y_1\|^2+\frac{1}{2}\lambda_1^{m_2}N_2(\|\nabla^{m_2}v\|^2)\|y_2\|^2-\\&\frac{\varepsilon^2}{2}N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1}u\|^2-\frac{\varepsilon^2}{2}N_2(\|\nabla^{m_2}v\|^2)\cdot\\&\|\nabla^{m_2}v\|^2,\end{aligned}\tag{15}$$

$$\begin{aligned}&(f_1,y_1)+(f_2,y_2)\leq\|f_1\|\|y_1\|+\|f_2\|\|y_2\|\leq\\&\frac{1}{2}\|y_1\|^2+\frac{1}{2}\|y_2\|^2+\frac{1}{2}\|f_1\|^2+\frac{1}{2}\|f_2\|^2,\end{aligned}\tag{16}$$

Inserting the above estimates into (13), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|y_1\|^2+\|y_2\|^2+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\mathrm{d}\tau\\&+2J(u,v)\Big]+\frac{1}{2}\lambda_1^{m_1}N_1(\|\nabla^{m_1}u\|^2)\|y_1\|^2-\\&(\frac{1}{2}-\varepsilon-\frac{\varepsilon^2}{2})\|y_1\|^2+\frac{1}{2}\lambda_1^{m_2}N_2(\|\nabla^{m_2}v\|^2)\|y_2\|^2-\\&(\frac{1}{2}-\varepsilon-\frac{\varepsilon^2}{2})\|y_2\|^2+\\&\varepsilon M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)\\&-(\frac{\varepsilon^2}{2}N_1(\|\nabla^{m_1}u\|^2)+\frac{\varepsilon^2}{2}\lambda_1^{-m_1})\|\nabla^{m_1}u\|^2-\end{aligned}$$

$$(\frac{\varepsilon^2}{2}N_2(\|\nabla^{m_2}v\|^2)+\frac{\varepsilon^2}{2}\lambda_1^{-m_2})\|\nabla^{m_2}v\|^2 \le -\varepsilon(g_1(u,v),u)-\varepsilon(g_2(u,v),v)+\frac{1}{2}\|f_1\|^2+\frac{1}{2}\|f_2\|^2. \quad (17)$$

According to (A_1),

$$\varepsilon M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2) \ge \frac{\varepsilon}{4}\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\mathrm{d}\tau+\frac{3\varepsilon}{4}M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)\cdot(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2), \quad (18)$$

and according to (A_2),

$$-\varepsilon(g_1(u,v),u)-\varepsilon(g_2(u,v),v)\le -\varepsilon C_1 J(u,v)+\varepsilon\mu(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)+\varepsilon C_\mu. \quad (19)$$

Inserting (18) and (19) into (17), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\|y_1\|^2+\|y_2\|^2+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\mathrm{d}\tau+2J(u,v)\big]+(\lambda_1^{m_1}N_1(\|\nabla^{m_1}u\|^2)-1-2\varepsilon-\varepsilon^2)\|y_1\|^2+(\lambda_1^{m_2}N_2(\|\nabla^{m_2}v\|^2)-1-2\varepsilon-\varepsilon^2)\|y_2\|^2+\frac{\varepsilon}{2}\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\mathrm{d}\tau+2\varepsilon C_1J(u,v)+(\frac{3\varepsilon}{2}M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)-2\varepsilon\mu-\varepsilon^2N_1(\|\nabla^{m_1}u\|^2)-\varepsilon^2\lambda_1^{-m_1})\|\nabla^{m_1}u\|^2+(\frac{3\varepsilon}{2}M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)-2\varepsilon\mu-\varepsilon^2N_2(\|\nabla^{m_2}v\|^2)-\varepsilon^2\lambda_1^{-m_2})\|\nabla^{m_2}v\|^2\le 2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2. \quad (20)$$

Let $H_1(t)=\|y_1\|^2+\|y_2\|^2+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\mathrm{d}\tau+2J(u,v)$ and $\sigma_1=\min\{\lambda_1^{m_1}N_{10}-1-2\varepsilon-\varepsilon^2,\lambda_1^{m_2}N_{20}-1-2\varepsilon-\varepsilon^2,\frac{\varepsilon}{2},\varepsilon C_1\}$, we can infer from (20) that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}H_1(t)+\sigma_1H_1(t)\le 2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2. \quad (21)$$

According to Gronwall's inequality, we have

$$H_1(t)\le H_1(0)\mathrm{e}^{-\sigma_1 t}+\tfrac{2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2}{\sigma_1}, \quad (22)$$

and

$$H_1(t)\ge\|y_1\|^2+\|y_2\|^2+M_0(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)+2J(u,v)\ge\|y_1\|^2+\|y_2\|^2+\frac{M_0}{2}(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)-2C'_\mu\ge C_5(\|y_1\|^2+\|y_2\|^2+\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)-2C'_\mu \quad (23)$$

according to $(A_1)(A_2)$, where $\mu=\frac{M_0}{4}, C_5=\min\{1,\frac{M_0}{2}\}$, then,

$$\|(u,y_1,v,y_2)\|^2_{X_{0\times0}}=\|\nabla^{m_1}u\|^2+\|y_1\|^2+\|\nabla^{m_2}v\|^2+\|y_2\|^2\le\frac{(H_1(t)+2C'_\mu)}{C_5}\le\frac{H_1(0)\mathrm{e}^{-\sigma_1t}+2C'_\mu}{C_5}+\frac{2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2}{\sigma_1C_5}, \quad (24)$$

i.e.,

$$\overline{\lim_{t\to\infty}}\|(u,y_1,v,y_2)\|^2_{X_{0\times0}}\le\frac{2C'_\mu}{C_5}+\frac{2\varepsilon C_\mu+\|f_1\|^2+\|f_2\|^2}{\sigma_1C_5}=R_0. \quad (25)$$

Therefore, there exist positive constants $C(R_0)$ and t_0 that when $t\ge t_0$,

$$\|(u,y_1,v,y_2)\|^2_{X_{0\times0}}=\|\nabla^{m_1}u\|^2+\|y_1\|^2+\|\nabla^{m_2}v\|^2+\|y_2\|^2\le C(R_0). \quad (26)$$

Thus, Lemma 3 is proved.

Lemma 4. Assume that assumptions $(A1)-(A4)$ hold, if $f_1\in V_{k_1}, f_2\in V_{k_2}, k_1=1,2,\cdots,m_1,\ k_2=1,2,\cdots,m_2$, and initial data $(u_0,u_1,v_0,v_1)\in X_{k_1\times k_2}$. Then, for $R_{k_1\times k_2}>0$, there exist positive constants $C(R_{k_1\times k_2})$ and $t_{k_1\times k_2}$ that when $t\ge t_{k_1\times k_2}$, (u,y_1,v,y_2) determined by problems (1)-(3) satisfies

$$\|(u,y_1,v,y_2)\|^2_{X_{k_1\times k_2}}=\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{k_1}y_1\|^2+\|\nabla^{m_2+k_2}v\|^2+\|\nabla^{k_2}y_2\|^2\le C(R_{k_1\times k_2}), \quad (27)$$

where $y_1=u_t+\varepsilon u, y_2=v_t+\varepsilon v$.

Proof: Multiplying the first equation of (1) by $(-\Delta)^{k_1}y_1$, $k_1=1,2,\cdots,m_1$ in H and the second one by $(-\Delta)^{k_2}y_2$, $k_2=1,2,\cdots,m_2$ in H and then integrating over Ω, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\nabla^{k_1}y_1\|^2+\|\nabla^{k_2}y_2\|^2+M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)\cdot(\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2)\big]+\varepsilon M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)\cdot(\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2)-\varepsilon(\|\nabla^{k_1}y_1\|^2+\|\nabla^{k_2}y_2\|^2)+\varepsilon^2((\nabla^{k_1}u,\nabla^{k_1}y_1)+(\nabla^{k_2}v,\nabla^{k_2}y_2))+N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1+k_1}y_1\|^2+N_2(\|\nabla^{m_2}v\|^2)\|\nabla^{m_2+k_2}y_2\|^2-\varepsilon N_1(\|\nabla^{m_1}u\|^2)(\nabla^{m_1+k_1}y_1,\nabla^{m_1+k_1}u)-\varepsilon N_2(\|\nabla^{m_2}v\|^2)(\nabla^{m_2+k_2}y_2,\nabla^{m_2+k_2}v)+(g_1(u,v),(-\Delta)^{k_1}y_1)+(g_2(u,v),(-\Delta)^{k_2}y_2)=\frac{\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)+(f_1,(-\Delta)^{k_1}y_1)+(f_2,(-\Delta)^{k_2}y_2). \quad (28)$$

According to Holder's inequality, Young's inequality, Poincare inequality, etc., we have

$$-\varepsilon(\|\nabla^{k_1}y_1\|^2+\|\nabla^{k_2}y_2\|^2)+\varepsilon^2((\nabla^{k_1}u,\nabla^{k_1}y_1)+(\nabla^{k_2}v,\nabla^{k_2}y_2))\ge(-\varepsilon-\frac{\varepsilon^2}{2})(\|\nabla^{k_1}y_1\|^2+\|\nabla^{k_2}y_2\|^2)-\frac{\varepsilon^2}{2}(\|\nabla^{k_1}u\|^2+\|\nabla^{k_2}v\|^2)\ge(-\varepsilon-\frac{\varepsilon^2}{2})(\|\nabla^{k_1}y_1\|^2+\|\nabla^{k_2}y_2\|^2)-\frac{\varepsilon^2}{2}\lambda_1^{-m_1}\|\nabla^{m_1+k_1}u\|^2-\frac{\varepsilon^2}{2}\lambda_1^{-m_2}\|\nabla^{m_2+k_2}v\|^2, \quad (29)$$

$$N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1+k_1}y_1\|^2+$$

$$N_2(\|\nabla^{m_2}v\|^2)\|\nabla^{m_2+k_2}y_2\|^2 - \varepsilon N_1(\|\nabla^{m_1}u\|^2)(\nabla^{m_1+k_1}y_1, \nabla^{m_1+k_1}u) - \varepsilon N_2(\|\nabla^{m_2}v\|^2)(\nabla^{m_2+k_2}y_2, \nabla^{m_2+k_2}v) \geq \frac{1}{2}N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1+k_1}y_1\|^2 + \frac{1}{2}N_2(\|\nabla^{m_2}v\|^2)\|\nabla^{m_2+k_2}y_2\|^2 - \frac{\varepsilon^2}{2}N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1+k_1}u\|^2 - \frac{\varepsilon^2}{2}N_2(\|\nabla^{m_2}v\|^2)\|\nabla^{m_2+k_2}v\|^2, \tag{30}$$

$$(g_1(u,v), (-\Delta)^{k_1}y_1) + (g_2(u,v), (-\Delta)^{k_2}y_2) \leq \|g_1(u,v)\|\|\nabla^{2k_1}y_1\| + \|g_2(u,v)\|\|\nabla^{2k_2}y_2\| \leq \frac{N_{10}}{4}\|\nabla^{m_1+k_1}y_1\|^2 + \frac{\lambda_1^{k_1-m_1}}{N_{10}}\|g_1(u,v)\|^2 + \frac{N_{20}}{4}\|\nabla^{m_2+k_2}y_2\|^2 + \frac{\lambda_1^{k_2-m_2}}{N_{20}}\|g_2(u,v)\|^2, \tag{31}$$

$$(f_1, (-\Delta)^{k_1}y_1) + (f_2, (-\Delta)^{k_2}y_2) \leq \|\nabla^{k_1}f_1\|\|\nabla^{k_1}y_1\| + \|\nabla^{k_2}f_2\|\|\nabla^{k_2}y_2\| \leq \frac{1}{2}\|\nabla^{k_1}y_1\|^2 + \frac{1}{2}\|\nabla^{k_2}y_2\|^2 + \frac{1}{2}\|\nabla^{k_1}f_1\|^2 + \frac{1}{2}\|\nabla^{k_2}f_2\|^2, \tag{32}$$

and

$$\|g_1(u,v)\|^2 = \int_\Omega |g_1(u,v)|^2 \mathrm{d}x \leq \int_\Omega |C_2(1+|u|^{p_1}+|v|^{q_1})|^2 \mathrm{d}x \leq C_6\int_\Omega (1+|u|^{2p_1}+|v|^{2q_1})\mathrm{d}x \leq C_7(1+\|u\|_{2p_1}^{2p_1}+\|v\|_{2q_1}^{2q_1}),$$
$$\|g_2(u,v)\|^2 \leq C_8(1+\|u\|_{2p_2}^{2p_2}+\|v\|_{2q_2}^{2q_2}) \tag{33}$$

according to (A_3). Furthermore, based on the Gagliardo-Nirenberg inequality, we can conclude that

$$\begin{cases} \|u\|_{2p_j}^{2p_j} \leq C_{9j}\|\nabla^{m_1}u\|^{\frac{n(p_j-1)}{m_1}}\|u\|^{\frac{2m_1p_j-n(p_j-1)}{m_1}}, \\ \|v\|_{2q_j}^{2q_j} \leq C_{10j}\|\nabla^{m_2}v\|^{\frac{n(q_j-1)}{m_2}}\|v\|^{\frac{2m_2q_j-n(q_j-1)}{m_2}}. \end{cases}$$

Thus, we have

$$\|g_1(u,v)\|^2 + \|g_2(u,v)\|^2 \leq C(R_0). \tag{34}$$

Inserting (30) to (32) and (34) into (28), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big[\|\nabla^{k_1}y_1\|^2 + \|\nabla^{k_2}y_2\|^2 + M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)\cdot(\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2)\big] + \frac{(2N_1(\|\nabla^{m_1}u\|^2)-N_{10})\lambda_1^{m_1}-2-4\varepsilon-2\varepsilon^2}{4}\|\nabla^{k_1}y_1\|^2 + \frac{(2N_2(\|\nabla^{m_2}v\|^2)-N_{20})\lambda_1^{m_2}-2-4\varepsilon-2\varepsilon^2}{4}\cdot \|\nabla^{k_2}y_2\|^2 + (\varepsilon M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2) - \frac{\varepsilon^2}{2}N_1(\|\nabla^{m_1}u\|^2) - \frac{\varepsilon^2}{2}\lambda_1^{-m_1})\|\nabla^{m_1+k_1}u\|^2 + (\varepsilon M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2) - \frac{\varepsilon^2}{2}N_2(\|\nabla^{m_2}v\|^2) - \frac{\varepsilon^2}{2}\lambda_1^{-m_2})\|\nabla^{m_2+k_2}v\|^2 \leq$$
$$\frac{\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2}{2}\cdot \frac{\mathrm{d}}{\mathrm{d}t}M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2) + \frac{1}{2}\|\nabla^{k_1}f_1\|^2 + \frac{1}{2}\|\nabla^{k_2}f_2\|^2 + C(R_0,\lambda_1) \leq (\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2)\cdot M'(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)\cdot((\nabla^{m_1}u,\nabla^{m_1}u_t)+(\nabla^{m_2}v,\nabla^{m_2}v_t)) + \frac{1}{2}\|\nabla^{k_1}f_1\|^2 + \frac{1}{2}\|\nabla^{k_2}f_2\|^2 + C(R_0,\lambda_1) \leq C_9(\|\nabla^{m_1}u_t\|+\|\nabla^{m_2}v_t\|)\cdot(\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2) + \frac{1}{2}\|\nabla^{k_1}f_1\|^2 + \frac{1}{2}\|\nabla^{k_2}f_2\|^2 + C(R_0,\lambda_1). \tag{35}$$

Let $H_2(t) = \|\nabla^{k_1}y_1\|^2 + \|\nabla^{k_2}y_2\|^2 + M(\|\nabla^{m_1}u\|^2 + \|\nabla^{m_2}v\|^2)(\|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{m_2+k_2}v\|^2)$ and $\sigma_2 = \min\{\frac{\lambda_1^{m_1}N_{10}-2-4\varepsilon-2\varepsilon^2}{2}, \frac{\lambda_1^{m_2}N_{20}-2-4\varepsilon-2\varepsilon^2}{2}, \frac{\varepsilon}{2}\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}H_2(t) + \sigma_2 H_2(t) \leq C_{10}(\|\nabla^{m_1}u_t\|+\|\nabla^{m_2}v_t\|)H_2(t) + \|\nabla^{k_1}f_1\|^2 + \|\nabla^{k_2}f_2\|^2 + C(R_0,\lambda_1). \tag{36}$$

Taking the scalar product of (1) in H with u_t, v_t, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\|u_t\|^2+\|v_t\|^2+\int_0^{\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2}M(\tau)\mathrm{d}\tau + 2J(u,v) - 2(f_1,u) - 2(f_2,v)\Big] + N_1(\|\nabla^{m_1}u\|^2)\|\nabla^{m_1}u_t\|^2 + N_2(\|\nabla^{m_2}v\|^2)\|\nabla^{m_2}v_t\|^2 = 0, \tag{37}$$

and integrating (37) in $\mathrm{d}t$ on $(0,t)$ derives

$$\int_0^t(\|\nabla^{m_1}u_t(\tau)\|^2+\|\nabla^{m_2}v_t(\tau)\|^2)\mathrm{d}\tau \leq \frac{1}{\min\{N_{10},N_{20}\}}\int_0^t(N_1(\|\nabla^{m_1}u(\tau)\|^2)\|\nabla^{m_1}u_t(\tau)\|^2 + N_2(\|\nabla^{m_2}v(\tau)\|^2)\|\nabla^{m_2}v_t(\tau)\|^2)\mathrm{d}\tau \leq \frac{1}{\min\{N_{10},N_{20}\}}(\|u_1\|^2+\|v_1\|^2 + \int_0^{\|\nabla^{m_1}u_0\|^2+\|\nabla^{m_2}v_0\|^2}M(\tau)\mathrm{d}\tau + 2J(u_0,v_0) - 2(f_1,u_0) - 2(f_2,v_0)) \leq C_{11}. \tag{38}$$

Then,

$$C_{10}\int_s^t((\|\nabla^{m_1}u_t(\tau)\|+\|\nabla^{m_2}v_t(\tau)\|))\mathrm{d}\tau \leq \frac{\sigma_2}{2}(t-s)+a \tag{39}$$

for $t > s \geq 0$ and some $a > 0$. Together with (36), (39) and Lemma 1, we can obtain that

$$H_2(t) \leq C_{12}H_2(0)\mathrm{e}^{-\frac{\sigma_2}{2}t} + C_{13}. \tag{40}$$

According to (A_1), we have

$$H_2(t) \geq \|\nabla^{k_1}y_1\|^2 + \|\nabla^{k_2}y_2\|^2 + M(\|\nabla^{m_1}u\|^2+\|\nabla^{m_2}v\|^2)\cdot(\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2) \geq C_{14}(\|\nabla^{k_1}y_1\|^2+\|\nabla^{k_2}y_2\|^2 + \|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2), \tag{41}$$

then,

$$\|(u,y_1,v,y_2)\|^2_{X_{k_1\times k_2}}=\|\nabla^{k_1}y_1\|^2+\|\nabla^{k_2}y_2\|^2+\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2\le\frac{C_{12}H_2(0)\mathrm{e}^{-\frac{\sigma_2}{2}t}+C_{13}}{C_{14}},\tag{42}$$

i.e.,

$$\overline{\lim_{t\to\infty}}\|(u,y_1,v,y_2)\|^2_{X_{k_1\times k_2}}\le R_{k_1\times k_2}.\tag{43}$$

Therefore, there exist positive constants $C(R_{k_1\times k_2})$ and $t_{k_1\times k_2}$ that when $t\ge t_{k_1\times k_2}$, (u,y_1,v,y_2) satisfies

$$\|(u,y_1,v,y_2)\|^2_{X_{k_1\times k_2}}=\|\nabla^{k_1}y_1\|^2+\|\nabla^{k_2}y_2\|^2+\|\nabla^{m_1+k_1}u\|^2+\|\nabla^{m_2+k_2}v\|^2\le C(R_{k_1\times k_2}),\quad k_1=1,2,\cdots,m_1,k_2=1,2,\cdots,m_2.\tag{44}$$

Thus, Lemma 4 is proved.

Proof of Theorem 1: According to previous findings [10] and the Faedo-Galerkin method, problems (1) to (3) have global solutions combining with Lemma 3 and Lemma 4.

Let u^1,v^1 and u^2,v^2 be two solutions of problems (1) to (3) corresponding to the same initial data, respectively, $w=u^1-u^2,z=v^1-v^2$. Then, (w,z) solves

$$\begin{cases}w_{tt}+\frac{1}{2}\sigma_{12}(t)(-\Delta)^{m_1}w_t+\frac{1}{2}\Phi_{12}(t)(-\Delta)^{m_1}w+G_1(u^1,u^2,v^1,v^2;t)=0,\\ z_{tt}+\frac{1}{2}\sigma_{34}(t)(-\Delta)^{m_2}z_t+\frac{1}{2}\Phi_{12}(t)(-\Delta)^{m_2}z+G_2(u^1,u^2,v^1,v^2;t)=0,\end{cases}\tag{45}$$

where $\sigma_{12}=\sigma_1(t)+\sigma_2(t)$, $\Phi_{12}(t)=\Phi_1(t)+\Phi_2(t)$, $\sigma_i(t)=N_1(\|\nabla^{m_1}u^i\|^2)$, $\Phi_i(t)=M(\|\nabla^{m_1}u^i\|^2+\|\nabla^{m_2}v^i\|^2)$,$i=1,2$, $\sigma_{34}=\sigma_3(t)+\sigma_4(t)$, $\sigma_j(t)=N_2(\|\nabla^{m_2}v^j\|^2)$, $j=3,4$, $G_1(u^1,u^2,v^1,v^2;t)=\frac{1}{2}\{[\sigma_1(t)-\sigma_2(t)](-\Delta)^{m_1}(u_t^1+u_t^2)+[\Phi_1(t)-\Phi_2(t)](-\Delta)^{m_1}(u^1+u^2)\}+g_1(u_1,v_1)-g_1(u_2,v_2)$, $G_2(u^1,u^2,v^1,v^2;t)=\frac{1}{2}\{[\sigma_3(t)-\sigma_4(t)](-\Delta)^{m_2}(v_t^1+v_t^2)+[\Phi_1(t)-\Phi_2(t)](-\Delta)^{m_2}(v^1+v^2)\}+g_2(u_1,v_1)-g_2(u_2,v_2)$. According to Lemma 3, $\sigma'_{12}\le C(R_0)(\|\nabla^{m_1}u_t^1\|+\|\nabla^{m_1}u_t^2\|),\sigma'_{34}\le C(R_0)(\|\nabla^{m_2}v_t^1\|+\|\nabla^{m_2}v_t^2\|)$.

Taking the scalar product of (45) in H with w_t,z_t, we can obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|w_t\|^2+\|z_t\|^2+\frac{1}{4}\Phi_0\cdot(\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2)]+\frac{1}{2}\sigma_{12}(t)\|\nabla^{m_1}w_t\|^2+\frac{1}{2}\sigma_{34}(t)\|\nabla^{m_2}z_t\|^2+(G_1(u^1,u^2,v^1,v^2;t),w_t)+(G_2(u^1,u^2,v^1,v^2;t),z_t)=0.\tag{46}$$

According to Lemma 3 and $(A1)$, $M_0\le M\le C(R_0,H_1(0))\equiv M_1$. When $\frac{\mathrm{d}}{\mathrm{d}t}(\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2)\ge0$, $\Phi_0=2M_0$; otherwise $\Phi_0=2M_1$.

Let $(G_1(u^1,u^2,v^1,v^2;t),w_t)=G_{11}+G_{12}+G_{13}$ and $(G_2(u^1,u^2,v^1,v^2;t),z_t)=G_{21}+G_{22}+G_{23}$, we have

$$G_{11}=\frac{1}{2}(\sigma_1(t)-\sigma_2(t))(\nabla^{m_1}(u_t^1+u_t^2),\nabla^{m_1}w_t)\le C(R_0)(\|\nabla^{m_1}u_t^1\|+\|\nabla^{m_1}u_t^2\|)\|\nabla^{m_1}w\|\|\nabla^{m_1}w_t\|\le\frac{\sigma_{120}}{8}\|\nabla^{m_1}w_t\|^2+\frac{2C(R_0)}{\sigma_{120}}(\|\nabla^{m_1}u_t^1\|^2+\|\nabla^{m_1}u_t^2\|^2)\|\nabla^{m_1}w\|^2,\tag{47}$$

$$G_{12}=\frac{1}{2}(\Phi_1(t)-\Phi_2(t))(\nabla^{m_1}(u^1+u^2),\nabla^{m_1}w_t)\le C(R_0)(\|\nabla^{m_1}w\|+\|\nabla^{m_2}z\|)\|\nabla^{m_1}w_t\|\le\frac{\sigma_{120}}{8}\|\nabla^{m_1}w_t\|^2+\frac{2C(R_0)}{\sigma_{120}}(\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2),\tag{48}$$

$$G_{13}=(g_1(u_1,v_1)-g_1(u_2,v_2),w_t)\le C(R_0)(\|w_t\|^2+\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2),\tag{49}$$

$$G_{21}=\frac{1}{2}(\sigma_3(t)-\sigma_4(t))(\nabla^{m_2}(v_t^1+v_t^2),\nabla^{m_2}z_t)\le C(R_0)(\|\nabla^{m_2}v_t^1\|+\|\nabla^{m_2}v_t^2\|)\cdot\|\nabla^{m_2}z\|\|\nabla^{m_2}z_t\|\le\frac{\sigma_{340}}{8}\|\nabla^{m_2}z_t\|^2+\frac{2C(R_0)}{\sigma_{340}}(\|\nabla^{m_2}v_t^1\|^2+\|\nabla^{m_2}v_t^2\|^2)\|\nabla^{m_2}z\|^2,\tag{50}$$

$$G_{22}=\frac{1}{2}(\Phi_1(t)-\Phi_2(t))(\nabla^{m_2}(v^1+v^2),\nabla^{m_2}z_t)\le C(R_0)(\|\nabla^{m_1}w\|+\|\nabla^{m_2}z\|)\|\nabla^{m_2}z_t\|\le\frac{\sigma_{340}}{8}\|\nabla^{m_2}z_t\|^2+\frac{2C(R_0)}{\sigma_{340}}(\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2),\tag{51}$$

$$G_{23}=(g_2(u_1,v_1)-g_2(u_2,v_2),z_t)\le C(R_0)(\|z_t\|^2+\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2),\tag{52}$$

where $\sigma_{120}=2N_{10},\sigma_{340}=2N_{20}$.

Inserting (46) to (52) into (45), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|w_t\|^2+\|z_t\|^2+\frac{1}{4}\Phi_0\cdot(\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2)]\le C_{14}(1+\|\nabla^{m_2}v_t^1\|^2+\|\nabla^{m_2}v_t^2\|^2)\cdot[\|w_t\|^2+\|z_t\|^2+\frac{1}{4}\Phi_0\cdot(\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2)].\tag{53}$$

Solving this differential inequality yields

$$[\|w_t\|^2+\|z_t\|^2+\frac{1}{4}\Phi_0\cdot(\|\nabla^{m_1}w\|^2+\|\nabla^{m_2}z\|^2)]\le[\|w_1\|^2+\|z_1\|^2+\frac{1}{4}\Phi_0\cdot(\|\nabla^{m_1}w_0\|^2+\|\nabla^{m_2}z_0\|^2)]\cdot\exp(\int_0^t C_{15}(1+\|\nabla^{m_2}v_t^1\|^2+\|\nabla^{m_2}v_t^2\|^2)\mathrm{d}s).\tag{54}$$

Thus, the uniqueness of the solution is proved.

Therefore, problems (1) to (3) possess a unique solution u,v. Theorem 1 is proved.

According to Theorem 1, we define a nonlinear operator $\{S(t)\}_{t\ge0}$ on space $X_{0\times0}$: $S(t)(u_0,u_1,v_0,v_1)=(u,u_t,v,v_t)$, for all $t\ge0$. Theorem 1 shows that $\{S(t)\}_{t\ge0}$ compose a continuous semi-group in $X_{0\times0}$. Before proving the family of global attractors, we first give their definition.

Definition 1. Let X_0 be a Banach space and $\{S(t)\}_{t\ge0}$ be a continuous operator semi-group, if there exists a compact set $A_{k_1\times k_2}$ satisfying

(i) Invariance: all $A_{k_1\times k_2}$ are invariant sets under the action of semi-group $\{S(t)\}_{t\ge0}$,

$$S(t)A_{k_1\times k_2}=A_{k_1\times k_2};\forall t\ge0;$$

(ii) Attractiveness: all $A_{k_1\times k_2}$ attract all bounded sets in X_0, i.e., for any bounded $B\subset X_0$,

$$dist(S(t)B,A_{k_1\times k_2})=\sup_{x\in B}\inf_{y\in A_{k_1\times k_2}}\|S(t)x-y\|_{X_0}\to0,t\to\infty.$$

In particular, when $t \to \infty$, all trajectories $S(t)u_0$ from u_0 converge to $A_{k_1\times k_2}$, i.e.,

$$dist(S(t)u_0, A_{k_1\times k_2}) \to 0, t \to \infty.$$

Then, compact set A_k is the global attractors of semi-group $\{S(t)\}_{t\geq 0}$. Let $\mathcal{A} = \{A_{k_1\times k_2} \subset X_0 : k_1 = 1, 2, \cdots, m_1, k_2 = 1, 2, \cdots, m_2\}$ be a subset family in X_0, $\mathcal{A}$ is the global attractor family in X_0.

Proof of Theorem 2: According to Lemma 3, for all $R_0 > 0$, $\|(u_0, u_1, v_0, v_1)\|_{X_{0\times 0}} \leq R_0$. Thus,

$$\|S(t)(u_0, u_1, v_0, v_1)\|^2_{X_{0\times 0}} = \|u\|^2_{V_{m_1}} + \|u_t\|^2_{V_0} + \|v\|^2_{V_{m_2}} + \|v_t\|^2_{V_0} \leq C(R_0),$$

indicating that $\{S(t)\}_{t\geq 0}$ are uniformly bounded in $X_{0\times 0}$; further, $B_{k_1\times k_2,0} = \{(u, u_t, v, v_t) \in X_{k_1\times k_2} : \|(u, u_t, v, v_t)\|^2_{X_{k_1\times k_2}} = \|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{k_1}u_t\|^2 + \|\nabla^{m_2+k_2}v\|^2 + \|\nabla^{k_2}v_t\|^2 \leq C(R_0) + C(R_{k_1\times k_2})\}$ are bounded absorbing sets of semi-group $\{S(t)\}_{t\geq 0}$ in $X_{0\times 0}$; because $X_{k_1\times k_2} \hookrightarrow\hookrightarrow X_{0\times 0}$ are compactly embedding, i.e., the bounded sets in $X_{k_1\times k_2}$ are compact sets in $X_{0\times 0}$, solution semi-group $\{S(t)\}_{t\geq 0}$ is a fully continuous operator. To sum up, we obtained the global attractor family $\mathcal{A} = \{A_{k_1\times k_2}\}$ of solution semi-group $\{S(t)\}_{t\geq 0}$ in $X_{0\times 0}$, and

$$A_{k_1\times k_2} = \omega(B_{k_1\times k_2,0}) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} S(t)B_{k_1\times k_2,0}},$$
$$A_{k_1\times k_2} \subset X_{0\times 0}, k_1 = 1, 2, \cdots, m_1, k_2 = 1, 2, \cdots, m_2.$$

Theorem 2 is proved.

Note 1 Lemma 4 and Theorem 2 show that bounded absorbing sets $B_{k_1\times k_2,0} = \{(u, u_t, v, v_t) \in X_{k_1\times k_2} : \|(u, u_t, v, v_t)\|^2_{X_{k_1\times k_2}} = \|\nabla^{m_1+k_1}u\|^2 + \|\nabla^{k_1}u_t\|^2 + \|\nabla^{m_2+k_2}v\|^2 + \|\nabla^{k_2}v_t\|^2 \leq C(R_0) + C(R_{k_1\times k_2})\}$ are compact bounded absorbing sets in $X_{0\times 0}$. Therefore, based on condition 3 in Lemma 2, the operator semi-group $\{S(t)\}_{t\geq 0}$ only needs to be a continuous operator. According to Theorem 1, semi-group $\{S(t)\}_{t\geq 0}$ is already a continuous semi-group. Thus, the global attractor family $\mathcal{A} = \{A_{k_1\times k_2}\}$ of problems (1) to (3) in $X_{0\times 0}$ can also be obtained.

IV. Summary and Prospect

In this paper, we studied higher-order coupled Kirchhoff systems with nonlinear strong damping. For the first time, we systematically defined the global attractor family of problems (1) to (3) and proved its existence. The findings enriched the related conclusions of higher-order coupled Kirchhoff models and laid a theoretical foundation for future practical applications.

Despite the defined and proven global attractor family of the higher-order coupled Kirchhoff system, many questions concerning such models still require further investigation.

1. The higher-order coupled Kirchhoff system in this paper is autonomous, while the relatively complex non-autonomous coupled Kirchhoff systems have not been studied. Thus, it is very meaningful to study the asymptotic behaviors of such systems;
2. This paper mainly studies the global attractor family of dynamic systems, while many other properties are not explored, such as the dimension estimation, the exponential attractor family, and the inertial manifold family. The scarce relevant theoretical results warrant further research efforts.

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